# One-Face Shortest Disjoint Paths with a Deviation Terminal

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**Abstract**

For an undirected graph $G$ and distinct vertices $s_1, t_1, \dots, s_k, t_k$ called terminals, the *shortest $k$-disjoint paths problem* asks for $k$ pairwise vertex-disjoint paths $P_1, \dots, P_k$ such that $P_i$ connects $s_i$ and $t_i$ for $i = 1, \dots, k$ and the sum of their lengths is minimized. This problem is a natural optimization version of the well-known $k$-disjoint paths problem, and its polynomial solvability is widely open. One of the best results on the shortest $k$-disjoint paths problem is due to Datta et al. [9], who present a polynomial-time algorithm for the case when $G$ is planar and all the terminals are on one face. In this paper, we extend this result by giving a polynomial-time randomized algorithm for the case when all the terminals except one are on some face of $G$. In our algorithm, we combine the arguments of Datta et al. with some results on the shortest disjoint $(A+B)$-paths problem shown by Hirai and Namba [15]. To this end, we present a non-trivial bijection between $k$ disjoint paths and disjoint $(A+B)$-paths, which is a key technical contribution of this paper.

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## 1 Introduction

### 1.1 Shortest Disjoint Paths Problem

The *$k$-disjoint paths problem* is a well-studied and important problem in algorithmic graph theory and combinatorial optimization. In the problem, we are given an undirected graph $G = (V, E)$ and $2k$ distinct vertices $s_1, t_1, \dots, s_k, t_k$, called *terminals*, and the objective is to find $k$ pairwise vertex-disjoint paths $P_1, \dots, P_k$ such that $P_i$ connects $s_i$ and $t_i$ for $i = 1, \dots, k$, if they exist. This problem has attracted attention since 1980s because of its applications to practical problems such as network routing [24, 34] and VLSI-design [13, 19].

The main focus in this topic is the polynomial solvability of the problem. When $k$ is a part of the input, the $k$-disjoint paths problem is shown to be NP-hard by Karp [16], and it remains NP-hard even if the input graph is restricted to be planar [22]. When $k = 2$, elementary polynomial-time algorithms are presented in [32, 33, 36], whereas the directed variant is NP-hard [12]. For the case when the graph is undirected and $k$ is a fixed constant, Robertson and Seymour's graph minor theory gives a polynomial-time algorithm [29], which is one of the biggest achievements in this area.

An interesting special case of the disjoint paths problem is when the input graph is planar or embedded on a fixed surface. For example, when $G$ is planar and all the terminals are on one face or two faces, a structural characterization and a polynomial algorithm are given in [27]. The $k$-disjoint paths problem on a fixed surface is solved in [28]. In the graph minor series, these special cases play important roles to obtain the algorithm for general graphs [29]. A faster algorithm for the planar case is presented in [1]. The directed variant of the problem can be solved in polynomial time if the input digraph is planar and $k$ is a fixed constant [8, 30].

The *shortest $k$-disjoint paths problem* is a natural optimization version of the $k$-disjoint paths problem. In the problem, we are given an undirected graph $G = (V, E)$ and terminals $s_1, t_1, \dots, s_k, t_k$, and the objective is to find $k$ pairwise vertex-disjoint paths $P_1, \dots, P_k$ such that $P_i$ connects $s_i$ and $t_i$ for $i = 1, \dots, k$ and the sum of their lengths is minimized. Here, the length of a path is defined as the number of edges in it. When $k$ is a part of the input, since the search problem is NP-hard, so is the optimization problem. Although the problem setting is natural and easy to understand, surprisingly, the polynomial solvability of the shortest $k$-disjoint paths problem is widely open when $k$ is a fixed constant. For the shortest 2-disjoint paths problem, Björklund and Husfeldt [5] present a randomized polynomial-time algorithm based on an algebraic approach, while a deterministic polynomial-time algorithm is still unknown. They also give a deterministic polynomial-time algorithm for counting optimal solutions of the shortest 2-disjoint paths problem if the input graph is cubic and planar [4].

Several positive results are obtained if $G$ is planar and the configuration of the terminals satisfies certain conditions. Colin de Verdière and Schrijver [7] devise an $O(kn \log n)$ time algorithm for the shortest $k$-disjoint paths problem when all sources $s_1, \dots, s_k$ are on one face and all sinks $t_1, \dots, t_k$ are on another face, where $n = |V|$. Kobayashi and Sommer [18] give a polynomial-time algorithm for the case when $k = 2$ and the terminals are on two faces in an arbitrary way. The problem is not easy even when all the terminals are on one face, which we call One-Face Shortest $k$-disjoint paths problem. Polynomial-time algorithms are presented for the cases when $k = 3$ [18] and $k = 4$ [11]. Borradaile et al. [6] give an $O(kn^5)$ time algorithm for the case when $s_1, t_1, s_2, t_2, \dots, s_k, t_k$ are on the boundary of some face counter-clockwise in this order. Datta et al. [9] generalize these results by presenting a polynomial-time algorithm for One-Face Shortest $k$-disjoint paths problem when $k$ is a fixed constant. Their idea is to compute a polynomial associated with $k$ disjoint paths by using determinants of several polynomial matrices, and a similar argument will be used in this paper.

## 1.2 Our Contribution

As described in the previous subsection, for the shortest $k$-disjoint paths problem, polynomial-time algorithms are devised only for very restricted cases. Among them, one of the strongest results is a polynomial-time algorithm for One-Face Shortest $k$-disjoint paths problem due to Datta et al. [9]. In this paper, we extend this result by dealing with the case when all the terminals except one are on some face of $G$, which we call One-Face Shortest $k$-Disjoint Paths Problem with a Deviation Terminal.

**One-Face Shortest $k$-Disjoint Paths Problem with a Deviation Terminal**

**Input:** A planar graph $G = (V, E)$ and distinct vertices $s_1, t_1, \dots, s_k, t_k$ called terminals such that $2k - 1$ of them are on the boundary of some face of $G$.

**Output:** Pairwise vertex-disjoint paths $P_1, \dots, P_k$ such that $P_i$ connects $s_i$ and $t_i$ for $i = 1, \dots, k$ and the sum of their lengths is minimized (if they exist).

The main contribution of this paper is to present a randomized polynomial-time algorithm for this problem.

▶ **Theorem 1.** *For a fixed positive integer $k$,* One-Face Shortest $k$-Disjoint Paths Problem with a Deviation Terminal *can be solved in randomized polynomial time.*

Although our problem looks similar to One-Face Shortest $k$-disjoint paths problem, there is a big gap between these two problems in the following sense. In One-Face Shortest $k$-disjoint paths problem, if $s_1, s_2, \dots, s_k, t_k, \dots, t_2$, and $t_1$ are on the boundary of a face in this order, then the problem can be solved easily (e.g., by a minimum cost flow algorithm or by a determinant computation). This special case acts as a "base case" in the algorithm of Datta et al. [9]. In contrast, in One-Face Shortest $k$-disjoint paths problem with a Deviation Terminal, such an easy "base case" does not exist. This difference makes the problem quite difficult and interesting, and reinforces the importance of Theorem 1.

We overcome the above difficulty by giving a non-trivial bijection between $k$ disjoint paths and disjoint $(A+B)$-paths introduced by Hirai and Namba [15] (see Section 2.2 for disjoint $(A+B)$-paths), which is a key ingredient in our algorithm. By combining several polynomials associated with disjoint $(A+B)$-paths, we compute a polynomial associated with the desired $k$ disjoint paths in a similar way to [9], which enables us to compute an optimal solution.

## 1.3 Related Work

The *$k$-disjoint shortest paths problem* is another variant of the $k$-disjoint paths problem that is actively studied recently. In the problem, an instance consists of a graph and terminals $s_1, t_1, \dots, s_k, t_k$ in the same way as the $k$-disjoint paths problem. The objective is to find pairwise vertex-disjoint paths $P_1, \dots, P_k$ such that $P_i$ is a *shortest* path between $s_i$ and $t_i$ for $i = 1, \dots, k$. It is obvious that if each $P_i$ is a shortest path, then the total length is minimized. Therefore, algorithms for the shortest $k$-disjoint paths problem can solve the $k$-disjoint shortest paths problem. Eilam-Tzoreff [10] introduces the $k$-disjoint shortest paths problem and devises a polynomial-time algorithm for the case of $k = 2$. For the case when $k$ is a fixed constant, polynomial-time algorithms are recently given by Lochet [21] and Bentert et al. [2]. When each edge has a non-negative length, Gottschau et al. [14] and Kobayashi and Sako [17] independently give polynomial-time algorithms for the 2-disjoint shortest paths problem. The polynomial solvability of the $k$-disjoint shortest paths problem with non-negative edge-length is still open for fixed $k$. For the directed variant with positive edge-length, Bérczi and Kobayashi [3] present a polynomial-time algorithm when $k = 2$.

A special case when (almost) all the terminals are on the boundary of some face has attracted attention also in the context of the edge-disjoint paths problem and the multicommodity flow problem. The most famous example will be Okamura-Seymour Theorem [26]. Suppose that all the terminals are on the boundary of some face of a planar graph and assume that the Euler condition holds. Then, Okamura-Seymour Theorem states that the existence of edge-disjoint paths connecting the terminal pairs can be characterized by the cut condition. This theorem is generalized by Okamura [25] to the case when the terminals are on two faces. A node-capacitated variant of Okamura-Seymour Theorem is studied by Lee et al. [20]. See [31, Chapter 74] for more related results.

## 1.4 Organization

The remaining of this paper is organized as follows. In Section 2, we give notation and describe some results on the shortest disjoint $(A+B)$-paths problem by Hirai and Namba [15]. In Sections 3.1 and 3.2, we show a bijection between $k$ disjoint paths and disjoint $(A+B)$-paths without giving a proof of a key lemma (Lemma 7). By using this bijection, in Section 3.3, we present a randomized algorithm for One-Face Shortest $k$-Disjoint Paths Problem with a Deviation Terminal and prove Theorem 1. The proof of Lemma 7 is given in Section 4.

# 2 Preliminaries

## 2.1 Basic Notation

For a positive integer $p$, we denote $[p] = \{1, \ldots, p\}$. For a finite set $X$, let $\#X$ and $|X|$ denote the cardinality of $X$, which is also called the *size* of $X$. In this paper, all graphs and paths are undirected unless stated otherwise. The *length* of a path in a graph is defined as the number of edges in it. For a collection $\mathcal{P}$ of paths, let $w(\mathcal{P})$ denote the total length of the paths in $\mathcal{P}$.

Let $k$ be a fixed positive integer. Suppose we are given an instance of One-Face Shortest $k$-Disjoint Paths Problem with a Deviation Terminal, which consists of a planar graph $G = (V, E)$ and terminals $s_1, t_1, \ldots, s_k, t_k$. Let $T$ denote the set of all the terminals. Suppose that all the terminals except one terminal, say $s_1$, are on the boundary of some face $F$ of $G$. Without loss of generality, we may assume that $G$ is 2-connected, since otherwise we can easily reduce to the 2-connected case. Then, the boundary of $F$ forms a cycle.

For a finite set $X$ of size $2p$, a partition of $X$ into $p$ disjoint sets of size two is called a *pairing of* $X$. That is, a pairing $M$ of $X$ is of the form $M = \{\{x_1, y_1\}, \ldots, \{x_p, y_p\}\}$, where $X = \{x_1, y_1, \ldots, x_p, y_p\}$. Although each element in $M$ is an unordered pair, by following the convention, each pair $\{x_i, y_i\}$ is denoted by $(x_i, y_i)$ in this paper. Thus, $(x_i, y_i)$ is identified with $(y_i, x_i)$. A pairing of $T$ is simply called a *pairing* if $T$ is clear from the context. The pairing $M^* = \{(s_i, t_i) \mid i \in [k]\}$ of $T$ is referred to as the *input pairing*.

In our argument, the ordering of the terminals along the boundary of $F$ is more important than the input pairing. Hence, we rename the terminals so that $T = \{u^*, u_0, u_1, \ldots u_{2k-2}\}$, $u^* = s_1$, $u_0 = t_1$, and $u_0, u_1, \ldots, u_{2k-2}$ lie on the boundary of $F$ counter-clockwise in this order (Figure 1). Let $I = \{0, 1, \ldots, 2k-2\}$ be the set of the indices of the vertices incident to $F$.

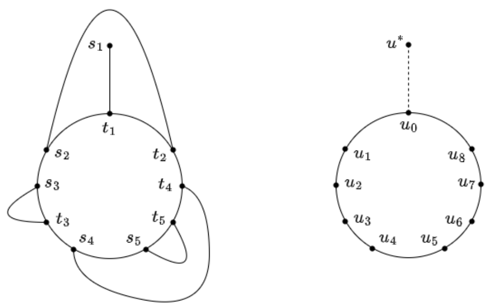


**Figure 1** Renaming the terminals.

## 2.2 The Shortest Disjoint $(A + B)$-Paths Problem

Hirai and Namba [15] introduce the *shortest disjoint* $(A+B)$*-paths problem* as a generalization of the shortest 2-disjoint paths problem. In the shortest disjoint $(A + B)$-paths problem, we are given a graph $G = (V, E)$ and two disjoint terminal sets $A, B \subseteq V$ of even size, and the task is to find $|A|/2 + |B|/2$ pairwise vertex-disjoint paths with endpoints both in $A$ or both in $B$ such that the sum of their lengths is minimized. Note that each feasible solution is called *disjoint* $(A + B)$*-paths*. For this problem, Hirai and Namba [15] design a randomized algorithm running in $|V|^{O(|A|+|B|)}$ time, which is polynomial if $|A| + |B|$ is fixed. For the case when the input graph is cubic and planar, Björklund and Husfeldt [4] give a deterministic polynomial-time algorithm.

We now describe the outline of the algorithm in [15], because some of their results will be used in our argument later. In their algorithm, they construct a univariate polynomial matrix from the instance, and compute its *hafnian*. Although computing the hafnian of a matrix is hard in general, they establish a technique to compute the hafnian modulo $2^\tau$ for a fixed positive integer $\tau$ based on a similar argument to [5]. Note that the hafnian of a $2n \times 2n$ symmetric matrix $S = (s_{ij})$ is defined as

$$\operatorname{haf} S = \sum_{M \in \mathcal{M}} \prod_{(i,j) \in M} s_{ij},$$

where $\mathcal{M}$ is the set of all pairings of $\{1, 2, 3, \ldots, 2n\}$. The key theorems in [15] are formally stated as follows.

▶ **Theorem 2** (Hirai and Namba [15, Lemma 2.4]). *For any instance of the shortest disjoint $(A + B)$-paths problem, in polynomial time, we can construct a matrix $S$ such that each element is a polynomial in $x$ with integer coefficients and*

$$\operatorname{haf} S = \sum_{\mathcal{P}} \pm 2^{\frac{|A|}{2} + \frac{|B|}{2}} x^{w(\mathcal{P})}(1 + x f_{\mathcal{P}}(x)),$$

*where $\mathcal{P}$ ranges over all disjoint $(A + B)$-paths, $f_{\mathcal{P}}(x)$ is some polynomial with integer coefficients, and the sign is determined by $A, B$, and $\mathcal{P}$.*

Note that $f_{\mathcal{P}}(x)$ depends only on $\mathcal{P}$, and it does not depend on how the end vertices of $\mathcal{P}$ are partitioned into $A$ and $B$. Recall that $w(\mathcal{P})$ is the number of edges in the paths in $\mathcal{P}$.

▶ **Theorem 3** (Hirai and Namba [15, Theorem 2.1]). *Let $\tau$ be a fixed positive integer. For a given univariate matrix with integer coefficients, we can compute its hafnian modulo $2^\tau$ in polynomial time.*

Note that "hafnian modulo $2^\tau$" means that the integer coefficients of the hafnian are modulo $2^\tau$. In their algorithm, after perturbing the length of edges to obtain an instance with a unique optimal solution, we construct a matrix $S$ as in Theorem 2. Then, we compute the hafnian of $S$ modulo $2^{\frac{|A|}{2} + \frac{|B|}{2} + 1}$ by Theorem 3. We can see that the lowest degree term of the hafnian corresponds to the shortest disjoint $(A + B)$-paths under the assumption that the instance has a unique optimal solution. Note that the randomization is required only in the perturbation step, and the algorithms in Theorems 2 and 3 are deterministic.

## 3 One-Face Shortest $k$-Disjoint Paths with a Deviation Terminal

In this section, we present our algorithm for One-Face Shortest $k$-disjoint paths problem with a Deviation Terminal. In our algorithm, we consider several partitions $(A, B)$ of the terminals $T$ and compute polynomials associated with disjoint $(A + B)$-paths by using Theorems 2 and 3. By using these polynomials, we compute a polynomial whose lowest degree term corresponds to the shortest $k$ disjoint paths. A key ingredient in our argument is to make a correspondence between a partition $(A, B)$ and a pairing of $T$. More precisely, we show that there is a bijection between a *good* partition and a *feasible* pairing including $(u^*, u_0)$, which will be defined in Section 3.1.

### 3.1 Good Partition

Let $M$ be a pairing of $T$. We say that $M$ is *infeasible* if there exist distinct pairs $(u_i, u_j), (u_{i'}, u_{j'}) \in M$ such that $i < i' < j < j'$, that is, $u_i, u_{i'}, u_j$, and $u_{j'}$ are on the boundary of $F$ counter-clockwise in this order. Otherwise, $M$ is called *feasible*. Observe that, for a pairing $M$ of $T$, if $G$ contains $k$ disjoint paths connecting the pairs in $M$, then $M$ is feasible. For a partition $(A, B)$ of $T$, a pairing $M$ of $T$ is called $(A, B)$-*compatible* if, for any $(s, t) \in M$, both $s$ and $t$ belong to the same set, either $A$ or $B$.

In our algorithm, for several partitions $(A, B)$ of $T$, we compute polynomials associated with disjoint $(A + B)$-paths by using Theorems 2 and 3. We focus on disjoint $(A + B)$-paths that contain a $u^*$-$u_0$ path, and so we consider partitions $(A, B)$ of $T$ such that every $(A, B)$-compatible pairing contains $(u^*, u_0)$. Such a partition is called *good*, which is formally defined as follows.

▶ **Definition 4.** *A partition $(A, B)$ of $T$ is said to be* good *if*

- $u^*, u_0 \in A$,
- *there exists a feasible $(A, B)$-compatible pairing $M$ with $(u^*, u_0) \in M$, and*
- *there exists no feasible $(A, B)$-compatible pairing $M$ with $(u^*, u_i) \in M$ for $i \in [2k - 2]$.*

## 3.2 Bijection Between Good Partitions and Feasible Pairings

In this subsection, we give a bijection between good partitions of $T$ and feasible pairings including $(u^*, u_0)$. For this purpose, we show that both good partitions and feasible pairings are related to *ballot sequences*. A sequence $(f(1), \ldots, f(2k))$ of integers is called a $k$-*ballot sequence* if $f(i) \in \{+1, -1\}$ for $i \in [2k]$, $\sum_{i=1}^{\ell} f(i) \geq 0$ for $\ell \in [2k - 1]$, and $\sum_{i=1}^{2k} f(i) = 0$. It is well-known that the number of $k$-ballot sequences is equal to the $k$-th Catalan number $c_k = \frac{1}{k+1}\binom{2k}{k}$ (see [35]).

First, we give a bijection between $(k - 1)$-ballot sequences and feasible pairings including $(u^*, u_0)$. For a $(k - 1)$-ballot sequence $(f(1), f(2), \ldots, f(2k - 2))$, we construct the corresponding pairing $g_1(f(1), f(2), \ldots, f(2k - 2))$ as follows.

▶ **Definition 5.** *Let $(f(1), f(2), \ldots, f(2k - 2))$ be a $(k - 1)$-ballot sequence. We initialize $M$ as $M = \{(u^*, u_0)\}$. For every $i \in [2k - 2]$ with $f(i) = +1$, let $i' \in [2k - 2]$ be the minimum index such that $\sum_{j=1}^{i-1} f(j) = \sum_{j=1}^{i'} f(j)$ and $i < i'$, and add $(u_i, u_{i'})$ to $M$. Then, define $g_1(f(1), f(2), \ldots, f(2k - 2)) = M$.*

This construction is the same as a well-known bijection between ballot sequences and *legal sequences of parentheses* (see [35]). Since a legal sequence of parentheses can be identified with a feasible pairing including $(u^*, u_0)$, we obtain the following lemma.

▶ **Lemma 6** (see [35]). *Mapping $g_1$ in Definition 5 is a bijection from $(k-1)$-ballot sequences to feasible pairings including $(u^*, u_0)$.*

We next construct a bijection from good partitions of $T$ to $(k - 1)$-ballot sequences. For a partition $(A, B)$ of $T$, define

$$f_{(A,B)}(i) = \begin{cases} -1 & \text{if } u_i \in A \text{ and } i \text{ is odd,} \\ +1 & \text{if } u_i \in A \text{ and } i \text{ is even,} \\ +1 & \text{if } u_i \in B \text{ and } i \text{ is odd,} \\ -1 & \text{if } u_i \in B \text{ and } i \text{ is even} \end{cases}$$

for $i \in [2k - 2]$. We show the following lemma, whose proof is given in Section 4.

▶ **Lemma 7.** *Let $(A, B)$ be a partition of $T$ with $u^*, u_0 \in A$. Then, $(A, B)$ is good if and only if $(f_{(A,B)}(1), \ldots, f_{(A,B)}(2k - 2))$ is a $(k - 1)$-ballot sequence.*

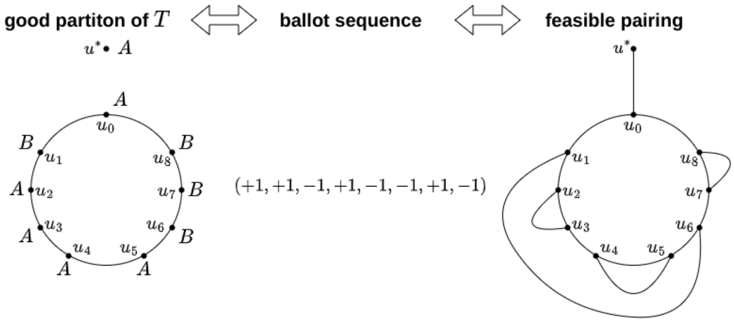


**Figure 2** Correspondence among good partitions, ballot sequences, and feasible pairings.

By this lemma, the construction of $f_{(A,B)}$ defines a bijection $g_2$ from good partitions of $T$ to $(k-1)$-ballot sequences. Therefore, by composing $g_1$ and $g_2$, we obtain a bijection $g$ from good partitions to feasible pairings including $(u^*, u_0)$; see Figure 2.

Since the number of ballot sequences is equal to the Catalan number, Lemma 7 implies the following result, which is of independent interest.

▶ **Corollary 8.** *The number of good partitions of $2k$ terminals $T$ is equal to the Catalan number $c_{k-1} = \frac{1}{k}\binom{2k-2}{k-1}$.*

We next show a property of bijection $g$, which plays an important role in our algorithm. For a pairing $M$ of $T$ with $(u^*, u_0) \in M$, let $d(M) = \sum_{(u_i,u_j)\in M\setminus\{(u^*,u_0)\}} |i-j|$.

▶ **Lemma 9.** *For any good partition $(A, B)$ of $T$, $g(A, B)$ is a feasible $(A, B)$-compatible pairing. Furthermore, among all feasible $(A, B)$-compatible pairings $M$, $g(A, B)$ is a unique minimizer of $d(M)$.*

**Proof.** By Lemmas 6 and 7, $g(A, B)$ is a feasible pairing with $(u^*, u_0) \in g(A, B)$. We first show that $g(A, B)$ is $(A, B)$-compatible. Suppose that $(u_i, u_{i'}) \in g(A, B)$ with $i < i'$. By the construction of bijection $g_1$ in Definition 5, we see that $i \in [2k-3]$ satisfies $f_{(A,B)}(i) = +1$, and $i' \in [2k-2]$ is the minimum index such that $\sum_{j=1}^{i-1} f_{(A,B)}(j) = \sum_{j=1}^{i'} f_{(A,B)}(j)$ and $i < i'$. Then, $f_{(A,B)}(i') = -1$ by the minimality of $i'$, and $i - 1 \equiv i' \pmod 2$. Since $f_{(A,B)}(i) \neq f_{(A,B)}(i')$ and $i \not\equiv i' \pmod 2$, the definition of $f_{(A,B)}$ shows that both $u_i$ and $u_{i'}$ belong to the same set, either $A$ or $B$. Therefore, $g(A, B)$ is $(A, B)$-compatible.

To show the latter half of the lemma, let $M$ be a feasible $(A, B)$-compatible pairing. Since $(A, B)$ is a good partition of $T$, we obtain $(u^*, u_0) \in M$. If $(u_i, u_{i'}) \in M$ for $i, i' \in [2k-2]$, then $i$ and $i'$ have different parities by the feasibility of $M$. Since both $u_i$ and $u_{i'}$ belong to the same set, either $A$ or $B$, the definition of $f_{(A,B)}$ shows that

$$f_{(A,B)}(i) + f_{(A,B)}(i') = 0. \tag{1}$$

Define $I^+ = \{i \in [2k-2] \mid f_{(A,B)}(i) = +1\}$ and $I^- = \{i \in [2k-2] \mid f_{(A,B)}(i) = -1\}$. Then, (1) means that each pair in $M \setminus \{(u^*, u_0)\}$ consists of $u_i$ with $i \in I^+$ and $u_{i'}$ with $i' \in I^-$. Furthermore, for any $(u_i, u_{i'}) \in g(A, B)$ with $i < i'$, the definition of $g_1$ shows that $i \in I^+$ and $i' \in I^-$. Therefore, we obtain

$$d(M) = \sum_{(u_i,u_{i'})\in M} |i'-i| \geq \sum_{\substack{(u_i,u_{i'})\in M \\ i\in I^+,\ i'\in I^-}} (i'-i) = \sum_{i'\in I^-} i' - \sum_{i\in I^+} i = d(g(A, B)). \tag{2}$$

This shows that $g(A, B)$ is a minimizer of $d(M)$.

To show the uniqueness of the minimizer, suppose that $M \neq g(A,B)$. It suffices to show that the inequality in (2) is strict, that is, there exists $(u_i, u_{i'}) \in M$ such that $i \in I^+$, $i' \in I^-$, and $i > i'$. For $(u_i, u_{i'}) \in M$, since the vertices between $u_i$ and $u_{i'}$ are partitioned into pairs in $M$ and (1) holds for any pair in $M \setminus \{(u^*, u_0)\}$, we obtain

$$\sum_{j=1}^{i-1} f_{(A,B)}(j) = \sum_{j=1}^{i'} f_{(A,B)}(j). \tag{3}$$

Since $M \neq g(A,B)$, there exist $i_0 \in I^+$ and $i'_0 \in I^-$ with $(u_{i_0}, u_{i'_0}) \in M \setminus g(A,B)$. If $i_0 > i'_0$, then we immediately conclude that the inequality in (2) is strict. In what follows, suppose that $i_0 < i'_0$. Let $i'_1 \in I^-$ be the minimum index such that $\sum_{j=1}^{i_0-1} f_{(A,B)}(j) = \sum_{j=1}^{i'_1} f_{(A,B)}(j)$ and $i_0 < i'_1$, that is, $(i_0, i'_1) \in g(A,B)$. By (3) and by the minimality of $i'_1$, we obtain $i'_1 < i'_0$. We now consider the pair $(u_{i_1}, u_{i'_1}) \in M$ containing $u_{i'_1}$, where $i_1 \in I^+$. Since $M$ is feasible and $i_0 < i'_1 < i'_0$, we obtain $i_0 < i_1$. Furthermore, since $\sum_{j=1}^{i_1-1} f_{(A,B)}(j) = \sum_{j=1}^{i'_1} f_{(A,B)}(j)$ by (3), the minimality of $i'_1$ shows that $i_1$ is not contained in the interval between $i_0$ and $i'_1$. Therefore, we obtain $i'_1 < i_1$, and hence the inequality in (2) is strict. This shows that $g(A,B)$ is a unique minimizer of $d(M)$. ◀

## 3.3 Algorithm

We are now ready to describe our algorithm, which is based on an idea similar to [9]. In this subsection, since all computations of polynomials are done modulo $2^{k+1}$, we regard polynomials with integer coefficients as elements in $\mathbb{Z}_{2^{k+1}}[x]$. For a set $\mathcal{P}$ of paths, define $f_{\mathcal{P}}(x)$ as in Theorem 2. For a feasible pairing $M$ of $T$, define a polynomial $\Phi(M)$ by

$$\Phi(M) = \sum_{\mathcal{P}} 2^k x^{w(\mathcal{P})}(1 + x f_{\mathcal{P}}(x)),$$

where $\mathcal{P}$ ranges over all sets of $k$ disjoint paths connecting the pairs in $M$.

Let $\mathcal{M}$ be the set of all feasible pairings including $(u^*, u_0)$. For a good partition $(A,B)$ of $T$, let $\mathcal{M}_{(A,B)}$ be the set of all feasible $(A,B)$-compatible pairings. Since $(A,B)$ is a good partition, we see that $\mathcal{M}_{(A,B)} \subseteq \mathcal{M}$. With these notations, Theorem 2 shows that we can obtain a polynomial matrix $S_{(A,B)}$ such that

$$\operatorname{haf} S_{(A,B)} = \sum_{M \in \mathcal{M}_{(A,B)}} \Phi(M),$$

where we note that the computation is done modulo $2^{k+1}$. Since $g(A,B)$ is a unique minimizer of $d(M)$ in $\mathcal{M}_{(A,B)}$ by Lemma 9, this shows that

$$\Phi(g(A,B)) = \operatorname{haf} S_{(A,B)} - \sum_{\substack{M \in \mathcal{M}_{(A,B)} \\ d(M) > d(g(A,B))}} \Phi(M) \tag{4}$$

for a good partition $(A,B)$ of $T$. By using this equation, for every $M \in \mathcal{M}$, we can compute $\Phi(M)$ in the decreasing order of $d(M)$ as follows. Note that the number of pairing is bounded by a fixed constant as $k$ is fixed, and hence $\mathcal{M}_{(A,B)}$ can be enumerated in a brute-force way.

First, suppose that $M \in \mathcal{M}$ is a maximizer of $d(M)$. Since $g$ is a bijection, there exists a good partition $(A,B)$ of $T$ with $g(A,B) = M$. Then, (4) implies $\Phi(g(A,B)) = \operatorname{haf} S_{(A,B)}$, and hence this value can be computed by Theorem 3. Recall again that the computation is done modulo $2^{k+1}$.

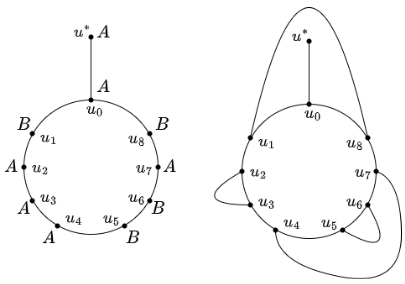


Figure 3 Partition $(A_1, B_1)$ and $M_1$.

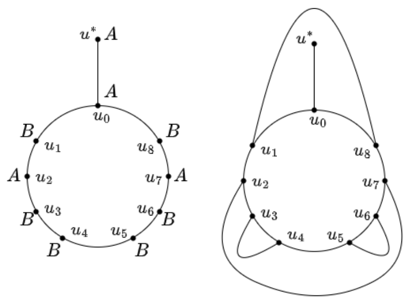


Figure 4 Partition $(A_2, B_2)$ and $M_2$.

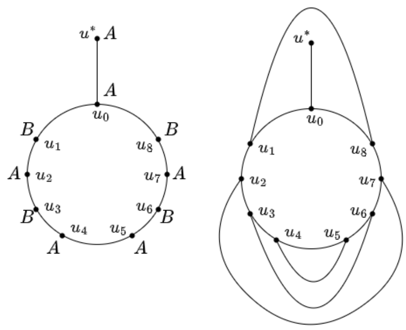


Figure 5 Partition $(A_3, B_3)$ and $M_3$.

Next, let $M \in \mathcal{M}$ be a feasible pairing and suppose that $\Phi(M')$ is computed for every $M' \in \mathcal{M}$ with $d(M') > d(M)$. Let $(A, B)$ be the good parition of $T$ with $g(A, B) = M$. Since haf $S_{(A,B)}$ can be computed by Theorem 3, we obtain $\Phi(M)$ by using (4).

By applying this procedure repeatedly, we obtain $\Phi(M)$ for every $M \in \mathcal{M}$. In particular, we obtain $\Phi(M^*)$, where $M^*$ is the input pairing. Since $|\mathcal{M}| = c_{k-1}$ is a constant for fixed $k$ and each step runs in polynomial time by Theorems 2 and 3, this algorithm runs in polynomial time.

▶ **Example 10.** Let $T = \{u^*, u_0, u_1, \ldots, u_8\}$, and suppose that

$$M_1 = \{(u^*, u_0), (u_1, u_8), (u_2, u_3), (u_4, u_7), (u_5, u_6)\},$$
$$M_2 = \{(u^*, u_0), (u_1, u_8), (u_2, u_7), (u_3, u_4), (u_5, u_6)\},$$
$$M_3 = \{(u^*, u_0), (u_1, u_8), (u_2, u_7), (u_3, u_6), (u_4, u_5)\}.$$

For $i = 1, 2, 3$, the partition $(A, B)$ corresponding to $M_i$, which we denote $(A_i, B_i)$, is as shown in Figures 3–5, respectively. Since

$$\text{haf } S_{(A_1,B_1)} = \Phi(M_1) + \Phi(M_2),$$
$$\text{haf } S_{(A_2,B_2)} = \Phi(M_2) + \Phi(M_3),$$
$$\text{haf } S_{(A_3,B_3)} = \Phi(M_3),$$

we can compute $\Phi(M_3)$, $\Phi(M_2)$, and $\Phi(M_1)$ in this order.

By using this procedure, we can obtain a polynomial-time algorithm if the given instance has at most one unique optimal solution.

▶ **Lemma 11.** ONE-FACE SHORTEST $k$-DISJOINT PATHS PROBLEM WITH A DEVIATION TERMINAL *can be solved in polynomial time under the assumption that a given instance has a unique optimal solution or has no feasible solution.*

**Proof.** If the given instance has a unique optimal solution, then the lowest degree term of $\Phi(M^*)$ is $2^k x^{\mathsf{opt}}$, where $\mathsf{opt}$ is the optimal value. If the instance has no feasible solution, then $\Phi(M^*) = 0$, i.e., the optimal value is $\mathsf{opt} = +\infty$. Since $\Phi(M^*)$ can be computed in polynomial time by the above argument, we obtain $\mathsf{opt}$.

We now describe how to obtain an optimal solution $\mathcal{P}^*$ when $\mathsf{opt} < +\infty$. For $e \in E$, we remove $e$ and compute the optimal value $\mathsf{opt}_e$ in the obtained instance by using the same algorithm as above. Then, $e$ is contained in $\mathcal{P}^*$ if and only if $\mathsf{opt}_e \neq \mathsf{opt}$. This shows that we can determine whether $e$ is contained in $\mathcal{P}^*$ or not in polynomial time. By applying this procedure for every $e \in E$, we obtain $\mathcal{P}^*$. ◀

When we do not assume the uniqueness of the optimal solutions, we perturb the length of edges so that the instance has a unique solution. The following lemma is derived from the *Isolation Lemma* [23], and the same argument is used in [5, 15].

▶ **Lemma 12** (See [5, 15, 23].). *Let $\mathcal{F}$ be a non-empty family of subsets of $E$ with $|E| = m$ such that $|F| \leq n$ for every $F \in \mathcal{F}$. If we assign for each $e \in E$, $w(e) \in \{2mn, 2mn+1, \ldots, 2mn+2m-1\}$ uniformly at random, then with probability greater than $1/2$, there exists a unique set $F^* \in \mathcal{F}$ with the minimum weight $\sum_{e\in F^*} w(e)$. Furthermore, $|F^*| = \min_{F\in\mathcal{F}} |F|$.*

**Proof of Theorem 1.** Let $G = (V, E)$ be the input graph, where $|V| = n$ and $|E| = m$. For each edge $e \in E$, choose $w(e) \in \{2mn, 2mn + 1, \ldots, 2mn + 2m - 1\}$ uniformly at random, and replace $e$ with a path of length $w(e)$. By applying Lemma 12 in which $\mathcal{F}$ consists of all edge sets of feasible solutions of the problem, we see that the obtained instance has a unique optimal solution with probability greater than $1/2$ (unless the original instance is infeasible). Therefore, by applying Lemma 11 to the obtained instance, we can solve the problem in polynomial time. ◀

## 4 Proof of Lemma 7

In this section, we give a proof of Lemma 7. To this end, we characterize when a feasible $(A, B)$-compatible pairing including $(u^*, u_0)$ exists.

▶ **Lemma 13.** *Let $(A, B)$ be a partition of $T$ with $u^*, u_0 \in A$. There is a feasible $(A, B)$-compatible pairing $M$ with $(u^*, u_0) \in M$ if and only if*

$$\frac{\#A - 2}{2} = \#\{j \in [2k-2] \mid j \textit{ is odd}, u_j \in A\} = \#\{j \in [2k-2] \mid j \textit{ is even}, u_j \in A\}. \quad (5)$$

**Proof.** We observe that if one equality in (5) holds, then the other equality also holds, because $\#A - 2 = \#\{j \in [2k-2] \mid j \text{ is odd}, u_j \in A\} + \#\{j \in [2k-2] \mid j \text{ is even}, u_j \in A\}$.

We first show the necessity ("only if" part). Suppose that there is a feasible $(A, B)$-compatible pairing $M$ with $(u^*, u_0) \in M$. For any pair $(u_i, u_{i'}) \in M \setminus \{(u^*, u_0)\}$, the parities of $i$ and $i'$ are different as $M$ is feasible. Thus, we obtain $\#\{j \in [2k-2] \mid j \text{ is odd}, u_j \in A\} = \#\{j \in [2k-2] \mid j \text{ is even}, u_j \in A\}$, and hence (5) holds.

We next show the sufficiency ("if" part) by induction on $|T|$. Suppose that (5) holds. A desired pairing obviously exists when $k = 1$, and so suppose that $k \geq 2$. Then, there exists an index $j \in [2k-3]$ such that both $u_j$ and $u_{j+1}$ belong to the same set, either $A$ or $B$, since otherwise terminals in $A$ and $B$ appear alternately along the boundary of $F$, which contradicts (5). Since removing $u_j$ and $u_{j+1}$ does not affect the parities of the indices of the other terminals, (5) holds after removing $u_j$ and $u_{j+1}$. By the induction hypothesis, we obtain a feasible pairing $M'$ of $T \setminus \{u_j, u_{j+1}\}$. Then, $M = M' \cup \{(u_j, u_{j+1})\}$ is a desired pairing, which shows the sufficiency. ◀

Even when $(u^*, u_0) \notin M$, we can obtain a similar characterization by shifting the indices in Lemma 13 as follows. Recall that $I = \{0, 1, \dots, 2k-2\} = [2k-2] \cup \{0\}$.

▶ **Lemma 14.** *Let $i \in I$ and let $(A, B)$ be a partition of $T$ with $u^*, u_i \in A$. There is a feasible $(A, B)$-compatible pairing $M$ with $(u^*, u_i) \in M$ if and only if*

$$\frac{\#A - 2}{2} = \#\{j \in I \mid j < i,\ j \text{ is even},\ u_j \in A\} + \#\{j \in I \mid j > i,\ j \text{ is odd},\ u_j \in A\}. \quad (6)$$

**Proof.** Let $v_0 = u_i$ and define

$$v_j = \begin{cases} u_{i+j} & \text{if } j \le 2k-2-i, \\ u_{i+j-2k+1} & \text{if } j \ge 2k-1-i \end{cases}$$

for $j \in [2k-2]$. That is, we relabel the terminals in $T \setminus \{u^*\}$ so that $v_0, v_1, v_2, \dots, v_{2k-2}$ appear counter-clockwise in this order and $v_0 = u_i$. Since the right-hand side of (6) is equal to either $\#\{j \in [2k-2] \mid j \text{ is odd}, v_j \in A\}$ or $\#\{j \in [2k-2] \mid j \text{ is even}, v_j \in A\}$, we obtain the lemma by Lemma 13. ◀

We are now ready to prove Lemma 7, which we restate here.

▶ **Lemma 7.** *Let $(A, B)$ be a partition of $T$ with $u^*, u_0 \in A$. Then, $(A, B)$ is good if and only if $(f_{(A,B)}(1), \dots, f_{(A,B)}(2k-2))$ is a $(k-1)$-ballot sequence.*

**Proof.** We first show the necessity ("only if" part). Suppose that $(A, B)$ is a good partition of $T$. Since there exists a feasible $(A, B)$-compatible pairing $M$ with $(u^*, u_0) \in M$, by Lemma 13, we obtain (5). By the definition of $f_{(A,B)}$, we obtain

$$\begin{aligned} &\sum_{j=1}^{2k-2} f_{(A,B)}(j) \\ &= \#\{j \in [2k-2] \mid j \text{ is even},\ u_j \in A\} - \#\{j \in [2k-2] \mid j \text{ is odd},\ u_j \in A\} \\ &\quad - \#\{j \in [2k-2] \mid j \text{ is even},\ u_j \in B\} + \#\{j \in [2k-2] \mid j \text{ is odd},\ u_j \in B\} \qquad (7) \\ &= 2(\#\{j \in [2k-2] \mid j \text{ is even},\ u_j \in A\} - \#\{j \in [2k-2] \mid j \text{ is odd},\ u_j \in A\}) \\ &= 0, \end{aligned}$$

where the last equality follows from (5).

To derive a contradiction, assume that $(f_{(A,B)}(1), \dots, f_{(A,B)}(2k-2))$ is not a $(k-1)$-ballot sequence. Then, there exists $i \in [2k-2]$ such that $\sum_{j=1}^{i} f_{(A,B)}(j) < 0$. Among such $i$, we choose the minimum one. By the minimality of $i$, we obtain $\sum_{j=1}^{i-1} f_{(A,B)}(j) = 0$ and $f_{(A,B)}(i) = -1$. Since $i - 1 \equiv \sum_{j=1}^{i-1} f_{(A,B)}(j) = 0 \pmod 2$, we see that $i$ is odd, and hence $u_i \in A$. By a similar calculation as (7), we obtain

$$0 = \sum_{j=1}^{i-1} f_{(A,B)}(j) = 2(\#\{j \in [i-1] \mid j \text{ is even},\ u_j \in A\} - \#\{j \in [i-1] \mid j \text{ is odd},\ u_j \in A\}),$$

which shows that $\#\{j \in [i-1] \mid j \text{ is even}, u_j \in A\} = \#\{j \in [i-1] \mid j \text{ is odd}, u_j \in A\}$. This together with (5) shows that

$$\begin{aligned}\frac{\#A-2}{2} &= \#\{j \in [2k-2] \mid j \text{ is odd}, u_j \in A\} \\ &= \#\{j \in [i-1] \mid j \text{ is odd}, u_j \in A\} + 1 + \#\{j \in I \mid j > i, j \text{ is odd}, u_j \in A\} \\ &= \#\{j \in [i-1] \mid j \text{ is even}, u_j \in A\} + 1 + \#\{j \in I \mid j > i, j \text{ is odd}, u_j \in A\} \\ &= \#\{j \in I \mid j < i, j \text{ is even}, u_j \in A\} + \#\{j \in I \mid j > i, j \text{ is odd}, u_j \in A\},\end{aligned}$$

where we note that $u_0, u_i \in A$ and $\{j \in I \mid j < i\} = [i-1] \cup \{0\}$. By Lemma 14, this shows that there exists a feasible $(A, B)$-compatible pairing $M$ with $(u^*, u_i) \in M$, which contradicts that $(A, B)$ is a good partition.

We next show the sufficiency ("if" part). Suppose that $(f_{(A,B)}(1), \ldots, f_{(A,B)}(2k-2))$ is a $(k-1)$-ballot sequence. Since $\sum_{j=1}^{2k-2} f_{(A,B)}(j) = 0$, by the same calculation as (7), we obtain $\#\{j \in [2k-2] \mid j \text{ is even}, u_j \in A\} = \#\{j \in [2k-2] \mid j \text{ is odd}, u_j \in A\}$, and hence (5) holds. Then, there exists a feasible $(A, B)$-compatible pairing $M$ with $(u^*, u_0) \in M$ by Lemma 13.

To derive a contradiction, assume that $(A, B)$ is not a good partition of $T$. Then, there exists a feasible $(A, B)$-compatible pairing $M$ with $(u^*, u_i) \in M$ for some $i \in [2k-2]$. By Lemma 14, this means that

$$\frac{\#A-2}{2} = \#\{j \in I \mid j < i, j \text{ is even}, u_j \in A\} + \#\{j \in I \mid j > i, j \text{ is odd}, u_j \in A\}. \quad (8)$$

Since

$$\begin{aligned}\frac{\#A-2}{2} &= \#\{j \in [2k-2] \mid j \text{ is odd}, u_j \in A\} \\ &= \#\{j \in I \mid j \leq i, j \text{ is odd}, u_j \in A\} + \#\{j \in I \mid j > i, j \text{ is odd}, u_j \in A\}\end{aligned}$$

by (5), this together with (8) shows that

$$\#\{j \in I \mid j < i, j \text{ is even}, u_j \in A\} = \#\{j \in I \mid j \leq i, j \text{ is odd}, u_j \in A\}. \quad (9)$$

If $i$ is odd, then

$$\begin{aligned}&\sum_{j=1}^{i} f_{(A,B)}(j) \\ &= \#\{j \in [i] \mid j \text{ is even}, u_j \in A\} - \#\{j \in [i] \mid j \text{ is odd}, u_j \in A\} \\ &\quad - \#\{j \in [i] \mid j \text{ is even}, u_j \in B\} + \#\{j \in [i] \mid j \text{ is odd}, u_j \in B\} \\ &= \#\{j \in [i] \mid j \text{ is even}, u_j \in A\} - \#\{j \in [i] \mid j \text{ is odd}, u_j \in A\} \\ &\quad - \Big(\frac{i-1}{2} - \#\{j \in [i] \mid j \text{ is even}, u_j \in A\}\Big) + \Big(\frac{i+1}{2} - \#\{j \in [i] \mid j \text{ is odd}, u_j \in A\}\Big) \\ &= 2(\#\{j \in [i] \mid j \text{ is even}, u_j \in A\} - \#\{j \in [i] \mid j \text{ is odd}, u_j \in A\}) + 1 \\ &= 2(\#\{j \in I \mid j < i, j \text{ is even}, u_j \in A\} - 1 - \#\{j \in I \mid j \leq i, j \text{ is odd}, u_j \in A\}) + 1 \\ &= -1,\end{aligned}$$

where the fourth equality follows from $u_0 \in A$ and the last equality is by (9). This contradicts that $f_{(A,B)}(j)$ is a ballot sequence.

Similarly, if $i$ is even, then

$$\begin{aligned}
&\sum_{j=1}^{i-1} f_{(A,B)}(j) \\
&= \#\{j \in [i-1] \mid j \text{ is even}, u_j \in A\} - \#\{j \in [i-1] \mid j \text{ is odd}, u_j \in A\} \\
&\quad - \#\{j \in [i-1] \mid j \text{ is even}, u_j \in B\} + \#\{j \in [i-1] \mid j \text{ is odd}, u_j \in B\} \\
&= 2(\#\{j \in [i-1] \mid j \text{ is even}, u_j \in A\} - \#\{j \in [i-1] \mid j \text{ is odd}, u_j \in A\}) + 1 \\
&= 2(\#\{j \in I \mid j < i, j \text{ is even}, u_j \in A\} - 1 - \#\{j \in I \mid j \leq i, j \text{ is odd}, u_j \in A\}) + 1 \\
&= -1,
\end{aligned}$$

which is a contradiction. ◀

## 5 Conclusion

We introduced One-Face Shortest $k$-Disjoint Paths Problem with a Deviation Terminal and gave a first randomized polynomial-time algorithm. In our algorithm, we combined the arguments by Datta et al. [9] and Hirai and Namba [15] with new insights on combinatorial properties of the problem.

It is natural to ask whether our results can be extended to the case when all the terminals except two or more are on the same face, which is still open. Another interesting unsolved case is when the terminals are on two faces in an arbitrary way.

Note that our algorithm can be derandomized if the input planar graph is cubic, because we can adopt a deterministic algorithm of Björklund and Husfeldt [4] for disjoint $(A+B)$-paths problem instead of the randomized one of Hirai and Namba [15]. It is open whether there exists a deterministic algorithm for non-cubic graphs.

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