A Finite Algorithm for the Realizabilty of a Delaunay Triangulation

Akanksha Agrawal
Indian Institute of Technology Madras, Chennai, India

Saket Saurabh
The Institute of Mathematical Sciences, HBNI, Chennai, India
University of Bergen, Norway

Meirav Zehavi
Ben-Gurion University of the Negev, Beer-Sheva, Israel

Abstract

The *Delaunay graph* of a point set $P \subseteq \mathbb{R}^2$ is the plane graph with the vertex-set P and the edge-set that contains $\{p, p'\}$ if there exists a disc whose intersection with P is exactly $\{p, p'\}$. Accordingly, a triangulated graph G is *Delaunay realizable* if there exists a triangulation of the Delaunay graph of some $P \subseteq \mathbb{R}^2$, called a *Delaunay triangulation* of P, that is isomorphic to G. The objective of Delaunay Realization is to compute a point set $P \subseteq \mathbb{R}^2$ that realizes a given graph G (if such a P exists). Known algorithms do not solve Delaunay Realization as they are non-constructive. Obtaining a constructive algorithm for Delaunay Realization was mentioned as an open problem by Hiroshima et al. [19]. We design an $n^{\mathcal{O}(n)}$-time constructive algorithm for Delaunay Realization. In fact, our algorithm outputs sets of points with *integer* coordinates.

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1 Introduction

We study Delaunay graphs – through the lens of the well-known Delaunay Realization problem – which are defined as follows. Given a point set $P \subseteq \mathbb{R}^2$, the *Delaunay graph, $\mathcal{DG}(P)$, of P* is the graph with vertex-set P and edge-set that consists of every pair (p, p') of points in P that satisfies the following condition: there exists a disc whose boundary intersects P only at p and p', and whose interior does not contain any point in P. The point set $P \subseteq \mathbb{R}^2$ is in general position if it contains no four points from P on the boundary of a disc. If P is in general position, $\mathcal{DG}(P)$ is a triangulation, called a *Delaunay triangulation*, denoted by $\mathcal{DT}(P)$.[1] Otherwise, Delaunay triangulation and the notation $\mathcal{DT}(P)$, may refer

[1] We assume that $|P| \geq 4$, as otherwise, the problem that we consider, is solvable in polynomial time.

to any triangulation obtained by adding edges to $\mathcal{DG}(P)$. Thus, Delaunay triangulation of a point set P is unique if and only if $\mathcal{DG}(P)$ is a triangulation. An alternate characterization of Delaunay triangulations is that in such a triangulation, for any three points of a triangle of an interior face, the unique disc whose boundary contains these three points does not contain any other point in P.

The Delaunay graph of a point set is a planar graph [7], and triangulations of such graphs form an important subclass of the class of triangulations of a point set, also known as the class of maximal planar sub-divisions of the plane. Accordingly, efficient algorithms for computing a Delaunay triangulation for a given point set have been developed (see [7, 9, 18]). One of the main reasons underlying the interest in Delaunay triangulations is that any angle-optimal triangulation of a point set is actually a Delaunay triangulation of the point set. Here, optimality refers to the maximization of the smallest angle [7, 12]. This property is particularly useful when it is desirable to avoid "slim" triangles – this is the case, for example, when approximating a geographic terrain. Another main reason underlying the interest in Delaunay triangulations is that these triangulations are the duals of "Voronoi diagrams" (see [27]).

We are interested in a well-known problem which, in a sense, is the "opposite" of computing a Delaunay triangulation for a given point set. Here, rather than a point set, we are given a triangulated graph G. The graph G is *Delaunay realizable* if there exists $P \subseteq \mathbb{R}^2$ such that $\mathcal{DT}(P)$ is isomorphic to G. Specifically, a point set $P \subseteq \mathbb{R}^2$ is said to *realize* G (as a Delaunay triangulation) if $\mathcal{DT}(P)$ is isomorphic to G.[2] The problem of finding a point set that realizes G is called Delaunay Realization. This problem is important not only theoretically, but also practically (see, e.g., [26, 32, 33]). Formally, it is defined as follows.

Delaunay Realization
Input: A triangulation G on n vertices.
Output: If G is realizable as a Delaunay triangulation, then output $P \subseteq \mathbb{R}^2$ that realizes G (as a Delaunay triangulation). Otherwise, output NO.

Dillencourt [14] established necessary conditions for a triangulation to be realizable as a Delaunay triangulation. On the other hand, Dillencourt and Smith [16] established sufficient conditions for a triangulation to be realizable as a Delaunay triangulation. Dillencourt [15] gave a constructive proof showing that any triangulation where all vertices lie on the outer face is realizable as a Delaunay triangulation. Their approach, which results in an algorithm that runs in time $\mathcal{O}(n^2)$, uses a criterion concerning angles of triangles in a hypothetical Delaunay triangulation. In 1994, Sugihara [31] gave a simpler proof that all outerplanar triangulations are realizable as Delaunay triangulations. Later, in 1997, Lambert [22] gave a linear-time algorithm for realizing an outerplanar triangulation as a Delaunay triangulation. More recently, Alam et al. [3] gave yet another constructive proof for outerplanar triangulations.

Hodgson et al. [20] gave a polynomial-time algorithm for checking if a graph is realizable as a convex polyhedron with all vertices on a common sphere. Using this, Rivin [30] designed a polynomial-time algorithm for testing if a graph is realizable as a Delaunay triangulation. Independently, Hiroshima et al. [19] found a simpler polynomial-time algorithm, which relies on the proof of a combinatorial characterization of Delaunay realizable graphs. Both these results are non-constructive, i.e., they cannot output a point set P that realizes the input as a Delaunay triangulation, but only answer YES or NO. It is a long standing open problem to design a finite time algorithm for Delaunay Realization.

[2] As G is triangulation, if $\mathcal{DT}(P)$ is isomorphic to G, then $\mathcal{DT}(P)$ is unique.

Obtaining a constructive algorithm for Delaunay Realization was mentioned as an open problem by Hiroshima et al. [19]. We give the *first* exponential-time algorithm for the Delaunay Realization problem. Our algorithm is based on the computation of two sets of polynomial constraints, defined by the input graph G. In both sets of constraints, the degrees of the polynomials are bounded by 2 and the coefficients are integers. The first set of constraints forces the points on the outer face to form a convex hull,[3] and the second set of constraints ensures that for each edge in G, there is a disc containing only the endpoints of the edge. Roughly speaking, we prove that a triangulation is realizable as a Delaunay triangulation if and only if a point set realizing it as a Delaunay triangulation satisfies every constraint in our two sets of constraints. We proceed by proving that if a triangulation is realizable as a Delaunay triangulation, then there is $P \subseteq \mathbb{Z}^2$ such that $\mathcal{DT}(P)$ is isomorphic to G. This result is crucial to the design of our algorithm, not only for the sake of obtaining an integer solution, but for the sake of obtaining any solution. In particular, it involves a careful manipulation of a (hypothetical) point set in $\mathbb{R}^2$, which allows to argue that it is "safe" to add new polynomials to our two sets of polynomials. Having these new polynomials, we are able to ensure that certain approximate solutions, which we can find in finite time, are actually exact solutions. We show that the special approximate solutions can be computed in polynomial time, and hence we actually solve the problem precisely. To find a solution satisfying our sets of polynomial constraints, our algorithm runs in time $n^{\mathcal{O}(n)}$. All other steps of the algorithm can be executed in polynomial time.

We believe that our contribution is a valuable step forward in the study of algorithms for geometric problems where one is interested in finding a solution rather than only determining whether one exists. Such studies have been carried out for various geometric problems (or their restricted versions) like Unit-Disc Graph Realization [23], Line-Segment Graph Realization [21], Planar Graph Realization (which is the same as Coin Graph Realization) [11], Convex Polygon Intersection Graph Realization [24], and Delaunay Realization. (The above list is not comprehensive; for more details we refer the readers to given citations and references therein.) We note that the higher dimension analogue of Delaunay Realization, called Delaunay Subdivisions Realization, is $\exists\mathbb{R}$-complete; for details on this generalization, see [1].

2 Preliminaries

In this section, we present basic concepts related to Geometry, Graph Theory and Algorithm Design, and establish some of the notation used throughout.

We refer the reader to the books [7, 28] for geometry-related terms that are not explicitly defined here. We denote the set of natural numbers by $\mathbb{N}$, the set of rational numbers by $\mathbb{Q}$ and the set of real numbers by $\mathbb{R}$. By $\mathbb{R}^+$ we denote the set $\{x \in \mathbb{R} \mid x > 0\}$. For $n \in \mathbb{N}$, we use $[n]$ as a shorthand for $\{1, 2, \cdots, n\}$. A point is an element in $\mathbb{R}^2$. We work on Euclidean plane and the Cartesian coordinate system with the underlying bijective mapping of points in the Euclidean plane to vectors in the Cartesian coordinate system. For $p, q \in \mathbb{R}^2$, by $\mathsf{dist}(p, q)$ we denote the distance between p and q in $\mathbb{R}^2$.

Graphs. We use standard terminology from the book of Diestel [13] for graph-related terms not explicitly defined here. For a graph G, $V(G)$ and $E(G)$ denote the vertex and edge sets of G, respectively. For a vertex $v \in V(G)$, $d_G(v)$ denotes the *degree* of v, i.e the number of edges

[3] The convex hull of a point set realizing G forms the outer face of its Delaunay triangulation.

incident on v, in the graph G. For an edge $(u, v) \in E(G)$, u and v are called the *endpoints* of the edge (u, v). For $S \subseteq V(G)$, $G[S]$, and $G - S$ are the subgraphs of G induced on S and $V(G) \setminus S$, respectively. For $S \subseteq V(G)$, we let $N_G(S)$ and $N_G[S]$ denote the open and closed neighbourhoods of S in G, respectively. That is, $N_G(S) = \{v \mid (u, v) \in E(G), u \in S\} \setminus S$ and $N_G[S] = N_G(S) \cup S$. We drop the sub-script G from $d_G(v), N_G(S)$, and $N_G[S]$ whenever the context is clear. A *path* in a graph is a sequence of distinct vertices $v_0, v_1, \ldots, v_\ell$ such that (v_i, v_{i+1}) is an edge for all $0 \leq i < \ell$. Furthermore, such a path is called a v_0 to v_ℓ path. A graph is *connected* if for all distinct $u, v \in V(G)$, there is a u to v path in G. A graph which is not connected is said to be *disconnected*. A graph G is called k*-connected* if for all $X \subseteq V(G)$ such that $|X| < k$, $G - X$ is connected. A *cycle* in a graph is a sequence of distinct vertices $v_0, v_1, \ldots, v_\ell$ such that $(v_i, v_{(i+1) \mod (\ell+1)})$ is an edge for all $0 \leq i \leq \ell$. A cycle C in G is said to be a *non-separating cycle* in G if $G - V(C)$ is connected.

Planar Graphs and Plane Graphs. A graph G is called *planar* if it can be drawn on the plane such that no two edges cross each other except possibly at their endpoints. Formally, an embedding of a graph G is an injective function $\varphi : V(G) \to \mathbb{R}^2$ together with a set $\mathcal{C}$ containing a continuous curve $C_{(u,v)}$ in the plane corresponding to each $(u, v) \in E(G)$ such that $\varphi(u)$ and $\varphi(v)$ are the endpoints of $C_{(u,v)}$. An embedding of a graph G is *planar* if distinct $C, C' \in \mathcal{C}$ intersect only at the endpoints – that is, any point in the intersection of C, C' is an endpoint of both C, C'. A graph that admits a planar embedding is a *planar graph*. Hereafter, whenever we say an embedding of a graph, we mean a planar embedding of it, unless stated otherwise. We often refer to a graph with a fixed embedding on the plane as a *plane graph*. For a plane graph G, the regions in $\mathbb{R}^2 \setminus G$ are called the *faces* of G. We denote the set of faces in G by $F(G)$. Note that since G is bounded and can be assumed to be drawn inside a sufficiently large disc, there is exactly one face in $F(G)$ that is unbounded, which is called the *outer face* of G. A face of G that is not the outer face is called an *inner face* of G. An embedding of a planar graph with the property that the boundary of every face (including the outer face) is a convex polygon is called a *convex drawing*. Below we state propositions related to planar and plane graphs that will be useful later.

▶ **Proposition 1** (Proposition 4.2.5 [13]). *For a* 2*-connected plane graph G, every face of G is bounded by a cycle.*

For a graph G and a face $f \in F(G)$, we let $V(f)$ denote the set of vertices in the cycle by which f is bounded. We often refer to $V(f)$ as the *face boundary* of f.

▶ **Proposition 2** (Proposition 4.2.10 [13]). *For a* 3*-connected planar graph, its face boundaries are precisely its non-separating induced cycles.*

Note that from Proposition 2, for a 3-connected planar graph and its planar embeddings $G_{\mathcal{P}}$ and $G_{\mathcal{P}'}$, it follows that $F(G_{\mathcal{P}}) = F(G_{\mathcal{P}'})$. (In the above we slightly abused the notation, and think of the sets $F(G_{\mathcal{P}})$ and $F(G_{\mathcal{P}'})$ in terms of their bounding cycles, rather than the regions of the plane.) Hence, it is valid to talk about $F(G)$ for a 3-connected planar graph G, even without knowing its embedding on the plane.

▶ **Proposition 3** (Tutte's Theorem [34], also see [8, 25]). *A* 3*-connected planar graph admits a convex embedding on the plane with any face as the outer face. Moreover, such an embedding can be found in polynomial time.*

For a plane graph G and a face $f \in F(G)$, by *stellating* f we mean addition of a new vertex v_f^* inside f and making it adjacent to all $v \in V(f)$. We note that stellating a face of a planar graph results in another planar graph [16].

Triangulations and Delaunay Triangulations. A *triangulation* is a plane graph where each inner face is bounded by a cycle on three vertices. A graph which is isomorphic to a triangulation is called a *triangulated graph*. We state the following simple but useful property of triangulations that will be exploited later.

▶ **Proposition 4.** *Let G be a triangulation with f^* being the outer face. Then, all the degree-2 vertices in G must belong to $V(f^*)$.*

▶ **Proposition 5** (Theorem 9.6 [7])**.** *For a point set $P \subseteq \mathbb{R}^2$ on n points, three points $p_1, p_2, p_3 \in P$ are vertices of the same face of the Delaunay graph of P if and only if the circle through p_1, p_2, p_3 contains no point of P in its interior.*

A *Delaunay triangulation* is any triangulation that is obtained by adding edges to the Delaunay graph. A Delaunay triangulation of a point set P is unique if and only if $\mathcal{DG}(P)$ is a triangulation, which is the case if P is in general position [7]. We refer to the Delaunay triangulation of a point set P by $\mathcal{DT}(P)$ (assuming it is unique, which is the case in our paper). A *triangulated graph* is *Delaunay realizable* if there exists a point set $P \subseteq \mathbb{R}^2$ such that $\mathcal{DT}(P)$ is isomorphic to G. If G has at most three points, then testing if it is Delaunay realizable is solvable in constant time. Also, we can compute an integer representation for it in constant time, if it exists. (Recall that while defining the general position assumption, we assumed that the point set has at least four points. This assumption does not cause any issues because we look for a realization of a graph which has at least four vertices.)

Polynomial Constraints. Let us now give some definitions and notation related to polynomials and sets of polynomial constraints (equalities and inequalities). We refer the reader to the books [5, 6] for algebra-related terms that are not explicitly defined here. For $t, n \in \mathbb{N}$ and a set C, a polynomial $\mathcal{P} = \Sigma_{i\in[t]} a_i \cdot (\Pi_{j\in[n]} X_j^{d_j^i})$ on n variables and t terms is said to be a polynomial over C if for all $i \in [t]$, $j \in [n]$ we have $a_i \in C$ and $d_j^i \in \mathbb{N}$. Furthermore, the *degree* of the polynomial $\mathcal{P}$ is defined to be $\max_{i\in[t]}(\Sigma_{j\in[n]} d_j^i)$. We denote the set of polynomials on n variables $X_1, X_2, \cdots X_n$ with coefficients in C by $C[X_1, X_2, \cdots X_n]$.

A *polynomial constraint* $\mathcal{C}$ on n variables with coefficients from $C \subseteq \mathbb{R}$ is a sequence $\mathcal{P}\Delta 0$, where $\mathcal{P} \in C[X_1, X_2, \cdots, X_n]$ and $\Delta \in \{=, \geq, >, \leq, <\}$. The *degree* of such a constraint is the degree of $\mathcal{P}$, and it is said to be an equality constraint if Δ is '='. We say that the constraint is *satisfied* by an element $(\bar{x}_1, \bar{x}_2, \cdots \bar{x}_n) \in \mathbb{R}^n$ if $\mathcal{P}(\bar{x}_1, \bar{x}_2, \cdots \bar{x}_n)\Delta 0$.[4] Given a set $\mathfrak{C}$ of polynomial constraints on n variables, $X_1, X_2, \cdots, X_n$, and with coefficients from $C \subseteq \mathbb{R}$, we say that an element $(\bar{x}_1, \bar{x}_2, \cdots \bar{x}_n) \in \mathbb{R}^n$ *satisfies* $\mathfrak{C}$ if for all $\mathcal{C} \in \mathfrak{C}$, we have that $(\bar{x}_1, \bar{x}_2, \cdots \bar{x}_n)$ satisfies $\mathcal{C}$. In this case, $(\bar{x}_1, \bar{x}_2, \cdots \bar{x}_n)$ is also called a *solution* of $\mathfrak{C}$. Furthermore, $\mathfrak{C}$ is said to be *satisfiable* (in $\mathbb{R}$) if there exists $(\bar{x}_1, \bar{x}_2, \cdots \bar{x}_n) \in \mathbb{R}^n$ satisfying $\mathfrak{C}$.

Below we state a result regarding a method for solving a finite set of polynomial constraints, which will be used by our algorithm. This result is a direct implication of Propositions 3.8.1 and 4.1 in [29] (see also [6]).

▶ **Proposition 6** (Propositions 3.8.1 and 4.1 in [29])**.** *Let $\mathfrak{C}$ be a set of m polynomial constraints of degree 2 on n variables with coefficients in $\mathbb{Z}$ whose bitsizes are bounded by $\mathcal{O}(1)$. Then, in time $m^{\mathcal{O}(n)}$ we can decide if $\mathfrak{C}$ is satisfiable in $\mathbb{R}$. Moreover, if $\mathfrak{C}$ is satisfiable in $\mathbb{R}$, then in time $m^{\mathcal{O}(n)}$ we can also compute a (satisfiable) set $\widehat{\mathfrak{C}}$ of n polynomial constraints, $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$, with coefficients in $\mathbb{Z}$, where for all $i \in [n]$, we have that $\mathcal{C}_i$ is an equality constraint on X_i (only), and a solution of $\widehat{\mathfrak{C}}$ is also a solution of $\mathfrak{C}$.*

[4] Here, $\mathcal{P}(\bar{x}_1, \bar{x}_2, \cdots \bar{x}_n)$ is the evaluation of $\mathcal{P}$, where the variable X_i is assigned the value $\bar{x}_i$, for $i \in [n]$.

3 Restricted-Delaunay Realization: Generating Polynomials

In this section, we generate a set of polynomials that encodes the realizability of a triangulation as a Delaunay triangulation in the case where the outer face of the Delaunay triangulation is known. More precisely, we suppose that the outer faces of G and the Delaunay triangulation are the same. For the general case where we might not know a priori which is the face in G that is supposed to be the outer face of the Delaunay triangulation (this is the case when G is a maximal planar graph), we will "guess" the outer face and then use our restricted version to solve the problem. Formally, we solve the following problem.

RESTRICTED-DELAUNAY REALIZATION (RES-DR)
Input: A triangulation G with outer face f^*.
Output: A set of polynomial constraints $\mathsf{Const}(G)$ such that $\mathsf{Const}(G)$ is satisfiable if and only if G is realizable as a Delaunay triangulation with f^* as the outer face.

Let (G, f^*) be an instance of RES-DR, and let n denote $|V(G)|$. We denote $V(G)$ by the set $\{v_1, v_2, \cdots v_n\}$. Note that except possibly f^*, each of the faces of G is bounded by a cycle on three vertices. With each $v_i \in V(G)$ we associate two variables, X_i and Y_i, which correspond to the values of the x and y coordinates of v_i in the plane. Furthermore, we let P_i denote the vector (X_i, Y_i). We let $\bar{X}$ denote the value that some solution of $\mathsf{Const}(G)$ assigns to the variable X. Accordingly, we denote $\bar{P}_i = (\bar{X}_i, \bar{Y}_i)$. For the sake of clarity, we sometimes abuse the notation $\bar{P}_i$ by letting it denote both $\bar{P}_i$ and P_i (this is done in situations where both interpretations are valid).

Our algorithm is based on the computation of two sets of polynomial constraints of bounded degree and integer coefficients. Informally, we have one set of inequalities which ensures that the points to which vertices of f^* are mapped are in convex position, and another set of inequalities which ensures that for each $(v_i, v_j) \in E(G)$, there exists a disc containing $(\bar{X}_i, \bar{Y}_i)$ and $(\bar{X}_j, \bar{Y}_j)$ on its boundary and excluding all other points $(\bar{X}_k, \bar{Y}_k)$. (While other sets of inequalities may be devised to ensure these properties, we subjectively found the two sets presented here the easiest to employ.)

3.1 Inequalities Ensuring that the Outer Face Forms the Convex Hull

We first generate the set of polynomial constraints ensuring that the points associated with the vertices in f^* form the convex hull of the output point set. Here, we also ensure that the vertices in f^* have the same cyclic ordering (given by the cycle bounding f^*) as the points corresponding to them have in the convex hull. Note that the edges of the convex hull are present in any Delaunay triangulation [7]. Moreover, the convex hull of a point set forms the outer face of its Delaunay triangulation. To formulate our equations, we rely on the notions of *left and right turns*. Their definitions are the same as those in the book [10], which uses *cross product* to determine whether a turn is a left turn or a right turn. For the sake of clarity, we also explain these notions below.

Left and Right Turns. Consider two vectors (or points) $\bar{P}_1$ and $\bar{P}_2$, denoting some (x_1, y_1) and (x_2, y_2), respectively. The cross product $\bar{P}_1 \times \bar{P}_2$ of $\bar{P}_1$ and $\bar{P}_2$ is defined as follows.

$$\bar{P}_1 \times \bar{P}_2 = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = x_1 y_2 - x_2 y_1.$$

If $\bar{P}_1 \times \bar{P}_2 > 0$, then $\bar{P}_1$ is said to be *clockwise* from $\bar{P}_2$ (with respect to the origin $(0,0)$). Else, if $\bar{P}_1 \times \bar{P}_2 < 0$, then $\bar{P}_1$ is said to be *counterclockwise* from $\bar{P}_2$. Otherwise (if $\bar{P}_1 \times \bar{P}_2 = 0$), $\bar{P}_1$ and $\bar{P}_2$ are said to be *collinear*. Given line segments $\overline{P_0 P_1}$ and $\overline{P_1 P_2}$, we would like to determine the type of turn taken by the angle $\angle P_0P_1P_2$. To this end, we check whether the directed segment $\overline{P_0P_2}$ is clockwise or counterclockwise from $\overline{P_0P_1}$. Towards this, we first compute the cross product $(\bar{P}_2 - \bar{P}_0) \times (\bar{P}_1 - \bar{P}_0)$. If $(\bar{P}_2 - \bar{P}_0) \times (\bar{P}_1 - \bar{P}_0) > 0$, then $\overline{P_0P_2}$ is clockwise from $\overline{P_0P_1}$, and we say that we take a *right turn* at $\bar{P}_1$. Else, if $(\bar{P}_2 - \bar{P}_0) \times (\bar{P}_1 - \bar{P}_0) < 0$, then $\overline{P_0P_2}$ is counterclockwise from $\overline{P_0P_1}$, and we say that we take a *left turn* at $\bar{P}_1$. Otherwise, we make *no turn* at $\bar{P}_1$. Note that the computation of $(\bar{P}_2 - \bar{P}_0) \times (\bar{P}_1 - \bar{P}_0)$ can be done as follows.

$$(\bar{P}_2 - \bar{P}_0) \times (\bar{P}_1 - \bar{P}_0) = \begin{vmatrix} x_2 - x_0 & x_1 - x_0 \\ y_2 - y_0 & y_1 - y_0 \end{vmatrix} = x_2y_1 - x_2y_0 - x_0y_1 - x_1y_2 + x_1y_0 + x_0y_2.$$

The Polynomials. For three vectors (or points) $\bar{P}_0 = (x_0, y_0), \bar{P}_1 = (x_1, y_1)$ and $\bar{P}_2 = (x_2, y_2)$, by $\mathsf{Con}(\bar{P}_0, \bar{P}_1, \bar{P}_2)$ we denote the polynomial $x_2y_1 - x_2y_0 - x_0y_1 - x_1y_2 + x_1y_0 + x_0y_2$. Note that $\mathsf{Con}(\bar{P}_0, \bar{P}_1, \bar{P}_2)$ determines whether we have a right, left or no turn at $\bar{P}_1$.

Before stating the constraints based on these polynomials, let us recall the well-known fact stating that a non-intersecting polygon is convex if and only if every interior angle of the polygon is less than 180°. While we ensure the non-intersecting constraint later, the characterization of each angle being less than 180° is the same as taking a *right* (or *left*) turn at P_j for every three consecutive points P_i, P_j and P_k of the polygon. We will use this characterization to enforce convexity on the points corresponding to the vertices in $V(f^*)$. Let us also recall that f^* is a cycle C^* in G. Next, whenever we talk about consecutive vertices in C^*, we always follow clockwise direction.

For every three consecutive vertices v_i, v_j and v_k in C^*, we add the following inequality:

$$\mathsf{Con}(P_i, P_j, P_k) > 0.$$

These inequalities ensure that in any output point set, the points corresponding to vertices in $V(f^*)$ are in convex position (together with the non-intersecting condition to be ensured later).

Next, we further need to ensure that all the points which correspond to vertices in $V(G) \setminus V(f^*)$ belong to the interior of the convex hull formed by the points corresponding to vertices in $V(f^*)$ and the polygon formed by the points corresponding to $V(f^*)$ is non-self intersecting. For this purpose, we crucially rely on the following property of convex hulls (or convex polygons): For any edge of the convex hull, it holds that all the points, except for the endpoints of the edge, are located in one of the sides of the edge. Using this property, we know that for any two consecutive vertices v_i and v_j in C^*, all points are on one side of the line associated with v_i and v_j. Since at each $v_i \in C^*$ we ensure that we turn right, we must have all the points located on the right of the line defined by the edge (v_i, v_j). This, in turn, implies that for every pair of consecutive vertices v_i and v_j in C^*, for any vertex $v_k \in V(G) \setminus V(f^*)$, we must be turning left at v_k (according to the ordered triplet (v_i, v_k, v_j)). Hence, we add the following inequalities:

$$\mathsf{Con}(P_i, P_k, P_j) < 0.$$

where v_i and v_j are consecutive vertices of C^* and $v_k \in V(G) \setminus \{v_i, v_j\}$.

We denote the set of inequalities generated above by $\mathsf{Con}(G)$.

3.2 Inequalities Guaranteeing Existence of Edges

For each edge $(v_i, v_j) \in E(G)$, we add two new variables, X_{ij} and Y_{ij}, to indicate the coordinates of the centre of a disc that realizes the edge (v_i, v_j). There might exist many discs that realize the edge (v_i, v_j), but we are interested in only one such disc, say C_{ij}.

Note that C_{ij} should contain $(\bar{X}_i, \bar{Y}_i)$ and $(\bar{X}_j, \bar{Y}_j)$ on its boundary, and it should not contain any $(\bar{X}_k, \bar{Y}_k)$ such that $k \notin \{i, j\}$. Towards this, for each edge $(v_i, v_j) \in E(G)$, we add a set of inequalities that we denote by $\mathsf{Dis}(v_i, v_j)$. Note that the radius r_{ij} of C_{ij} is given by $r_{ij}^2 = (X_i - X_{ij})^2 + (Y_i - Y_{ij})^2$ (if (X_i, Y_i) lies on the boundary) and by $r_{ij}^2 = (X_j - X_{ij})^2 + (Y_j - Y_{ij})^2$ (if (X_j, Y_j) lies on the boundary). Therefore, we want to ensure the following.

$$(X_j - X_{ij})^2 + (Y_j - Y_{ij})^2 = (X_i - X_{ij})^2 + (Y_i - Y_{ij})^2$$

$$\Rightarrow X_i^2 - X_j^2 + Y_i^2 - Y_j^2 - 2X_{ij}X_i - 2Y_{ij}Y_i + 2X_{ij}X_j + 2Y_{ij}Y_j = 0.$$

Hence, we add the above constraint to $\mathsf{Dis}(v_i, v_j)$. Further, we want to ensure that for each $k \in [n] \setminus \{i, j\}$, (X_k, Y_k) does not belong to C_{ij}. Therefore, for each $k \in [n] \setminus \{i, j\}$, the following must hold.

$$(X_k - X_{ij})^2 + (Y_k - Y_{ij})^2 - (X_i - X_{ij})^2 - (Y_i - Y_{ij})^2 > 0$$

$$\Rightarrow X_k^2 - X_i^2 + Y_k^2 - Y_i^2 - 2X_{ij}X_k - 2Y_{ij}Y_k + 2X_{ij}X_i + 2Y_{ij}Y_i > 0$$

Hence, we also add the above constraint to $\mathsf{Dis}(v_i, v_j)$ for $k \in [n] \setminus \{i, j\}$. Overall, we denote $\mathsf{Dis}(G) = \bigcup_{(v_i, v_j) \in E(G)} \mathsf{Dis}(v_i, v_j)$. This completes the description of all inequalities relevant to this section.

3.3 Correctness

Let us denote $\mathsf{Const}(G) = \mathsf{Con}(G) \cup \mathsf{Dis}(G)$. We begin with the following observation. Here, to bound the number of variables, we rely on the fact that G is a planar graph, its number of edges is upper bounded by $3n$, and hence in total we introduced less than $8n$ variables.

▶ **Observation 7.** *The number of constraints in $\mathsf{Const}(G)$ is bounded by $\mathcal{O}(n^2)$ and the total number of variables is bounded by $\mathcal{O}(n)$. Moreover, each constraint in $\mathsf{Const}(G)$ is of degree 2, and its coefficients belong to $\{-2, -1, 0, 1, 2\}$.*

Now, we state the central lemma establishing the correctness of our algorithm for Res-DR.

▶ **Lemma 8.** *A triangulation G with outer face f^* is realizable as a Delaunay triangulation with f^* as its outer face if and only if $\mathsf{Const}(G)$ is satisfiable.*

Proof. Let G be a triangulation realizable as a Delaunay triangulation with f^* as the outer face of the Delaunay triangulation. Then, there exists $P \subseteq \mathbb{R}^2$ such that $\mathcal{DT}(P)$ is isomorphic to G and f^* is the outer face of $\mathcal{DT}(P)$. Furthermore, for each $(\bar{P}_i, \bar{P}_j) \in E(\mathcal{DT}(P))$, there exists a disc C_{ij} which contains $\bar{P}_i$ and $\bar{P}_j$ on its boundary, and which contains no point $\bar{P}_k$, $k \in [n] \setminus \{i, j\}$, on neither its boundary nor its interior. We let $\bar{P}_{ij}$ denote the centre of C_{ij}. Let $\mathcal{P}$ be the vector assigning $\bar{P}_i$ to the vertex $v_i \in V(G)$ and $\bar{P}_{ij}$ to the centre of the disc C_{ij}. We note that the vertices of f^* are in convex position in $\mathcal{DT}(P)$. Clearly, we then have that $\mathcal{P}$ satisfies $\mathsf{Const}(G)$. This concludes the proof of the forward direction.

In the reverse direction, consider some $\mathcal{P}$ that satisfies $\mathsf{Const}(G)$. By our polynomial constraints, $\mathcal{P}$ assigns some $\bar{P}_i$ to each vertex $v_i \in V(G)$, such that for each edge $(v_i, v_j) \in E(G)$, it lets $\bar{P}_{ij}$ be the centre of a disc C_{ij} containing $\bar{P}_i$ and $\bar{P}_j$ (on its boundary) and no point $\bar{P}_k$ where $k \notin \{i, j\}$. Further, we let $P = \{\bar{P}_i \mid i \in [n]\}$. By the construction of $\mathsf{Const}(G)$, it follows that if $(v_i, v_j) \in E(G)$, then $(\bar{P}_i, \bar{P}_j) \in \mathcal{DT}(P)$ and the points in P corresponding to vertices in $V(f^*)$ form the convex hull of P. This implies that the points corresponding to the vertices in $V(f^*)$ are on the outer face of $\mathcal{DT}(P)$. From Theorem 9.1 in [7], it follows that $|E(G)| \leq |E(\mathcal{DT}(P))|$. Thus, $E(G) = E(\mathcal{DT}(P))$. This concludes the proof of the reverse direction. ◀

The next theorem follows from the construction of $\mathsf{Const}(G)$, Observation 7 and Lemma 8.

▶ **Theorem 9.** *Let G be a triangulation on n vertices with f^* as the outer face. Then, in time $\mathcal{O}(n^2)$, we can output a set of polynomial constraints $\mathsf{Const}(G)$ such that G is realizable as a Delaunay triangulation with f^* as its outer face if and only if $\mathsf{Const}(G)$ is satisfiable. Moreover, $\mathsf{Const}(G)$ consists of $\mathcal{O}(n^2)$ constraints and $\mathcal{O}(n)$ variables, where each constraint is of degree 2 and with coefficients only from $\{-2, -1, 0, 1, 2\}$.*

4 Restricted-Delaunay Realization: Replacing Points by Discs

Let G be a triangulation on n vertices with f^* as its outer face. Suppose that G is realizable as a Delaunay triangulation where the points corresponding to vertices in $V(f^*)$ belong to the outer face. By Theorem 9, it follows that $\mathsf{Const}(G)$ is satisfiable. Let n^* denote the number of variables of $\mathsf{Const}(G)$. Since $\mathsf{Const}(G)$ is satisfiable, there exists $\mathcal{Q}$ satisfying $\mathsf{Const}(G)$. Let $\bar{Q}_i$ be the value assigned to the vertex $v_i \in V(G)$ for $i \in [n]$. Let $Q = \{\bar{Q}_i \mid v_i \in V(G)\}$. Recall that apart from assigning points in the plane to vertices in $V(G)$, $\mathcal{Q}$ assigns to each $(v_i, v_j) \in E(G)$, a point $\bar{Q}_{ij}$ corresponding to the centre of some disc, say C'_{ij}, containing $\bar{Q}_i, \bar{Q}_j$ on its boundary and excluding all other points in Q.

In this section, we prove that for any given $\beta \in \mathbb{R}^+$, there exists a set of discs of radius β, one for each vertex in $V(G)$, with the following property. If for every $v_i \in V(G)$, we choose some point $\bar{P}_{C_i}$ inside or on the boundary of its disc C_i, we get that $\mathcal{DT}(Q)$ and the Delaunay triangulation of our set of chosen points are isomorphic.

We start with two simple observations, where the second directly follows from the definition of the constraints in $\mathsf{Const}(G)$.

▶ **Observation 10.** *Let $(a, b), (x, y) \in \mathbb{R}^2$ be two points and $\alpha \in \mathbb{R}^+$. Then, $\mathsf{dist}((\alpha a, \alpha b), (\alpha x, \alpha y)) = \alpha \cdot \mathsf{dist}((a, b), (x, y))$.*

▶ **Observation 11.** *Let G be a triangulation on n vertices with f^* as its outer face. If $\mathcal{Q}$ is a solution of $\mathsf{Const}(G)$, then for any $\alpha \in \mathbb{R}^+$, it holds that $\alpha\mathcal{Q}$ also satisfies $\mathsf{Const}(G)$.*

In what follows, we create a point set P such that $\mathcal{DT}(P)$ is isomorphic to G, where the points corresponding to vertices in $V(f^*)$ form the outer face of $\mathcal{DT}(P)$. We then show that this point set defines a set of discs with the desired property – for each $v_i \in V(G)$, it defines one disc C_i with $\bar{P}_i$ as centre and with radius $r^* \geq \beta > 0$ (to be determined), such that, roughly speaking, each point of C_i is a valid choice for v_i. For this purpose, we first define the real numbers, d_N, d_C, and d_A, which are necessary to determine r^* and P. Informally, d_N ensures that the discs we create around vertices do not intersect, d_C will be used to ensure existence of specific edges, d_A will be used to ensure that "convex hull property" is satisfied. These (positive) real numbers are defined as follows.

- Let $d_N = \min_{i,j \in [n], i \neq j} \{\mathsf{dist}(\bar{Q}_i, \bar{Q}_j)\}$, i.e., d_N is the minimum distance between any pair of distinct points in Q.
- Let $d_C = \min_{i,j,k \in [n], i \neq j, i \neq k, j \neq k} \{\mathsf{dist}(C'_{ij}, \bar{Q}_k)\}$, i.e., d_C denotes the minimum distance between a point corresponding to a vertex in $V(G)$ and a disc realizing an edge non-incident to it. (Recall that C'_{ij} is defined at the beginning of this section.) Note that $d_C > 0$ because in the above definition of d_C, we have only considered those disc and point pairs where the point lies outside the disc.

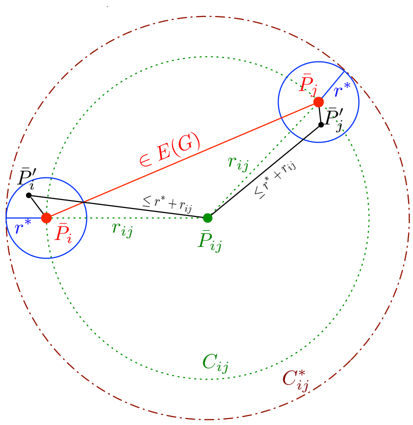

Figure 1 Proving that the points $\bar{P}'_i$ and $\bar{P}'_j$ lie inside the disc C^*_{ij} (Lemma 12).

- For each edge (v_i, v_j) of the cycle corresponding to the outer face f^*, let L^s_{ij} be the line containing $\bar{Q}_i$ and $\bar{Q}_j$. Moreover, let $s_{ij} = \min_{k\in[n]\setminus\{i,j\}}\{\mathsf{dist}(L^s_{ij}, \bar{Q}_k)\}$, i.e., the minimum distance between a line of the convex hull and another point. Finally, $d_A = \min_{(v_i,v_j)\in E(f^*)}\{s_{ij}\}$. We note that $d_A > 0$. This follows from the definition of $\mathsf{Con}(G)$ in Section 3.1.

Define $r = \frac{1}{3}\min\{d_N, d_C, d_A\}$. Notice that $r, \beta > 0$. Now, we compute r^* and P according to three cases:

1. If $r \geq \beta$, then $r^* = r$ and $\mathcal{P} = \mathcal{Q}$ (thus, $P = Q$).
2. Else if $1 \leq r < \beta$, then $\mathcal{P} = \beta\mathcal{Q}$, where $\beta\mathcal{Q} = \{(\beta\bar{X}, \beta\bar{Y}) \mid (\bar{X}, \bar{Y}) \in \mathcal{Q}\}$ and $r^* = \beta r$.
3. Otherwise ($r < 1$ and $r < \beta$), $\mathcal{P} = \frac{\beta}{r}\mathcal{Q}$ and $r^* = \frac{\beta}{r}r = \beta$.

By Observation 11, in each of the cases described above, we have that $\mathcal{P}$ satisfies $\mathsf{Const}(G)$. Hereafter, we will be working only with $\mathcal{P}$ and r^* as defined above. We let $\bar{P}_i$ be the point assigned to the vertex v_i, and $P = \{\bar{P}_i \mid i \in [n]\}$. Moreover, we let $\bar{P}_{ij}$ be the centre of the disc C_{ij} for the edge $(v_i, v_j) \in E(G)$ that is assigned by $\mathcal{P}$.

Next, we define d^*_N, d^*_C, and d^*_A in a manner similar to the one used to define d_N, d_C and d_A. Let $d^*_N = \min_{i,j\in[n],i\neq j}\{\mathsf{dist}(\bar{P}_i, \bar{P}_j)\} \geq 3r^*$, and $d^*_C = \min_{i,j,k\in[n],i\neq j,i\neq k,j\neq k}\{\mathsf{dist}(C_{ij}, \bar{P}_k)\}$ $\geq 3r^*$. For each edge (v_i, v_j) of the cycle corresponding to the outer face f^*, let L^s_{ij} be the line containing $\bar{P}_i$ and $\bar{P}_j$. Further, let $s_{ij} = \min_{k\in[n]\setminus\{i,j\}}\{\mathsf{dist}(L^s_{ij}, \bar{P}_k)\}$. Finally, let $d^*_A = \min_{(v_i,v_j)\in E(F)}\{s_{ij}\}$. Note that by Observation 10, we have that $d^*_N \geq 3r^*$, $d^*_C \geq 3r^*$, and $d^*_A \geq 3r^*$.

For each $v_i \in V(G)$, let C_i be the disc of radius r^* and centre $\bar{P}_i$. We now prove that if for each vertex $v_i \in V(G)$, we choose a point $\bar{P}'_i$ inside or on the boundary of C_i, then we obtain a point set P' such that $\mathcal{DT}(P)$ and $\mathcal{DT}(P')$ are isomorphic. Furthermore, the points on the outer face of $\mathcal{DT}(P)$, and also $\mathcal{DT}(P')$, correspond to the vertices in $V(f^*)$.

▶ **Lemma 12** (♠). [5] *$\mathcal{DT}(P)$ is isomorphic to $\mathcal{DT}(P')$ and the outer face of $\mathcal{DT}(P')$ consists of all the points corresponding to vertices in $V(f^*)$.*

[5] Proofs of results marked with ♠ is relegated to the full version of the paper [2].

▶ **Theorem 13.** *Let G be a triangulation on n vertices with f^* as its outer face, realizable as a Delaunay triangulation where the points corresponding to vertices of f^* lie on the outer face. Moreover, let $\mathcal{Q}$ be a solution of* $\mathsf{Const}(G)$ *and $\beta \in \mathbb{R}^+$. Then, there is a solution $\mathcal{P}$ of* $\mathsf{Const}(G)$*, assigning a set of points $P \subseteq \mathbb{R}^2$ to vertices of G, such that for each $v_i \in V(G)$, there exists a disc C_i with centre $\bar{P}_i$ and radius at least β for which the following condition holds. For any $P' = \{\bar{P}'_i \mid \bar{P}'_i \in C_i, i \in [n]\}$, it holds that $\mathcal{DT}(P')$ is isomorphic to $\mathcal{DT}(P)$, and the points corresponding to vertices of f^* lie on the outer face of $\mathcal{DT}(P')$.*

Proof. The proof of theorem follows directly from the construction of r^*, the discs C_i for $i \in [n]$, and Lemma 12. ◀

5 Delaunay Realization: Integer Coordinates

In this section, we prove our main theorem:

▶ **Theorem 14.** *Given a triangulation G on n vertices, in time $n^{\mathcal{O}(n)}$ we can either output a point set $P \subseteq \mathbb{Z}^2$ such that G is isomorphic to $\mathcal{DT}(P)$, or correctly conclude that G is not Delaunay realizable.*

The Outer Face of the Output. First, we explain how to identify the outer face f^* of the output (in case the output should not be NO). For this purpose, let f_{out} denote the outer face of G (according to the embedding of the triangulation G, given as the input). Recall our assumption that $n \geq 4$. Let us first consider the case where G is not a maximal planar graph, i.e., f_{out} consists of at least four vertices. Suppose that the output is not NO. Then, for any point set $P \subseteq \mathbb{R}^2$ that realizes G as a Delaunay triangulation, it holds that the points corresponding to the vertices of f_{out} form the outer face of $\mathcal{DT}(P)$. Thus, in this case, we simply set $f^* = f_{\mathsf{out}}$. Next, consider the case where G is a maximal planar graph. Again, suppose that the output is not NO. Then, for a point set $P \subseteq \mathbb{R}^2$ that realizes G as a Delaunay triangulation, the outer face of $\mathcal{DT}(P)$ need not be the same as f_{out}. To handle this case, we "guess" the outer face of the output (if it is not NO). More precisely, we examine each face f of G separately, and attempt to solve the "integral version" of RES-DR with f^* set to f, and where G is embedded with f^*, rather than f_{out}, as its outer face. Here, note that a maximal planar graph is 3-connected [35], and therefore, by Proposition 3, we can indeed compute an embedding of G with f^* as the outer face.

The number of iterations is bounded by $\mathcal{O}(n)$ (since the number of faces of G is bounded by $\mathcal{O}(n)$). Thus, from now on, we may assume that we seek only Delaunay realizations of G where the outer face is the same as the outer face of G (that we denote by f^*).

Sieving NO-Instances. We compute the set $\mathsf{Const}(G)$ as described in Section 3. From Theorem 9, we know that G is realizable as a Delaunay triangulation with the points corresponding to f^* on the outer face if and only if $\mathsf{Const}(G)$ is satisfiable. Using Proposition 6, we check whether $\mathsf{Const}(G)$ is satisfiable, and if the answer is negative, then we return NO. Thus, we next focus on the following problem.

INTEGRAL DELAUNAY REALIZATION (INT-DR) **Input:** A triangulation G with outer face f^* that is realizable as a Delaunay triangulation with outer face f^*. **Output:** A point set $P \subseteq \mathbb{Z}^2$ realizing G as a Delaunay triangulation with outer face f^*.

Similarly, we define the intermediate RATIONAL DELAUNAY REALIZATION (RATIONAL-DR) problem – here, however, $P \subseteq \mathbb{Q}^2$ rather than $\mathbb{Z}^2$. To prove Theorem 14, it is sufficient to prove the following result, which is the objective of the rest of this paper.

▶ **Lemma 15.** *Int-DR is solvable in time* $n^{\mathcal{O}(n)}$.

In what follows, we crucially rely on the fact that by Theorem 13, for all $\beta \in \mathbb{R}^+$, there is a solution $\mathcal{P}$ of $\mathsf{Const}(G)$ that assigns a set of points $P \subseteq \mathbb{R}^2$ to the vertices of G, such that for each $v_i \in V(G)$, there exists a disc C_i with radius at least β, satisfying the following condition: For any $P' = \{\bar{P}'_i \mid P'_i \in C_i, i \in [n]\}$, it holds that $\mathcal{DT}(P')$ is isomorphic to G with points corresponding to the vertices in f^* on the outer face (in the same order as in f^*).

As it would be cleaner to proceed while working with squares, we need the next observation.

▶ **Observation 16.** *Every disc C with radius at least 2 contains a square of side length at least 2 and with the same centre.*

We next extend $\mathsf{Const}(G)$ to a set $\mathsf{ConstSqu}(G)$, which explicitly ensures that there exists a *square* around each point in the solution such that the point can be replaced by any point in the square. Thus, rather than discs of radius 2 (whose existence, in some solution, is proven by choosing $\beta = 2$), we consider squares with side length 2 given by Observation 16, and force our constraints to be satisfied at the corner points of the squares. For this purpose, for each $v_i \in V(G)$, apart from adding constraints for the point $P_i = (X_i, Y_i)$ (which can be regarded as a disc of radius 0 in the previous setting), we also have constraints for the corner points of the square of side length 2 whose centre is P_i. For technical reasons, we also add constraints for the intersection points of perpendicular bisectors. For any constraint where P_i appears, we make copies for the points $P_i = (X_i, Y_i)$, $P_i^1 = (X_i - 1, Y_i - 1)$, $P_i^2 = (X_i - 1, Y_i + 1)$, $P_i^3 = (X_i + 1, Y_i - 1), P_i^4 = (X_i + 1, Y_i + 1), P_i^5 = (X_i - 1, Y_i), P_i^6 = (X_i, Y_i + 1), P_i^7 = (X_i + 1, Y_i), P_i^8 = (X_i, Y_i - 1)$.[6]

Inequalities that ensure the outer face forms the convex hull. We generate the set of constraints that ensure the points corresponding to vertices in $V(f^*)$ form a convex hull of the output point set. Let C^* be the cycle of the outer face f^*. Whenever we say consecutive vertices in C^*, we always follow clockwise direction. For three consecutive vertex v_i, v_j and v_k in C^*, for every $Z_i \in \{P_i\} \cup \{P_i^\ell \mid \ell \in [8]\}$, $Z_j \in \{P_j\} \cup \{P_j^\ell \mid \ell \in [8]\}$ and $Z_k \in \{P_k\} \cup \{P_k^\ell \mid \ell \in [8]\}$, we add the inequality $\mathsf{Con}(Z_i, Z_j, Z_k) > 0$. This ensures that the points corresponding to vertices in $V(f^*)$ are in convex position in any output point set. Further, we want all the points which correspond to the vertices in $V(G) \setminus V(f^*)$ to be in the interior of the convex hull formed by the points corresponding to vertices in $V(f^*)$. To achieve this, for each pair of vertices v_i, v_j that are consecutive vertices of C^*, $v_k \in V(G) \setminus \{v_i, v_j\}$, $Z_i \in \{P_i\} \cup \{P_i^\ell \mid \ell \in [8]\}$, $Z_j \in \{P_j\} \cup \{P_j^\ell \mid \ell \in [8]\}$ and $Z_k \in \{P_k\} \cup \{P_k^\ell \mid \ell \in [8]\}$, we add $\mathsf{Con}(Z_i, Z_k, Z_j) < 0$. We call the above set of polynomial constraints $\mathsf{ConSqu}(G)$.

Inequalities that guarantee existence of edges. For each edge $(v_i, v_j) \in E(G)$, we add three new variables, namely X_{ij}, Y_{ij} and r_{ij}. These newly added variables will correspond to the centre and radius of a disc that realizes the edge (v_i, v_j). There might exist many such discs, but we are interested in only one such disc. In particular, (X_{ij}, Y_{ij}) corresponds to centre of one such discs, say C_{ij}, with radius r_{ij}, containing all the points in $\{P_i\} \cup \{P_i^\ell \mid \ell \in [8]\}$ and $\{P_j\} \cup \{P_j^\ell \mid \ell \in [8]\}$ but none of the points in $\{P_k \mid k \in [n] \setminus \{i, j\}\} \cup \{P_k^\ell \mid \ell \in [8], k \in [n] \setminus \{i, j\}\}$. Towards this, we add a set of inequalities for each edge $(v_i, v_j) \in E(G)$, which we will denote by $\mathsf{DisSqu}(v_i, v_j)$. For each $Z \in \{P_i, P_j\} \cup \{P_i^\ell, P_j^\ell \mid \ell \in [8]\}$, we add the following inequalities to $\mathsf{DisSqu}(v_i, v_j)$, ensuring that C_{ij} contains $Z = (Z_X, Z_Y)$.

$$Z_X^2 + X_{ij}^2 - 2Z_X X_{ij} + Z_Y^2 + Y_{ij}^2 - 2Z_Y Y_{ij} - r_{ij}^2 \leq 0.$$

[6] We remark that we do not create new variables for the corresponding x- and y-coordinates for points P_i^ℓ, for $i \in [n]$.

Further, we want to ensure that for each $k \in [n] \setminus \{i, j\}$, $Z \in \{P_k\} \cup \{P_k^\ell \mid \ell \in [8]\}$ does not belong to C_{ij}. Hence, for each such $Z = (Z_X, Z_Y)$, the following must hold.

$$Z_X^2 + X_{ij}^2 - 2Z_X X_{ij} + Z_Y^2 + Y_{ij}^2 - 2Z_Y Y_{ij} - r_{ij}^2 > 0.$$

Hence, we add the above constraint to $\mathsf{DisSqu}(v_i, v_j)$ for $k \in [n] \setminus \{i, j\}$. We denote $\mathsf{DisSqu}(G) = \bigcup_{(v_i,v_i)\in E(G)} \mathsf{DisSqu}(v_i, v_j)$.

This completes the description of all the constraints we need. We let $\mathsf{ConstSqu}(G) = \mathsf{ConSqu}(G) \cup \mathsf{DisSqu}(G)$. We let n^* denote the number of variables appearing in $\mathsf{ConstSqu}(G)$. Note that $n^* = \mathcal{O}(n)$ and the number of constraints in $\mathsf{ConstSqu}(G)$ is bounded by $\mathcal{O}(n^2)$.

▶ **Theorem 17.** *Let G be a triangulation on n vertices with f^* as the outer face. Then, in time $\mathcal{O}(n^2)$ we can find a set of polynomial constraints $\mathsf{ConstSqu}(G)$ such that G is realizable as a Delaunay triangulation with f^* as its outer face if and only if $\mathsf{ConstSqu}(G)$ is satisfiable. Moreover, $\mathsf{ConstSqu}(G)$ consists of $\mathcal{O}(n^2)$ constraints and $\mathcal{O}(n)$ variables, where each constraint is of degree 2 and with coefficients only from $\{-10, -9, \ldots, 10\}$.*

Proof. Follows from the construction of $\mathsf{ConstSqu}(G)$, Lemma 8, and Theorems 13. ◀

Having proved Theorem 17, we use Proposition 6 to decide in time $n^{\mathcal{O}(n)}$ if $\mathsf{ConstSqu}(G)$ is satisfiable. Recall that if the answer is negative, then we returned NO. We compute a "good" approximate solution as we describe next. First, by Proposition 6, in time $n^{\mathcal{O}(n)}$ we compute a (satisfiable) set $\mathfrak{C}$ of n^* polynomial constraints, $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{n^*}$, with coefficients in $\mathbb{Z}$, where for all $i \in [n]$, we have that $\mathcal{C}_i$ is an equality constraint on the variable indexed i (only), and a solution of $\mathfrak{C}$ is also a solution of $\mathsf{ConstSqu}(G)$. Next, we would like to find a "good" rational approximation to the solution of $\mathfrak{C}$. Later we will prove that such an approximate solution is actually an *exact* solution to our problem.

For $\delta > 0$, a δ rational approximate solution $\mathcal{S}$ for a set of polynomial equality constraints is an assignment to the variables, for which there exists a solution $\mathcal{S}^*$, such that for any variable X, the (absolute) difference between the assignment to X by $\mathcal{S}$ and the assignment to X by $\mathcal{S}^*$ is at most δ.[7] We follow the approach of Arora et al. [4] to find a δ rational approximation to a solution for a set of polynomial equality constraints with $\delta = 1/2$. This approach states that using Renegar's algorithm [29] together with binary search, with search range bound given by Grigor'ev and Vorobjov [17], we can find a rational approximation to a solution of a set of polynomial equality constraints with accuracy up to δ in time $(\tau + n' + m'^{n'} + \log(1/\delta))^{\mathcal{O}(1)}$ where τ is the maximum bitsize of a coefficient, n' is the number of variables and m' is the number of constraints. In this manner, we obtain in time $n^{\mathcal{O}(n)}$ a rational approximation $\mathcal{S}$ to the solution of $\mathfrak{C}$ with accuracy 1/2. By Theorem 17, $\mathcal{S}$ is also a rational approximation to a solution of $\mathsf{ConstSqu}(G)$ with accuracy 1/2. We let $\bar{P}_{S_i} = (\bar{X}_{S_i}, \bar{Y}_{S_i})$ denote the value that $\mathcal{S}$ assigns to (X_i, Y_i) (corresponding to the vertex $v_i \in V(G)$). Further we let $P_S = \{\bar{P}_{S_i} \mid i \in [n]\}$. In the following lemma, we analyze $\mathcal{DT}(P_S)$.

▶ **Lemma 18 (♠).** *The triangulation G is isomorphic to $\mathcal{DT}(P_S)$ where points corresponding to vertices in f^* form the outer face (in that order). Here, P_S is the point set described above.*

Towards the proof of Lemma 15, we first consider our intermediate problem.

[7] We note that $\mathcal{S}$ may not be a solution in the sense that it may not satisfy all constraints (but it is close to some solution that satisfies all of them).

▶ **Lemma 19.** *Rational-DR is solvable in time $n^{\mathcal{O}(n)}$.*

Proof. Our algorithm first computes the set of polynomial constraints $\mathsf{ConstSqu}(G)$ in time $\mathcal{O}(n^2)$. Then, it computes a 1/2 accurate approximate solution for $\mathsf{ConstSqu}(G)$ by using the approach of Arora et al. [4] in time $n^{\mathcal{O}(n)}$. In Lemma 18, we have shown that such an approximate solution is an exact solution. This concludes the proof. ◀

Finally, we are ready to prove Lemma 15, and thus conclude the correctness of Theorem 14.

Proof of Lemma 15. We use the algorithm given by Lemma 19 to output a point set $P \subseteq \mathbb{Q}^2$ in time $n^{\mathcal{O}(n)}$ such that G is isomorphic to $\mathcal{DT}(P)$ and the points corresponding to vertices in $V(f^*)$ lie on the outer face of $\mathcal{DT}(P)$ in the order in which they appear in the cycle of f^*. We denote by $\bar{P}_i = (\bar{X}_i, \bar{Y}_i)$ the value P assigns to the vertex $v_i \in V(G)$. For $i \in [n]$, since $\bar{X}_i, \bar{Y}_i \in \mathbb{Q}$, we let the representation be $\bar{X}_i = \bar{X}_i^a / \bar{X}_i^b$ and $\bar{Y}_i = \bar{Y}_i^a / \bar{Y}_i^b$, where $\bar{X}_i^a, \bar{X}_i^b, \bar{Y}_i^a, \bar{Y}_i^b \in \mathbb{Z}$. For each edge $(\bar{P}_i, \bar{P}_j) \in E(\mathcal{DT}(P))$, there exists a disc C_{ij} with a centre, say $\bar{P}_{ij}$, containing only $\bar{P}_i$ and $\bar{P}_j$ from P. These assignments satisfy the constraints $\mathsf{Con}(G)$ and $\mathsf{Dis}(G)$ presented in Section 3. It thus follows that $\mathcal{P}$ satisfies $\mathsf{Const}(G)$. From Observation 11 it follows that for any $\alpha \in \mathbb{R}^+$, we have that $\alpha\mathcal{P}$ satisfies $\mathsf{Const}(G)$. We let $\beta = \Pi_{i \in [n]} \bar{X}_i^b \bar{Y}_i^b$. But then $\beta\mathcal{P}$ satisfies $\mathsf{Const}(G)$, and hence $\beta P = \{(\beta\bar{X}_i, \beta\bar{Y}_i \mid i \in [n]\}$ is a point set such that G is isomorphic to $\mathcal{DT}(\beta P)$ where the points corresponding to vertices in $V(f^*)$ lie on the outer face of $\mathcal{DT}(\beta P)$. Therefore, we output a correct point set, βP, with only integer coordinates. This concludes the proof. ◀

6 Conclusion

In this paper, we gave an $n^{\mathcal{O}(n)}$-time algorithm for the Delaunay Realization problem. We have thus obtained the first exact exponential-time algorithm for this problem. Still, the existence of a practical (faster) exact algorithm for Delaunay Realization is left for further research. In this context, it is not even clear whether a significantly faster algorithm, say a polynomial-time algorithm, exists. Perhaps one of the first questions to ask in this regard is whether there exist instances of graphs that are realizable but for which the integers in any integral solution need to be exponential in the input size? If yes, does even the representation of these integers need to be exponential in the input size?

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