Popular Edges with Critical Nodes

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── Abstract ──────────

In the *popular edge problem*, the input is a bipartite graph $G = (A \cup B, E)$ where A and B denote a set of men and a set of women respectively, and each vertex in $A \cup B$ has a strict preference ordering over its neighbours. A matching M in G is said to be *popular* if there is no other matching M' such that the number of vertices that prefer M' to M is more than the number of vertices that prefer M to M'. The goal is to determine, whether a given edge e belongs to some popular matching in G. A polynomial-time algorithm for this problem appears in [5].

We consider the popular edge problem when some men or women are prioritized or critical. A matching that matches all the critical nodes is termed as a feasible matching. It follows from [15, 20, 25, 24] that, when G admits a feasible matching, there always exists a matching that is popular among all feasible matchings.

We give a polynomial-time algorithm for the popular edge problem in the presence of critical men or women. We also show that an analogous result does not hold in the many-to-one setting, which is known as the Hospital-Residents Problem in literature, even when there are no critical nodes.

1 Introduction

The stable marriage problem is well-studied in literature. The input instance is a bipartite graph $G = (A \cup B, E)$ where A and B denote the sets of men and women respectively, and each vertex has a strict preference ordering on its neighbors. The preference ordering is referred to as the *preference list* of the vertex. Such an instance is referred to as a *marriage instance*. A matching M in G is said to be *stable* if there is no pair $(a, b) \in E$ such that both a and b prefer each other over their respective partners in M, denoted as $M(a)$ and $M(b)$ respectively. A matching is called *unstable* if such a pair (a, b) exists, and such a pair (a, b) is called a *blocking pair*. In their seminal paper, Gale and Shapley showed that stable matchings always exist and can be computed in linear time [8]. However, all the stable matchings match the same set of vertices [9] and they can be as small as half the size of a maximum matching [13]. Hence popularity has been considered as an alternative to stability.

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[1] Work was done when the author was a Master's student in Chennai Mathematical Institute.

33rd International Symposium on Algorithms and Computation (ISAAC 2022).
Editors: Sang Won Bae and Heejin Park; Article No. 54; pp. 54:1–54:14
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

▶ **Definition 1** (Popular Matching). *A matching M is said to be* popular *in a marriage instance G if, for all matchings N in G, the number of vertices that prefer N over M is no more than the number of vertices that prefer M over N. We denote the number of vertices preferring the matching M over N using the symbol $\phi(M, N)$.*

In other words, M is popular if it does not lose a head-to-head election with any other matching N where votes are cast by vertices. This notion was introduced by Gärdenfors [10] and has been well-studied since then (see Section 1.3). Popular matchings always exist since stable matchings are popular, in fact, stable matchings are minimum size popular matchings [13]. A subclass of maximum size popular matchings called *dominant matchings* was identified in [5].

▶ **Definition 2** (Dominant Matching). *A matching M in a marriage instance G is called a* dominant matching *if M is popular, and for each N such that $|N| > |M|$, the number of vertices that prefer M to N is more than the number of vertices that prefer N to M.*

Informally, a matching M is a dominant matching if M is popular and M wins against any other matching N which is larger than M. Note that a dominant matching is clearly a maximum size popular matching but a maximum size popular matching need not be a dominant matching.

Cseh and Kavitha [5] addressed the problem of determining whether there is a popular matching containing a given edge e, referred to as the *popular edge problem*. They gave a polynomial-time algorithm for this problem. This is surprising since, in [7], it was shown that stable matchings and dominant matchings are the only two tractable subclasses of popular matchings, and it is NP-hard to find a popular matching which is neither stable nor dominant.

Popular matchings find applications in situations where certain nodes are prioritized or critical and they are required to be matched. A real-life example of this scenario is assignment of sailors to billets in the US Navy [26, 29, 20] where certain billets are required to be matched. Rural hospitals often face the problem of understaffing in the National Resident Matching Program in the USA [27, 28]. Thus marking some positions in these hospitals as critical and finding a critical matching provides a way to address this issue. While matching students to mentors, it may be required to assign mentors to all the students whose past performance is below a certain threshold. In several other applications, a subset of people needs to be prioritized based on their economic, ethnic, geographic, or medical backgrounds. A matching that matches all the prioritized or *critical* nodes is termed as a *feasible* matching. Such a scenario has been considered in [24] and [25] in the many-to-one setting, and it is shown that there always exists a matching that is popular within the set of feasible matchings. In [20], a matching that matches as many critical nodes as possible has been referred to as a *critical matching*. It is shown in [20] that a matching that is popular in the set of critical matchings, called a *popular critical matching*, always exists and a polynomial time algorithm is given for the same. A special case of this is addressed in [15], where all the nodes are critical, and hence a critical matching is a matching that is popular amongst all maximum size matchings. A polynomial-time algorithm is given in [15] for this problem. In the presence of critical men or women, popular edge problem for feasible matchings is a natural question that arises in this context. Thus, given a marriage instance $G = (A \cup B, E)$, a set of critical nodes $C \subseteq A$, and an edge e, the problem is to determine whether there is a feasible matching containing e that is popular within the set of feasible matchings. We call this the *popular feasible edge problem*. In our work we only look for the case when $C \subseteq A$ we keep the case when $C \subseteq (A \cup B)$ as an interesting open problem.

▶ **Definition 3** (Popular feasible matching). *Given a marriage instance $G = (A \cup B, E)$, and a set of critical nodes C, a feasible matching that is popular among all the feasible matchings is called a* popular feasible matching.

We also define dominant feasible matchings below.

▶ **Definition 4** (Dominant feasible matching). *Given a marriage instance $G = (A \cup B, E)$ and a set of critical nodes C, a matching M is called a* dominant feasible matching *if M is a popular feasible matching, and for all the feasible matchings N such that $|N| > |M|$, M gets strictly more votes than N.*

1.1 Our contributions

We show the following main result in this paper:

▶ **Theorem 5.** *Given a marriage instance $G = (A \cup B, E)$ along with a set of critical nodes $C \subseteq A$, an edge $e \in E$ belongs to a popular feasible matching in G if and only if e belongs to a minimum size popular feasible matching or a dominant feasible matching in G.*

Theorem 5, along with the following results, leads to a polynomial-time algorithm for the popular critical edge problem.

▶ **Theorem 6.** *There are polynomial-time reductions from a given instance G with a set of critical nodes C to marriage instances G' and G'' such that there is a surjective map from stable matchings in G' to minimum size popular feasible matchings in G and there is a surjective map from stable matchings in G'' to dominant feasible matchings in G.*

The reductions are similar to those in [24, 25, 20], however, the surjectivity of the maps is not shown there. In [21], a similar reduction is given and the surjectivity of the map is shown using dual certificates, whereas our proofs of surjectivity are combinatorial.

Counter-example for the many-to-one setting. We show that a result analogous to Theorem 5 does not generalize to the many-to-one setting referred to as the *Hospital-Residents problem* in literature, even when there are no critical nodes. Figure 1 shows such an example. Informally, popularity in the many-to-one setting is defined as follows. To compare two matchings M and N, a hospital casts as many votes as its upper quota. It compares the sets of residents $M(h)$ and $N(h)$ that it gets in the matchings M and N respectively by fixing any correspondence function between $M(h) \setminus N(h)$ and $N(h) \setminus M(h)$. For the formal definition of popularity in the many-to-one setting in the presence of critical nodes, we refer the reader to [25, 24], where it is shown that the respective algorithms output a matching that is popular under any choice of the correspondence function.

1.2 Overview of our algorithm

We give a brief outline of our algorithm. After proving Theorem 5, the algorithm to determine whether an edge e belongs to a popular feasible matching goes as follows:
   (i) Check whether e belongs to a minimum size popular feasible matching. If so, output yes and stop, otherwise go to the next step.
  (ii) Check whether e belongs to a dominant feasible matching. If so, then output yes and stop. If not, then conclude that e does not belong to a popular feasible matching in that instance by Theorem 5.
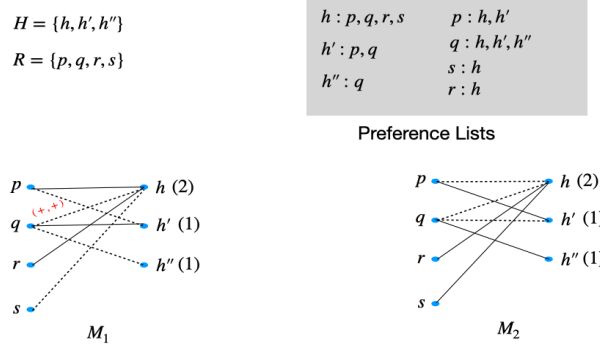
Figure 1 Here H and R are the sets of hospitals and residents respectively, h has upper quota or capacity 2, other hospitals have upper quota 1. The only stable matching is $M = \{(p, h), (q, h)\}$ of size 2 whereas the only dominant matching is M_2, of size 4. The edge (q, h') belongs to a popular matching M_1 of size 3, but does not belong to the stable matching M or to the dominant matching M_2. Thus Theorem 5 does not hold for this instance.

For steps (i) and (ii) above, we use the reductions mentioned in Theorem 6. For an edge e in G, there are multiple edges in G' and G'' corresponding to e. The stable edge algorithm of [23] can be used to determine whether any of the edges that correspond to e in G' or G'' is contained in some stable matching in G' or G''. The details are given in Section 3. To prove Theorem 5, we assume that e is contained in a popular feasible matching M which is neither a minimum size popular feasible matching nor a dominant feasible matching. We give a *Partition Method* in Section 3.1 which partitions the given instance into three parts. We call the restrictions of M on the three parts as M_d, M_m and M_r. Since e is contained in M, e must belong to one of the three parts viz. M_d, M_m and M_r. If $e \in (M_d \cup M_r)$, we convert the matching M_m to another matching M'_d which is a dominant feasible matching in that part, and show that $M_d^* = (M_d \cup M'_d \cup M_r)$ is a dominant feasible matching in the whole instance. Thus e is contained in a dominant feasible matching, namely M_d^*. Similarly, if $e \in (M_m \cup M_r)$ then we convert the matching M_d to another matching M'_m which is a minimum size popular feasible matching in the respective part, and moreover, $M_m^* = (M'_m \cup M_m \cup M_r)$ is a minimum size popular feasible matching in the whole instance. Thus e belongs to the minimum size popular feasible matching M_m^*.

1.3 Related Results

Gale and Shapley proposed an algorithm to find a stable matching in a marriage instance in their seminal paper [8]. The notion of popular matching was introduced by Gärdenfors [10]. Popular matchings in the marriage instance have been considered first in [1, 13, 15]. An $O(|E|)$-time algorithm to find a dominant matching in a marriage instance is given in [15]. In [15], a size-popularity tradeoff has been considered, and a polynomial-time algorithm for finding a maximum matching that is popular among all maximum matchings is given. The popular edge problem is inspired by the *stable edge problem*. The stable edge problem involves deciding whether a given edge e belongs to a stable matching in a Stable Marriage instance. A polynomial-time algorithm for the stable edge problem is given in the book by Knuth[23]. Cseh and Kavitha [5] addressed the popular edge problem and gave an $O(|E|)$ time algorithm for the same. Later, Faenza et. al [7] show that the problem of deciding whether an instance admits a popular matching containing a set of two or more edges is NP-Hard. In that paper, the authors also show that finding a popular matching in a stable marriage instance, which is neither stable nor dominant is NP-Hard.

In [24], the authors showed that a popular feasible matching always exists in an HRLQ instance. This has been further generalized by Nasre et al. [25] to the HRLQ case with critical residents. While our work and [24, 25] deal with instances that admit a feasible matching, the work of Kavitha [20] contains an algorithm to find a popular critical matching i.e., a matching that matches maximum possible number of critical nodes and is popular among all such matchings. Problems related to HRLQ have also been considered in [12] and [2] in different settings. Besides this, there has been a lot of recent work on various aspects of popular matchings and their generalizations e.g. weighted popular matchings, quasi-popular matchings, extended formulations, popular matchings with one-sided bias, dual certificates to popularity, popular matchings polytope and its extension complexity, hardness and algorithms for popular matchings in case of ties in preferences etc. [21, 18, 19, 22, 6, 17, 14, 16, 4, 11].

A comparison with [5]. Cseh and Kavitha in their paper [5] presented an $O(|E|)$-time algorithm for the popular edge problem. Our result follows a similar template as theirs, although unlike that in [5] where nodes are divided into two levels, we have nodes divided into a number of levels proportional to the number of critical nodes. Also, we need to partition the given instance into three parts, all of which can have blocking pairs, whereas in [5], all the blocking pairs can be put into only one of the two parts straight away.

1.4 Organization of the paper

In Section 2, we give the reductions from a marriage instance with critical men to marriage instances without critical nodes. In Section 3, we prove Theorem 5 and discuss the popular edge algorithm.

2 The Reductions

We describe the reductions from a marriage instance $G = (A \cup B, E)$ with a critical node set $C \subseteq A$ to marriage instances G' and G'' such that there is a surjective map from the set of stable matchings in G' to the set of minimum size popular feasible matchings in G, and a surjective map from the set of stable matchings in G'' to the set of dominant feasible matchings in G, thereby proving Theorem 6. We recall some notation below, that is standard in popular matchings literature (e.g. [13, 15, 5] etc.)

▶ **Definition 7** (Edge labels). *Given a matching M in G, a vertex u assigns a label $+1$ (respectively -1) to an edge (u, v) incident on it if $(u, v) \notin M$ and u prefers v over its partner in M denoted by $M(u)$ (respectively $M(u)$ over v). Thus each edge (u, v) gets two labels, one from u and the other from v.*

By above definition, an edge not present in a given matching M can get one of the four labels $(+1, +1), (+1, -1), (-1, +1), (-1, -1)$. We use the convention that the first label in the pair is from a vertex in A and the second label is from a vertex in B. Any vertex prefers to be matched to one of its neighbors over remaining unmatched.

2.1 Reduction from G to G'

Given an instance G, the instance G' is constructed as follows. Let $C \subseteq A$ be the set of critical nodes and $\ell = |C|$.

- **The set A':** For each $m \in C$, A' has $(\ell + 1)$ copies of m, denoted by the set $A'_m = \{m^0, m^1, ..., m^\ell\}$. We refer to $m^i \in A'$ as the *level i* copy of $m \in A$. For each $m \in A \setminus C$, A' has only one copy of m, denoted $A'_m = \{m^0\}$. Now, $A' = \bigcup_{m \in A} A'_m$.

- **The set B':** All the women in B are present in B'. Additionally, corresponding to each $m \in C$, B' contains ℓ dummy women denoted by the set $D_m = \{d^1_m, d^2_m, d^3_m, ..., d^\ell_m\}$. We call d^i_m as the *level i dummy woman for m*. For $m \in A \setminus C$, $D_m = \emptyset$. Now, $B' = B \cup \bigcup_{m \in A} D_m$.

We denote by $\mathrm{Pref}(m)$ and $\mathrm{Pref}(w)$ the preference lists of $m \in A$ and $w \in B$ respectively. Let $\mathrm{Pref}(w)^i$ be the list of level i copies of men present in $\mathrm{Pref}(w)$, if these copies exist. We now describe the preference lists in G'. Here $\circ$ denotes the concatenation of two lists.

$$
\begin{array}{lll}
m^0 \text{ s.t. } m \in A \setminus C & : & \mathrm{Pref}(m) \\
m \in C, i \in \{0, \ldots, \ell\}: & & \\
m^0 & : & \mathrm{Pref}(m), d^1_m \\
m^i & : & d^i_m, \mathrm{Pref}(m), d^{i+1}_m, i \in \{1, \ell - 1\}) \\
m^\ell & : & d^\ell_m, \mathrm{Pref}(m) \\
w \text{ s.t. } w \in B & : & \mathrm{Pref}(w)^\ell \circ \mathrm{Pref}(w)^{\ell-1} \circ \ldots \circ \mathrm{Pref}(w)^0 \\
d^i_m, i \in \{1, \ldots, \ell\} & : & m^{i-1}, m^i
\end{array}
$$

2.2 Correctness of the reduction

After constructing G', the mapping of a stable matching M' in G' to a minimum size popular feasible matching M in G is a simple and natural one: For $m \in A$, define the set $M(m) = B \cap \bigcup_{m \in A} M'(m^i)$, which is the set of non-dummy women matched to any copy of m in A'. In the rest of this section, the term *image* always refers to the image under this map. It remains to prove that M is a minimum size popular feasible matching i.e., M is a matching in G, it is feasible, popular, and no matching smaller than M is popular. We define some terminology first. A man $m \in A$ and his matched partner $w \in B$ in M are said to be at level i if $(m^i, w) \in M'$. A man $m \in A$ which is unmatched in M is said to be at level i if m^i in A' is unmatched in M'. All unmatched women are said to be at level 0. Now we give a sufficient condition for a minimum size popular feasible matching in G, the proof appears in [3].

▶ **Theorem 8.** *The image M of a stable matching M' in G' is a minimum size popular feasible matching in G and it satisfies the following conditions. Moreover, any matching M that satisfies the following conditions for some assignment of levels to vertices of G is a minimum size popular feasible matching.*

1. *All $(+1, +1)$ edges are present in between a man at level i and a woman w at level j where $j > i$.*
2. *All edges between a man at level i and a woman at level $(i-1)$ are $(-1, -1)$ edges.*
3. *No edge is present between a man at level i and a woman at level j where $j \le (i - 2)$, and all the edges of M are between vertices at the same level.*
4. *All unmatched men are at level 0.*

2.3 Surjectivity of the map

In this section, the goal is to prove the following theorem:

▶ **Theorem 9.** *For every minimum size popular feasible matching M in G, there exists a stable matching M' in G' such that M is the image of M'.*

To show the surjectivity i.e. the fact that every minimum size popular feasible matching M in G has a stable matching M' in G' as its pre-image, we first assign levels to nodes in G with respect to M. From the assignment of levels to nodes in G, the pre-image M' is then immediate. The assignment of levels is described in Algorithm 1. In the pseudocode for Algorithm 1, we denote the level of a vertex v by $level(v)$, and the matched partner of v in M as $M(v)$. The proof of Theorem 9 is immediate from the correctness of Algorithm 1, proved below.

■ **Algorithm 1** Leveling Algorithm for minimum size popular feasible matching.

Input: A marriage instance G, set of critical nodes $C \subseteq A$, a minimum size popular feasible matching M in G

Output: Assignment of levels to the vertices in G based on the matching M.

1: Initially all the men and the women are assigned level 0
2: flag = true
3: **while** flag = true **do**
4:     check1 = 0, check2 = 0, check3 = 0
5:     **while** $\exists\, m \in A, w \in B$ s.t. $level(m) = i$, $level(w) = j$, $j \leq i$, and (m, w) is a $(+1, +1)$ edge **do**
6:         Set $level(w) = level(M(w)) = i + 1$ ▷ Note that w cannot be unmatched in M because then $M \setminus (m, M(m)) \cup (m, w)$ is more popular than M and hence M would not be a popular feasible matching.
7:         check1 = 1
8:     **while** $\exists\, m \in A, w \in B$, s.t. $level(m) = i$, $level(w) = j$, $j < i$ and (m, w) is a $(+1, -1)$ or a $(-1, +1)$ edge **do**
9:         Set $level(w) = level(M(w)) = i$ ▷ Note that w cannot be unmatched in M because then M would not be a popular feasible matching.
10:         check2 = 1
11:     **while** $\exists m \in A, w \in B$ s.t. $level(m) = i$, $level(w) = j$, $j \leq (i - 2)$ and (m, w) is a $(-1, -1)$ edge **do**
12:         Set $level(w) = level(M(w)) = i - 1$ ▷ Note that w cannot be unmatched in M because then M would not be a PFM.
13:         check3 = 1
14:     **if** check1 = 0 and check2 = 0 and check3 = 0 **then**
15:         flag = false

In Algorithm 1, the Boolean variables check1, check2 and check3 are used to check whether the assignment of levels at any point violates one of the conditions of Theorem 8. If not, then we set flag to false and the algorithm terminates. In Theorem 10 below, we show that no level is empty. Since level of a vertex never reduces during the execution of Algorithm 1, it implies that the algorithm terminates.

▶ **Theorem 10.** *For a man m at level i there exists (i) either a woman w at each level j, where $j < i$, or (ii) an unmatched man m_0, if $j = 0$ such that there is an alternating path from w to m or from m_0 to m which consists of $(i - j)$ more $(+1, +1)$ edges than $(-1, -1)$ edges.*

The following corollary is a straight forward consequence of Theorem 10 and the fact that M is a minimum size popular feasible matching.

▶ **Corollary 11.** *All non-critical men and unmatched women are assigned level zero and the critical men are assigned level less than or equal to $|C|$ by Algorithm 1.*

Proof of Corollary 11. Suppose there is a non-critical man m_i at level $i, i > 0$. Now, from Theorem 10, there is an alternating path, say ρ, from a woman w_0 at level 0 or from an unmatched man m_0 to m_i which has i more $(+1, +1)$ edges than $(-1, -1)$ edges. Let $N = M \oplus \rho$. Observe that Algorithm 1 assigns level 0 to all the unmatched men in M, so m_i is matched. Now, it is easy to see that N is also a popular feasible matching. But $|N| < |M|$, so this contradicts the assumption that M is a minimum size popular feasible matching. ◀

After assigning levels to the vertices in G, the pre-image of M i.e. a stable matching M' in G' is constructed as follows. If a man m in G gets assigned level i then $M'(m_i) = M(m)$. If m is unmatched in M, then $m \notin C$ by feasibility of M, and m gets level 0 by Corollary 11. Then we leave m unmatched in M' as well. For $j < i$, m^j gets matched to the dummy woman d_m^{j+1} and for $j > i$, m^j gets matched to the dummy woman d_m^j. We give the proof of Theorem 12 in [3], where we prove that M' is stable in G'.

▶ **Theorem 12.** *The matching M' is a stable matching in G'.*

Similar reduction and proofs for dominant feasible matching appear in [3].

3   The Popular Edge Algorithm

Now we are ready to prove Theorem 5, from which, the popular edge algorithm is as follows. For a given edge $e = (m, w)$, we construct G', G'' using reductions from Section 2 and check if any of the edges (m^i, w) in G' or G'' belong to a stable matching in the respective instance using Knuth's algorithm for stable edges [23] If there is a minimum size popular feasible matching or a dominant feasible matching containing e then there is nothing to prove. So we need to prove the theorem for an edge e that is contained in a popular feasible matching M that is neither a minimum size popular feasible matching nor a dominant feasible matching, and show that there is also a minimum size popular feasible matching or a dominant feasible matching containing e. The proof of Theorem 5 involves the following two results:

▶ **Theorem 13.** *If M is neither a minimum size popular feasible matching or a dominant feasible matching, then $A \cup B$ can be partitioned into three parts $A_d \cup B_d$, $A_m \cup B_m$ and $A_r \cup B_r$ such that no edge of M is present in $A_i \times B_j$, $i \neq j$, where $i, j \in \{d, m, r\}$.*

We prove Theorem 13 in Section 3.1. Because of Theorem 13, it follows that the partition of $A \cup B$ also induces a partition of M into three parts, say M_d, M_m, M_r respectively. The following theorem shows that either M_d or M_m can be transformed into another matching so that the resulting matching is a minimum size popular feasible matching or a dominant feasible matching in G.

▶ **Theorem 14.** *There exist algorithms to transform:*
1. *the matching M_d to another matching M'_m in $A_d \cup B_d$ such that $M_m^* = M'_m \dot\cup M_m \dot\cup M_r$ is a minimum size popular feasible matching in G*
2. *the matching M_m to another matching M'_d in $A_m \cup B_m$ such that $M_d^* = M_d \dot\cup M'_d \dot\cup M_r$ is a dominant feasible matching in G.*

A proof of Theorem 14 is given in Section 3.2. From Theorems 13 and 14, Theorem 5 follows:

Proof of Theorem 5. Depending on the part that contains the given edge e, one of the two transformations mentioned in Theorem 14 can be applied: If $e \in M_m$ (respectively $e \in M_d$), apply the first (respectively, second) transformation from Theorem 14 i.e. convert the

matching M_d to M'_m (M_m to M'_d). Then, by Theorem 14, the resulting matching M^*_m (M^*_d) is a minimum size popular feasible matching (dominant feasible matching) in G containing e. If $e \in M_r$, we can apply any one of the two transformations mentioned in Theorem 14. Thus, in all the three cases, we get a minimum size popular feasible matching or a dominant feasible matching containing e. This completes the proof of Theorem 5. ◀

3.1 Partition Method

We prove Theorem 13 now. For partitioning $A \cup B$ and M, we first assign levels to the vertices of $A \cup B$ using Algorithm 1 described in Section 2.1. We refer to the level of a vertex $u \in A \cup B$ as $level(u)$. Since M is a popular feasible matching but not a minimum size popular feasible matching by assumption, all the non-critical men may not be at level 0. However, the following holds:

▶ **Lemma 15.** *After applying Algorithm 1 on a popular feasible matching M all non-critical men are assigned levels 0 or 1.*

Proof. If there is a non-critical man m who is assigned level $i \geq 2$, then according to Theorem 10 there exists a woman at level 0 such that m has an alternating path ρ from w with i more $(+1, +1)$ edges than $(-1, -1)$ edges. Since $i \geq 2$, the matching $M \oplus \rho$ is feasible and is more popular than M, contradicting the assumption that M is a popular feasible matching . Hence, all non-critical men are assigned levels 0 or 1. ◀

The following notions will be used in the partitioning procedure.

▶ **Definition 16** (Size Reducing Alternating Path (SRAP)). *An alternating path ρ with respect to a matching M is called as SRAP if the following conditions are satisfied:*
1. *The number of $(+1, +1)$ edges in ρ is one more than the number of $(-1, -1)$ edges in ρ,*
2. *It starts in a matched woman at level 0.*
3. *It ends in a matched non-critical man at level 1.*

▶ **Definition 17** (Size Increasing Alternating Path (SIAP)). *An alternating path ρ with respect to a matching M is called an SIAP if the following conditions are satisfied.*
1. *There are an equal number of $(+1, +1)$ and $(-1, -1)$ edges in ρ.*
2. *Its end-points are an unmatched man and an unmatched woman.*
Intuitively, if ρ is an SIAP (respectively an SRAP), then $M \oplus \rho$ gives a larger (respectively, smaller) popular feasible matching. Theorem 18 below shows that an SRAP and an SIAP must exist if M is not a minimum size popular feasible matching or a dominant feasible matching. We give the detailed proof in [3].

▶ **Theorem 18.** *If a popular feasible matching M is neither a minimum size popular feasible matching nor a dominant feasible matching, then G must contain an SRAP and an SIAP with respect to M.*

Proof. Suppose M_{min} is a minimum size popular feasible matching. Consider $M \oplus M_{min}$ which is a disjoint union of alternating paths and cycles. Since $|M_{min}| < |M|$, there exists an alternating path ρ in $M \oplus M_{min}$ whose both end-points are matched in M. By popularity of M and M_{min}, ρ must have one more $(+1, +1)$ edge than $(-1, -1)$ edges so that $\phi(M \oplus \rho, M) = \phi(M, M \oplus \rho)$. By feasibility of M_{min}, ρ must have a non-critical man m as one of its end-points, since m is unmatched in M_{min}. The level assigned to m has to be 1 because non-critical men can only be assigned levels 0 or 1 by Lemma 15. Moreover, since ρ has 1 more $(+1, +1)$ edge than $(-1, -1)$ edges, the level assigned to the other end-point w is 0. Recall that w is matched in M and unmatched in M_{min}. Hence ρ is an SRAP.

Now suppose M_d is a dominant feasible matching in G. The graph $M \oplus M_d$ is a disjoint union of alternating paths and cycles. Since $|M| < |M_d|$, there must exist an alternating path ρ in $M \oplus M_d$ whose end-points are unmatched in M and matched in M_d. Here, ρ must have an equal number of $(+1, +1)$ and $(-1, -1)$ edges, otherwise $(M \oplus \rho)$ becomes a more popular matching than M. Hence M has an SIAP.   ◀

The partitioning is based on SIAP and SRAP, so the following theorem is essential for the partitioning to be well-defined. Theorem 19 below shows that no vertex belongs to both an SRAP and an SIAP:

▶ **Theorem 19.** *Given a popular feasible matching M, no vertex in G belongs to both an* SIAP *and an* SRAP.

Proof. Note that if a man m belongs to both an SIAP ρ and an SRAP σ, then his matched partner $M(m)$ must belong to both ρ and σ too. Also note that no man or woman unmatched in M can belong to both ρ and σ because all the men and women in an SRAP are matched in M. Suppose a matched pair (m, w) in M belongs to both ρ and σ. Let m_I and w_I be the end-points of ρ and m_R and w_R be the end-points of σ. Let the level assigned to the pair (m, w) be $i_{(m,w)}$. Since m_I and w_I are unmatched in M, both of them are assigned level 0 by Corollary 11. Now, the alternating subpath ρ_I of ρ from w_I to m must contain $i_{(m,w)}$ more $(-1, -1)$ edges than $(+1, +1)$ edges. This is because all the adjacent vertices of the women present in level i must be present at a level j where $j \leq (i+1)$. So, ρ_I can go up by only one level that is from a level i woman it can only go to a level $i+1$ man and, from the properties of Algorithm 1, we also know that all the edges between level i women and level $(i+1)$ men are $(-1, -1)$ edges. Now, since an SIAP has an equal number of $(+1, +1)$ and $(-1, -1)$ edges, we have that the alternating subpath $\rho_{I'}$ of ρ from m_I to m must contain $i_{(m,w)}$ more $(+1, +1)$ edges than $(-1, -1)$ edges.

By a similar argument, the alternating subpath σ_R of σ starting from m to m_R consists of $(i_{(m,w)} - 1)$ more $(-1, -1)$ edges than $(+1, +1)$ edges. Thus the path $\beta = \rho_{I'} \circ \sigma_R$, where $\circ$ denotes concatenation, contains more $(+1, +1)$ edges than $(-1, -1)$ edges. This is because $\rho_{I'}$ contains $i_{(m,w)}$ more $(+1, +1)$ edges than $(-1, -1)$ edges, and then ρ_R has $(i_{(m,w)} - 1)$ more $(-1, -1)$ edges than $(+1, +1)$ edges. The matching $M \oplus \beta$ is more popular than M because β has one more $(+1, +1)$ edge than the number of $(-1, -1)$ edges, and $M \oplus \beta$ has the same size as that of M. This contradicts the fact that M is a popular feasible matching. Hence, no vertex in G can belong to both an SIAP and an SRAP for a given matching M.   ◀

Now we give the method to partition $A \cup B$ below, as required by Theorem 13.

3.1.1   Partitioning $A \cup B$

(a) Initialize A_d, A_m, A_r, B_d, B_m, B_r to empty sets.

(b) For all unmatched men (m_u) and unmatched women (w_u) in M we do: $A_m = A_m \cup \{m_u\}$ and $B_m = B_m \cup \{w_u\}$

(c) From Theorem 18, we know that M must have an SRAP and an SIAP. For all men m_d and women w_d in each SRAP do: $A_d = A_d \cup \{m_d\}$ and $B_d = B_d \cup \{w_d\}$

(d) For all men m and women w in each SIAP do: $A_m = A_m \cup \{m\}$ and $B_m = B_m \cup \{w\}$

(e) While there exists a $(+1, +1)$ edge (m, w) such that $m \in A \backslash (A_d \cup A_m)$, $level(m) = i$, $level(w) = j$, $j \leq i + 1$ do: $A_d = A_d \cup \{m\}$ and $B_d = B_d \cup \{M(m)\}$

(f) While there exists a $(+1, +1)$ edge (m, w) such that $m \in A_m$, $level(m) = i$, $w \in B \backslash (B_d \cup B_m)$, $level(w) = j$, $j \leq i + 1$, do: $B_m = B_m \cup \{w\}$ and $A_m = A_m \cup \{M(w)\}$

(g) $A_r = A \backslash (A_d \cup A_m)$ and $B_r = B \backslash (B_d \cup B_m)$. Let M_d, M_m, M_r be the parts of M present in the induced subgraph on $A_d \cup B_d$, $A_m \cup B_m$, and $A_r \cup B_r$ respectively.

To complete the proof of Theorem 13, we need to show that the above procedure partitions $A \cup B$ i.e., the three sets $A_d \cup B_d, A_m \cup B_m, A_r \cup B_r$ are disjoint. The partition procedure always puts a vertex and its matched partner in the same partition. So it is immediate that M_d, M_m, M_r partition M. Lemmas 20, 21, 22,23 are useful in showing that the partitioning is well-defined, and in proving correctness of the transformations mentioned in Theorem 14. We retain the same assignment of levels to all the vertices as was done before partitioning. We state Lemmas here. The proofs of the lemmas are given in [3].

▶ **Lemma 20.** *For a man $m \in A_m$ at level i, there is an alternating path with i more $(+1, +1)$ edges than $(-1, -1)$ edges which starts at m_I and ends in m where m_I is an unmatched man and also an endpoint of an* SIAP *with respect to M. Analogously, for a woman $w \in B_d$ at level i there is an alternating path with $(i - 1)$ more $(-1, -1)$ edges than $(+1, +1)$ edges which starts at m_R and ends in w where m_R is a non-critical man and also an endpoint of an* SRAP *with respect to M.*

▶ **Lemma 21.** *For an edge $(m, w) \in A_m \times B_d$ in G we have the following*
 (i) *If m is at level i and w is at level $(i + 1)$ then the edge (m, w) is not a $(+1, +1)$ edge.*
 (ii) *If m is at level i then w cannot be at level $(i - 1)$ or below.*
 (iii) *If m is at level i and w is at level i then (m, w) is a $(-1, -1)$ edge.*

▶ **Lemma 22.** *For an edge $(m, w) \in A_r \times B_d$ in G we have the following*
 (i) *If m is at level i and w is at level $(i + 1)$ then the edge (m, w) is not a $(+1, +1)$ edge.*
 (ii) *If m is at level i then w cannot be at level $(i - 1)$ or below.*
 (iii) *If m is at level i and w is at level i then (m, w) is a $(-1, -1)$ edge.*

▶ **Lemma 23.** *For a pair $(m, w) \in A_m \times B_r$ we have the following*
 (i) *If m is at level i and w is at level $(i + 1)$ then the edge (m, w) is not a $(+1, +1)$ edge.*
 (ii) *If m is at level i then w cannot be at level $(i - 1)$ or below.*
 (iii) *If m is at level i and w is at level i then (m, w) is a $(-1, -1)$ edge.*

3.2 Transformation Procedures

We prove Theorem 14 now. Recall that, before partitioning $A \cup B$, we have assigned levels, denoted by $level(u)$ to all the vertices $u \in A \cup B$ according to M using Algorithm 1, and that $level(u) = level(M(u))$. Our transformation procedures use these levels.

3.2.1 Transformation of M_d to M'_m

Following are the steps involved in the transformation, we refer to this as *Transformation* 1.
(a) For $m \in A_d, M(m) \in B_d$, if $level(m) = level(M(m)) = i$, $i \geq 1$, then set $level(m) = level(M(m)) = i - 1$
(b) Mark the matched edges present among level 0 vertices as unmatched edges. So all the level 0 men in A_d are not assigned to any partner now.
(c) Execute a proposal algorithm now. The men at level 0 start proposing from the beginning of their preference lists. A woman prefers a man at level j more than a man at level i where $j > i$. If a man m proposes to a woman w then w will accept m's proposal iff w is unmatched or if w prefers m more than her matched partner.
(d) If a critical man m at level i where $i < |C|$ exhausts his preference list while proposing and remains unmatched then we assign level $(i + 1)$ to m and m starts proposing again from the beginning of his preference list.
Let M'_m be the matching obtained after applying the above steps on the induced subgraph on $A_d \cup B_d$, and let $M^*_m = M'_m \dot\cup M_m \dot\cup M_r$ be the resulting matching in G.

3.2.2   Transformation of M_m to M_d'

This is referred to as Transformation 2 here onwards, and involves promoting all the unmatched men to level 1 and executing a similar proposal algorithm as above. Men that get unmatched during the course of the proposal algorithm continue proposing to women further down in their preference list. If they exhaust their preference list without getting matched, then they are promoted to the next higher level and continue proposing, however, non-critical men are not promoted beyond level 1. We give the formal transformation in [3]. Let the resulting matching in G be $M_d^* = M_m' \dot\cup M_d \dot\cup M_r$.

The following property is crucially used in proving that M_m^* is a minimum size popular feasible matching in G whereas M_d^* is a dominant feasible matching in G.

▶ **Lemma 24.** *For a man $m \in A_d$, if $level(m) = i, i > 0$ before applying Transformation 1 on $A_d \cup B_d$ then, after applying Transformation 1, $level(m) \in \{i - 1, i\}$. The same holds for a woman $w \in B_d$. Similarly, for a man $m \in A_m$, if $level(m) = i$ before applying Transformation 2 on $A_m \cup B_m$ then, after applying Transformation 2, $level(m) \in \{i, i + 1\}$. The same holds for a woman $w \in B_m$.*

Proof. We prove the property for Transformation 2. The proof for Transformation 1 is analogous.

Suppose there exists a man $m \in A_m$ who was assigned level i before applying Transformation 2 on $A_m \cup B_m$ but after applying Transformation 2 suppose m is assigned level $(i + 2)$ or more. In Transformation 2 we convert the matching M_m to M_d^*. If m is unmatched in M_m then the level of m in M_m was 0 and it can be at most at level 1 in M_d^* because m is a non-critical man. So, m cannot be an unmatched man because it contradicts our assumption that the level of m in M_d^* is $(i + 2)$ or more. Suppose m is matched to a woman w in M_m and the level of m is i. While applying Transformation 2 w rejected m because w got a proposal from some man m' who is better than m and is at level i. Note that w must have rejected m while applying Transformation 2 because after applying Transformation 2 level of m changes to $(i + 2)$ or more from i. Now, since m is assigned level $(i + 2)$ or more so m exhausts his preference list while proposing and remains unmatched at level i. So, m gets promoted to level $(i + 1)$ and he starts proposing from the beginning of his preference list. Again since m is assigned level $(i + 2)$ or more so m exhausts his preference list while proposing and remains unmatched at level $i + 1$ but this is not possible because in the worst case m can propose to w and get matched to her. This is because w would reject m' which is at level i and m is at level $(i + 1)$. Hence w would accept m's proposal and the level of w changes to $(i + 1)$. Note that no man m'' can get promoted from level i to level $(i + 1)$ and breaks the engagement of m and w because if this happens then in M_m the edge (m'', w) is a $(+1, +1)$ edge but since both m'' and w are in the same level i in M_m hence (m'', w) cannot be a $(+1, +1)$ edge. Hence, m does not exhaust his preference list while proposing at level $i + 1$. So, after applying Transformation 2 the level of m can be either i or $(i + 1)$. ◀

Theorems 25 and 26 show that the matchings output by the transformations are a minimum size popular feasible matching and a dominant feasible matching in G respectively. We prove Theorem 26 in [3].

▶ **Theorem 25.** $M_m^* = (M_m' \cup M_m \cup M_r)$ *is a minimum size popular feasible matching.*

Proof. The four conditions given in Theorem 8 are sufficient to show that a matching is a minimum size popular feasible matching. We show that M' satisfies all of them.

Before applying the Transformation 1, M satisfied conditions 1 to 3 of Theorem 8 because of the way Algorithm 1 assigns levels.After applying Transformation 1, the matching M_d changes to M_m' and the levels of the vertices in $A_d \cup B_d$ decrease by at most 1 (Lemma 24).

So, if M' does not satisfy conditions $1 - 3$ of Theorem 8 then it has to be because of the pairs present in $A_m \times B_d$ and $A_r \times B_d$. Now we show that the conditions are still satisfied. Below the proofs are given only for the pairs in $A_m \times B_d$. Proofs for the pairs in $A_r \times B_d$ are similar.

Let (m, w) be an edge in $A_m \times B_d$. From Lemma 21 (ii), if the level of w is i with respect to M, then m has level $j \leq i$. Now, after applying the Transformation 1, the level of w either remains i or becomes $(i - 1)$. In the former case, the first condition of Theorem 8 is satisfied.In the later case, we have three possibilities: (a) either $j < (i - 1)$ or (b) $j = (i - 1)$, or (c) $j = i$. In case (b), (m, w) is not a $(+1, +1)$ edge (Lemma 21 (i)), in case (c), (m, w) is a $(-1, -1)$ edge (Lemma 21 (iii)). Hence, there is no $(+1, +1)$ edge in between a pair $(m, w) \in A_m \times B_d$ in the matching M' where m is at level j and w is at level i such that $j \leq i$. Hence, condition 1 is satisfied.

From Lemma 21 (ii), if the level of w is i in M, then m has level $j \leq i$. If level of w changes to $(i - 1)$ after applying the Transformation 1, and if level of m is i, then due to Lemma 21(iii), (m, w) is a $(-1, -1)$ edge. Thus the condition 2 of Theorem 8 is satisfied.

From Lemma 21 (i), if w is at level i with respect to M, then level of m is $j \leq i$. If level of w changes to $(i - 1)$, the conditions of Theorem 8 are still satisfied because no man in A_m adjacent to w is present at level $(i + 1)$ or above.

We know that all the unmatched men are non-critical men. In the first step of Transformation 1, we decrease the level of each vertex by 1. Since the level of a non-critical man is at most 1 to begin with, and they are never promoted to a higher level in the Transformation 1, all the vertices unmatched in M_m^* remain at level 0. Since all the conditions of Theorem 8 are satisfied, M_m^* is a minimum size popular feasible matching. ◀

The following is an analogous result for Transformation 2. We refer the proof to [3].

▶ **Theorem 26.** $M_d^* = (M_d \cup M_d' \cup M_r)$ *is a dominant feasible matching in* G.

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