

Vertex Cover and Feedback Vertex Set Above and Below Structural Guarantees

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Abstract

VERTEX COVER parameterized by the solution size k is the quintessential fixed-parameter tractable problem. FPT algorithms are most interesting when the parameter is small. Several lower bounds on k are well-known, such as the maximum size of a matching. This has led to a line of research on parameterizations of VERTEX COVER by the difference of the solution size k and a lower bound. The most prominent cases for such lower bounds for which the problem is FPT are the matching number or the optimal fractional LP solution. We investigate parameterizations by the difference between k and other graph parameters including the feedback vertex number, the degeneracy, cluster deletion number, and treewidth with the goal of finding the border of fixed-parameter tractability for said difference parameterizations. We also consider similar parameterizations of the FEEDBACK VERTEX SET problem.

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1 Introduction

Given an undirected graph G and an integer k , the VERTEX COVER problem asks whether there is a set of at most k vertices that contains at least one endpoint of each edge. VERTEX COVER is arguably the most well-studied problem in parameterized complexity. After significant efforts, the state-of-the-art FPT algorithm parameterized by the solution size k runs in time $\mathcal{O}(1.2738^k + kn)$ [4], where n is the number of vertices. Very recently, Harris and Narayanaswamy [22] have announced an even faster algorithm with running time $\mathcal{O}(1.2540^k \cdot n^{\mathcal{O}(1)})$.

The aforementioned FPT algorithm is only useful when the parameter k is small. In practice, however, the minimum vertex cover size is often large. For this reason, many recent studies look into VERTEX COVER where the parameterization is k minus a lower bound on k .



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For instance, if the maximum matching size m is greater than k , then this would be a trivial no instance, since a vertex cover must contain at least one endpoint of each edge in any matching. This naturally gives rise to the “above guarantee” [33] parameter $k - m$. VERTEX COVER is FPT with respect to $k - m$ [37]. It has also been shown that VERTEX COVER is FPT for even smaller above guarantee parameters such as $k - r$ [8, 31] and $k - 2r + m$ [19], where r is the optimal LP relaxation value of VERTEX COVER. Kernelization with respect to these parameters has also been studied [28, 29].

This work considers above guarantee parameterizations of VERTEX COVER where the lower bounds are structural parameters not related to the matching number, such as feedback vertex number, degeneracy, and cluster vertex deletion number. We also study similar above guarantee parameterizations of the FEEDBACK VERTEX SET problem: Given a graph G and an integer k , it asks whether there is a set of at most k vertices whose deletion from G results in a forest. In this work, we do not deeply look into the “below guarantee” parameterization (where the number of vertices n is the most obvious upper bound) because VERTEX COVER and FEEDBACK VERTEX SET are known to be W[1]-hard when parameterized by $n - k$.¹

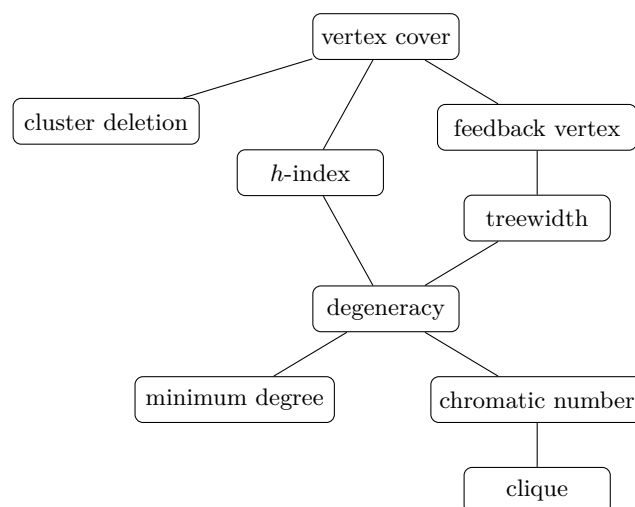
Motivation. We believe that FPT algorithms with above guarantee parameterizations may help explain the efficiency of some branching algorithms in practice. Consider an instance $I = (G, k)$ of VERTEX COVER for a complete graph G . This instance is trivial to solve: I has a solution if and only if $k \geq n - 1$. This triviality, however, is overlooked by the worse-case running time bound of FPT algorithms parameterized by the solution size k or the aforementioned smaller parameters $k - m$, $k - \ell$, or $k - 2\ell + m$, all of which amount to $n/2$ (for even n). Now consider another above guarantee parameter $k - (\omega - 1)$, where ω is the maximum clique size. (Note that for a clique C , a vertex cover contains at least $|C| - 1$ vertices of C .) As we will see, VERTEX COVER is FPT parameterized by this parameter (even if a maximum clique is not given). This gives us a theoretical reasoning as to why VERTEX COVER is indeed trivial to solve on complete graphs.

We also believe that above structural guarantee parameterizations are of theoretical interest, because they are closely related to identifying graphs in which two of its parameters coincide. Structural characterizations of such graphs have been extensively studied where the two parameters are maximum matching size and minimum vertex cover size [32], maximum matching size and minimum edge dominating set size [30], maximum matching size and induced matching size [3], maximum independent set size and minimum dominating set [35, 36], and minimum dominating set and minimum independent dominating set [21]. The corresponding computational complexity questions, i.e. whether these graphs can be recognized in polynomial time, have been studied as well [5, 11, 12, 16, 20, 40].

Finally, our parameterizations can be seen as “dual” of parameters studied in literature. There has been significant work (especially in the context of kernelization) on VERTEX COVER parameterized by structural parameters smaller than the solution size k such as feedback vertex set number [2, 7, 23, 26, 34].

Our contribution. In Section 3, we show that VERTEX COVER is FPT when parameterized by $k - h$ for the h -index h . This parameter is greater than or equal to many graph parameters such as degeneracy d , chromatic number χ , and clique number ω (See Figure 1). Thus, VERTEX COVER is FPT for $k - d$, $k - \chi$, and $k - \omega$ as well. Using a similar approach, we

¹ These two parameterized problem are essentially the W[1]-hard problems INDEPENDENT SET [14] and MAXIMUM INDUCED FOREST [27], respectively.



■ **Figure 1** A Hasse diagram of graph parameters. There is a line between two parameters p (above) and q (below) if $p + 1 \geq q$ holds for any graph G .

show in Section 4 that FEEDBACK VERTEX SET is FPT for $k - d$. We also show that on planar graphs, fixed-parameter tractability of VERTEX COVER with respect to $k - d$ can be strengthened: VERTEX COVER is FPT parameterized by $k - tw$ for the treewidth tw (Section 5). In the remaining sections, we prove hardness results. In Section 6, we show that VERTEX COVER admits no kernel of size polynomial in $k - \delta$ (δ is the minimum degree) and neither VERTEX COVER nor FEEDBACK VERTEX SET admit a kernel of size polynomial in $k - \omega$. We also show that VERTEX COVER is W[1]-hard for $k - fvs$ (Section 7) and NP-hard for $k - cd = 0$ (Section 8), where fvs and cd are the size of a minimum feedback vertex set and of a minimum cluster deletion set, respectively. Finally, we prove that FEEDBACK VERTEX SET NP-hard for $vc - k = 2$ in Section 9 where vc is the size of a minimum vertex cover.

2 Preliminaries

Graphs. For standard graph terminology, we refer to Diestel [13]. All graphs we consider are finite, undirected, and loopless. We call a function p that maps any graph G to an integer $p(G)$ a *graph parameter*. In the following, we will define several graph parameters that are of interest in this work. Let G be a graph. We denote the vertex set and edge set of G by $V(G)$ and $E(G)$, respectively. We denote the *minimum degree* of G by $\delta(G)$ and the *maximum degree* by $\Delta(G)$. The *vertex cover number* $vc(G)$ of G is the size of a smallest set $X \subseteq V(G)$ such that $G - X$ is edgeless. The *feedback vertex number* $fvs(G)$ of G is the size of a smallest set $X \subseteq V(G)$ such that $G - X$ is acyclic. The *h -index* $h(G)$ of a graph G is the largest integer k such that G contains at least k vertices each of degree at least k . The *degeneracy* of a graph G is $d(G) := \max_{V' \subseteq V(G)} \delta(G[V'])$. A subset V' of $V(G)$ that maximizes $\delta(G[V'])$ is a *core* of G . The *clique number* $\omega(G)$ of G is the size of a largest clique in G . The *chromatic number* $\chi(G)$ is the minimum integer k such that G can be properly k -colored. The *cluster deletion number* $cd(G)$ of G is the size of a smallest set $X \subseteq V(G)$ such that $G - X$ does not contain a P_3 as an induced subgraph. A pair $(T = (W, F), \beta)$ where T is a tree and $\beta: W \rightarrow 2^{V(G)}$ is a *tree decomposition* of G if (i) $\bigcup_{w \in W} \beta(w) = V(G)$, (ii) $\{w \in W \mid v \in \beta(w)\}$ induces

a connected subgraph of T for all $v \in V(G)$ and (iii) for all $\{u, v\} \in E(G)$, there exists a $w \in W$ with $u, v \in \beta(w)$. The *width* of $(T = (W, F), \beta)$ is $\max_{w \in W} |\beta(w)| - 1$. The *treewidth* $\text{tw}(G)$ of G is the minimum width over all tree decompositions of G .

If p and q are graph parameters, then we will say that p is *smaller* than q (and write $p \preceq q$), if there is a constant c such that $p(G) \leq q(G) + c$ for all graphs G . This differs from the way the boundedness relation between graph parameters is usually defined [25, 42], but this stricter definition is necessary in the context of difference parameterizations. This is because with this stricter definition the following is true (and easy to prove): If p, q, r are graph parameters such that $p \preceq q \preceq r$, then $r - q \preceq r - p$. Figure 1 depicts the graph parameters relevant to this work and the relationships between them.

Parameterized complexity. A *parameterized problem* is a pair (L, κ) where $L \subseteq \Sigma^*$ for a finite alphabet Σ and $\kappa: \Sigma^* \rightarrow \mathbb{N}$ is the parameter. The problem is *fixed-parameter tractable* (FPT) if it can be decided by an algorithm with running time $\mathcal{O}(f(\kappa(I)) \cdot |I|^c)$ where $I \in \Sigma^*$, f is a computable function and c is a constant. Note that if (L, κ) is FPT and $\kappa \preceq \kappa'$, then (L, κ') is also FPT. A *kernel* for this problem is a polynomial-time algorithm that takes the instance I and outputs a second instance I' such that (i) $I \in (L, \kappa) \iff I' \in (L, \kappa)$ and (ii) $|I'| \leq f(\kappa(I))$ for a computable function f . The *size* of the kernel is f . There is a hierarchy of computational complexity classes for parameterized problems: $\text{FPT} \subseteq \text{W}[1] \subseteq \dots \subseteq \text{XP}$. To show that a parameterized problem (L, κ) is (presumably) not FPT one may use a *parameterized reduction* from a W[1]-hard problem to L . A parameterized reduction from a parameterized problem (L, κ) to another parameterized problem (L', κ') is a function that acts as follows: For computable functions f and g , given an instance I of L , it computes in $f(\kappa(I)) \cdot |I|^{\mathcal{O}(1)}$ time an instance I' of L' so that $I \in (L, \kappa) \iff I' \in (L', \kappa')$ and $\kappa(I') \leq g(\kappa(I))$. For more details on parameterized algorithms and complexity, we refer to the standard literature [6, 15, 17].

Problem definitions. We are interested in above guarantee parameterizations of VERTEX COVER of the following form. Let $p \preceq \text{vc}$ be a graph parameter. Then, we define:

VERTEX COVER ABOVE p
Input: A graph G and an integer k .
Question: Does G contain a vertex cover of order at most k ?
Parameter: $\ell := k - p(G)$.

Similarly, we also consider above (below) guarantee parameterizations of FEEDBACK VERTEX SET. Now, let p be a graph parameter with $p \preceq \text{fvs}$ ($\text{fvs} \preceq p$). We consider the following problem:

FEEDBACK VERTEX SET ABOVE (BELOW) p
Input: A graph G and an integer k .
Question: Does G contain a feedback vertex set of order at most k ?
Parameter: $\ell := k - p(G)$ ($\ell := p(G) - k$).

3 Vertex Cover above h -Index

We start by proving that VERTEX COVER is FPT when parameterized by the difference between k and the h -index of the graph. Recall that the state-of-the-art algorithm for VERTEX COVER parameterized by the solution size k has running time $\mathcal{O}(1.274^k + kn)$ [4].

► **Theorem 1.** *VERTEX COVER ABOVE h -INDEX is FPT.*

Proof. Let (G, k) be an instance of VERTEX COVER where G is a graph with an h -index of h . Let $v_1, \dots, v_h \in V(G)$ with $\deg(v_i) \geq h$. We branch into the following $h + 1$ cases:

- (1) The solution contains all of the vertices v_1, \dots, v_h . Hence, we test the instance $(G - \{v_1, \dots, v_h\}, k - h)$ in time $\mathcal{O}(1.274^{k-h} + (k - h)n)$.
- (2) The solution does not contain v_i for some $i \in \{1, \dots, h\}$. Then, the solution must contain all of v_i 's neighbors. Hence, we test the instance $(G - N(v_i), k - |N(v_i)|)$. Since $|N(v_i)| \geq h$, this is possible in time $\mathcal{O}(1.274^{k-h} + (k - h)n)$.

In all, we get a running time of $\mathcal{O}(1.274^{k-h}h + (k - h)hn)$. ◀

This algorithm can also be used to obtain a Turing kernelization (cf. [18, Ch. 22]) by simply computing a kernel for each of the $h + 1$ instances of VERTEX COVER parameterized by the solution size that the algorithm branches into.

4 Feedback Vertex Set above Degeneracy

A similar approach to the one used in the previous section, branching once to lower k and then applying a known algorithm for the standard parameterization, can also be employed to show that FEEDBACK VERTEX SET ABOVE DEGENERACY is FPT. The fastest presently known deterministic algorithm for the standard parameterization of FEEDBACK VERTEX SET runs in time $\mathcal{O}(3.460^k \cdot n)$ [24].

► **Theorem 2.** *FEEDBACK VERTEX SET ABOVE DEGENERACY is FPT.*

Proof. Let (G, k) be an instance for FEEDBACK VERTEX SET where d is the degeneracy of G . It is well-known that the degeneracy and the core of a graph can be computed in polynomial time by iteratively deleting a minimum-degree vertex and storing the largest degree of a vertex at the time it is deleted. We start by computing a core V' of G . We branch into the following $|V'|^2 + 1$ cases.

- (1) The entire core is contained in the minimum feedback vertex set. The core must contain at least $d + 1$ vertices. Hence, we test the instance $(G - V', k - |V'|)$ in time $\mathcal{O}(3.460^{k-|V'|} \cdot n) = \mathcal{O}(3.460^{k-d} \cdot n)$.
- (2) The entire core is not contained in the minimum feedback vertex set. Let X denote the minimum feedback vertex and let $F := V(G) \setminus X$ be the maximum induced forest.
 - a. If $G[V' \cap F]$ contains an isolated vertex u , then all neighbors of u in $G[V']$, of which there are at least d , must be in X . Hence, for each $u \in V'$, we test the instance $(G - N_{G[V']}(u), k - \deg_{G[V']}(u))$ in time $\mathcal{O}(3.460^{k-\deg_{G[V']}(u)} \cdot n) = \mathcal{O}(3.460^{k-d} \cdot n)$.
 - b. If $G[V' \cap F]$ does not contain an isolated vertex, it must still contain a leaf u , since it is a forest. Then, all but one of the neighbors of u in $G[V']$ must be in X . Hence, for each pair $u \in V'$ and $v \in N_{G[V']}(u)$, we test the instance $(G - (N_{G[V']}(u) \setminus \{v\}), k - \deg_{G[V']}(u) + 1)$ in time $\mathcal{O}(3.460^{k-\deg_{G[V']}(u)+1} \cdot n) = \mathcal{O}(3.460^{k-d} \cdot n)$.

In all, this makes for a running time of $\mathcal{O}(3.460^{k-d} \cdot n^3)$. ◀

Like the algorithm in the previous section, this one can also be easily converted to a Turing kernelization.

5 Vertex Cover above Treewidth

In this section, we show that on planar graphs VERTEX COVER is also FPT with respect to $\ell = k - \text{tw}$, which is smaller than $k - d$.

► **Theorem 3.** *VERTEX COVER ABOVE TREewidth on planar graphs is FPT.*

Proof. Given a planar graph G , we compute the branchwidth β of G . This is possible in polynomial time, because G is planar [41]. Moreover, $\beta \leq \text{tw}(G) + 1 \leq \frac{3}{2}\beta$ [39, Theorem 5.1]. Any planar graph with treewidth w contains a $g \times g$ -grid with $g \geq \frac{w+4}{6}$ as a minor [38, Theorem 6.2]. Hence, having computed β , we know that G must contain a $g \times g$ -grid with $g \geq \frac{\beta+3}{6}$ as a minor. Any vertex cover of the $g \times g$ -grid must contain at least $\lfloor \frac{g}{2} \rfloor$ in each row, for a total of at least $g \cdot \lfloor \frac{g}{2} \rfloor \geq \frac{g(g-1)}{2}$ vertices. Since $\text{vc}(H_1) \leq \text{vc}(H_2)$, if H_1 is a minor of H_2 , it follows that $\text{vc}(G) \geq \frac{\beta^2-9}{72} =: r$. Hence, if $k < r$, we may reject the input. Otherwise, $\ell = k - \text{tw}(G) \geq k - \frac{3}{2}\beta \geq k - \frac{3}{2}\sqrt{72k-9}$. This means that ℓ is bounded from below by a function in k and, therefore, fixed-parameter tractability with respect to k implies fixed-parameter tractability with respect to ℓ . ◀

This algorithm relies on two properties of planar graphs: (i) large treewidth guarantees the existence of a $g \times g$ -grid where $g \in \Omega(\text{tw}^{1/2+\varepsilon})$ for $\varepsilon > 0$ and (ii) branchwidth can be computed in polynomial time on planar graphs. In any graph class that excludes a minor, (i) still holds true [9]. Although it is not clear that (ii) can be generalized, we remark that a constant approximation algorithm is known for graphs excluding single-crossing graphs as minors [10]. In fact, our result can be extended to any class of graphs that do not contain a single-crossing graph as a minor.

We leave open whether or not VERTEX COVER ABOVE TREewidth is FPT on graph classes that exclude a minor (other than planar graphs) or even on arbitrary graphs.

6 Kernelization Lower Bounds

In this section we show that, while there is a Turing kernel when parameterized above h -index, VERTEX COVER presumably does not admit a polynomial kernel when parameterized above the minimum degree or the clique number.

► **Theorem 4.** *VERTEX COVER ABOVE MINIMUM DEGREE and VERTEX COVER ABOVE CLIQUE NUMBER do not admit a polynomial kernel unless $\text{NP} \subseteq \text{coNP/poly}$.*

Proof. We prove the statement by giving a linear parametric transformation from CLIQUE parameterized by maximum degree and parameterized by the vertex cover number. Unless $\text{NP} \subseteq \text{coNP/poly}$, under neither parameterization does CLIQUE admit a polynomial kernel. This is folklore for maximum degree and was shown by Bodlaender et al. [1] for vertex cover number. The underlying reduction takes the CLIQUE instance (G, k) and transforms it into the instance (\bar{G}, \bar{k}) of VERTEX COVER where \bar{G} is the complement graph of G , that is $V(\bar{G}) := V(G)$ and $E(\bar{G}) := \binom{V(G)}{2} \setminus E(G)$, and $\bar{k} := |V| - k$. The reduction is obviously correct and computable in $\mathcal{O}(|V|^2)$ time. As for the parameterizations, observe that $\bar{k} - \delta(\bar{G}) = (|V| - k) - (|V| - 1 - \Delta(G)) \leq \Delta(G)$. Since $\omega(\bar{G}) \geq |V(G)| - \text{vc}(G)$, we also have $\bar{k} - \omega(\bar{G}) \leq (|V(G)| - k) - (n - \text{vc}(G)) \leq \text{vc}(G)$. This yields the claimed transformations. ◀

Using a standard reduction from VERTEX COVER to FEEDBACK VERTEX SET, we obtain the following.

► **Corollary 5.** *FEEDBACK VERTEX SET ABOVE CLIQUE NUMBER does not admit a polynomial kernel unless $\text{NP} \subseteq \text{coNP}/\text{poly}$.*

Proof. We provide a linear parametric transformation from VERTEX COVER ABOVE CLIQUE NUMBER. Given an instance (G, ℓ) of VERTEX COVER, we use the following folklore construction to obtain an instance (G', ℓ) of FEEDBACK VERTEX SET. After initializing G' as a copy of G , we add for each edge $\{u, v\} = e \in E(G)$ the vertex w_{uv} and the edges $\{u, w_{uv}\}$ and $\{v, w_{uv}\}$ to G' , so that for each edge $e \in E(G)$ there exists a unique triangle in G' . Clearly, unless $\omega(G) = 2$, we have $\omega(G') = \omega(G)$. Hence, the parameter $\ell - \omega(G')$ is upper-bounded by the parameter of the input problem, and we are done. ◀

We leave open whether FEEDBACK VERTEX SET ABOVE MINIMUM DEGREE admits a polynomial kernel.

7 Vertex Cover above Feedback Vertex Number

In this section we prove that, when parameterizing above feedback vertex number, VERTEX COVER is $\text{W}[1]$ -hard. First, we prove $\text{W}[1]$ -hardness with respect to a related parameter, namely above the *distance to K_r -free* for every constant $r \geq 3$, that is, the minimum number of vertices one needs to remove such that the remaining graph does not contain a clique of order r . Distance to K_3 -free is a lower bound on the feedback vertex number, so our proof also implies hardness for VERTEX COVER ABOVE FEEDBACK VERTEX NUMBER.

► **Theorem 6.** *VERTEX COVER ABOVE DISTANCE TO K_r -FREE is $\text{W}[1]$ -hard.*

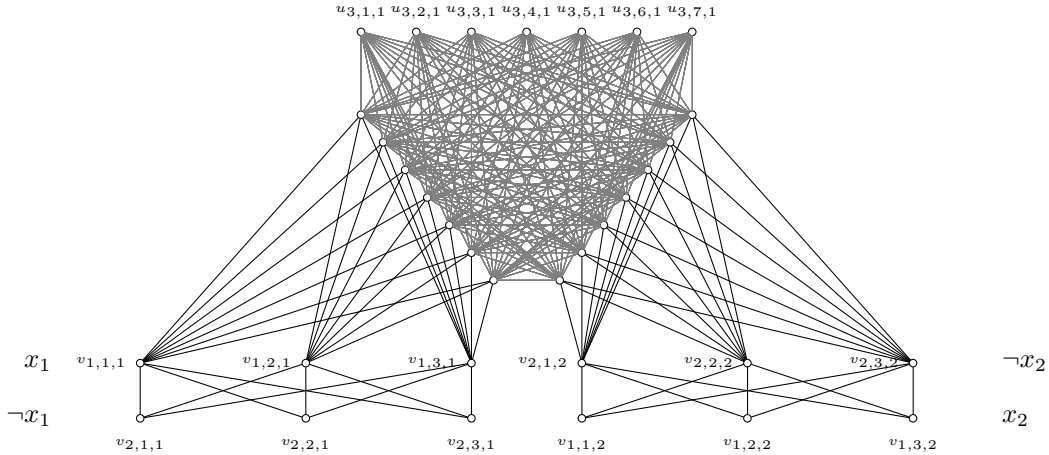
Proof. We provide a parameterized reduction from INDEPENDENT SET parameterized by the solution size k . Let $I = (G, k)$ be an instance of INDEPENDENT SET with $V(G) = [n]$ and $E(G) = \{e_1, \dots, e_m\}$. We create an instance (G', ℓ) as follows. First, for each $i \in [k]$ we add a clique on the vertex set $V_i = \{w_j^i \mid j \in [n]\}$ to G' . For each $i, j \in [k]$, we add the edge between w_q^i and w_q^j for each $q \in [n]$ and the edge between w_p^i and w_q^i for each $\{p, q\} \in E$. Next, for each $i \in [k]$ we add a set A_i of $r - 2$ vertices which form a clique, attach a leaf to each vertex in A_i , and make each vertex in A_i adjacent to each vertex in V_i . Let $A := \bigcup_{i \in [k]} A_i$. Then we add $k + 1$ cliques on $r - 1$ vertices. Call the set of these vertices B , attach a leaf to each $v \in B$, and make each $v \in B$ adjacent to each vertex in $\bigcup_{i \in [k]} V_i$. Denote by L the set of leaves in G' . Lastly, set $\ell := (n - 1)k + |A| + |B| = (n - 1)k + |L|$.

For the correctness, observe that G' contains a vertex cover of size k' if and only if G' contains an independent set of size

$$|V'| - \ell = nk + |A| + |B| + |L| - ((n - 1)k + |A| + |B|) = |L| + k.$$

Let $Y \subseteq V(G')$ be an independent set in G' . As it is always optimal to take leaves into an independent set, we may assume that Y contains all of L . Hence, we may assume that $Y \cap (A \cup B) = \emptyset$. Furthermore, Y contains at most one vertex of each clique on V_i , $i \in [k]$, and, by the construction of the edges in G' , Y can contain only such vertices in the cliques whose corresponding vertices in G are pairwise nonadjacent. Hence, G' contains an independent set of size $|L| + k$ if and only if G contains an independent set of size k , and the reduction is correct.

Finally, we will show that the distance to K_r -free of G' is exactly nk . This implies that the parameter of the output instance, if d is the distance to K_r -free of G' , is $k' = \ell - d = (n - 1)k + |A| + |B| - nk = |A| + |B| - k = (r - 2)k + (r - 1)(k + 1) - k$. Since r is a constant, this implies that k' is bounded in k . Let $D \subseteq V'$ be of minimum set such



■ **Figure 2** An excerpt of the graph G output by Construction 9: At the bottom are the vertex gadgets for the variables x_1 and x_2 . At the top is a clause gadget representing a clause that contains both x_1 and $\neg x_2$.

that $G' - D$ is K_r -free. Clearly, $D \cap L = \emptyset$ as L does not intersect any K_r . Furthermore, as every vertex $u \in A_i$ intersects a subset of the cliques that any vertex $v \in V_i$ intersects, we may exchange each vertex in $D \cap A_i$ with a vertex in V_i . As there are fewer than r vertices in each A_i we may assume that $D \cap A_i = \emptyset$. But then D must contain $n - 1$ vertices of each set V_i , hence, D contains all but k vertices from $\bigcup_{i \in [k]} V_i$. If however $v \notin D$ for one such $v \in \bigcup_{i \in [k]} V_i$, then D must contain at least one vertex from each clique in B . As there are $k + 1$ such cliques, $v \notin D$ contradicts D being minimum. Hence, $D := \bigcup_{i \in [k]} V_i$ is a K_r -deletion set of size nk . ◀

For $r = 3$, the deletion set D in the proof above is also a feedback vertex set. Hence, we obtain the following.

► **Corollary 7.** VERTEX COVER ABOVE FEEDBACK VERTEX NUMBER is $W[1]$ -hard.

Observe that in the proof of Theorem 6 we can specify a minimum deletion set. Hence, our hardness results also hold if a minimum deletion set is given as part of the input.

8 Vertex Cover above Cluster Deletion Number

Recall that the cluster deletion number is the minimum size of a set X such that $G - X$ is a cluster graph, i.e., every connected component of $G - X$ is a clique. We show that VERTEX COVER ABOVE CLUSTER DELETION NUMBER is NP-hard even if the parameter is zero.

► **Theorem 8.** VERTEX COVER ABOVE CLUSTER DELETION is NP-hard even if $\ell = 0$.

We will prove this theorem by reduction from 3-SAT. In fact, we prove a slightly stronger claim: VERTEX COVER is NP-hard when restricted to graphs G with $\text{cd}(G) = \text{vc}(G)$. The following construction is illustrated in Figure 2.

► **Construction 9.** Let φ be a Boolean formula in 3-CNF consisting of the clauses C_1, \dots, C_m over the variables x_1, \dots, x_n . We may assume that each clause of φ contains exactly three literals. For each $i \in \{1, \dots, m\}$, let $C_i = (L_i^1 \vee L_i^2 \vee L_i^3)$ where L_i^1, L_i^2 , and L_i^3 are literals.

We construct a graph G and an integer $k := 14m + 3n$ such that $\text{cd}(G) = \text{vc}(G)$ and $\text{vc}(G) \leq k$ if and only if φ is satisfiable. Each variable x_j is represented by a *variable gadget* consisting of six vertices $(v_{r,s,j})_{r \in \{1,2\}, s \in \{1,2,3\}}$ that induce a complete bipartite graph with three vertices in each color class. Each clause C_i is represented by a *clause gadget* consisting of twenty-one vertices $(u_{r,s,i})_{r \in \{1,2,3\}, s \in \{1, \dots, 7\}}$ that induce a complete tripartite graph with seven vertices in each color class. The two sides of the bipartition of a variable gadget correspond to its positive and negative literals and the three sides of the tripartition of a clause gadget correspond to the three literals the clause contains. All vertices in a side of a clause gadget are connected to all vertices in the side of a variable gadget if these two sides correspond to the literal of opposite sign. Formally, we let:

$$\begin{aligned} V(G) &:= \{u_{r,s,i} \mid r \in \{1,2,3\}, s \in \{1, \dots, 7\}, i \in \{1, \dots, m\}\} \\ &\quad \cup \{v_{r,s,j} \mid r \in \{1,2\}, s \in \{1,2,3\}, j \in \{1, \dots, n\}\} \text{ and} \\ E(G) &:= \{\{u_{r,s,i}, u_{r',s',i}\} \mid r, r' \in \{1,2,3\}, r \neq r', s, s' \in \{1, \dots, 7\}, i \in \{1, \dots, m\}\} \\ &\quad \cup \{\{v_{1,s,j}, v_{2,s',j}\} \mid s, s' \in \{1,2,3\}, j \in \{1, \dots, n\}\} \\ &\quad \cup \{\{u_{r,s,i}, v_{1,s',j}\} \mid s \in \{1, \dots, 7\}, s' \in \{1,2,3\}, L_i^r = x_j\} \\ &\quad \cup \{\{u_{r,s,i}, v_{2,s',j}\} \mid s \in \{1, \dots, 7\}, s' \in \{1,2,3\}, L_i^r = \neg x_j\}. \end{aligned}$$

► **Lemma 10.** *Let G be the graph output by Construction 9 and $X \subseteq V(G)$ be a minimum cluster deletion set. Then, in each clause gadget of G , X contains all vertices in two of the sides of the gadget and none of the vertices of the third side.*

Proof. For every $i \in \{1, \dots, m\}$, the deletion set X must contain either (i) all vertices from two sides of the clause gadget for C_i or (ii) all but one vertex from each side of this clause gadget, because if X omits two vertices from one side and an additional vertex from a second side these three vertices induce a P_3 .

In case (i), it only remains to show that X does not contain any of the vertices in the third side. Let $A := \{u_{r,1,i}, \dots, u_{r,7,i}\}$ with $r \in \{1,2,3\}$ be the vertices in this side. Let $\alpha_r := 1$, if $L_i^r = x_j$, and $\alpha_r := 2$, if $L_i^r = \neg x_j$. If X contains all of the vertices in $B := \{v_{\alpha_r,1,j}, v_{\alpha_r,2,j}, v_{\alpha_r,3,j}\}$, then $X \setminus A$ is a cluster deletion set strictly smaller than X . If X does not contain all vertices in B , then it must contain all but one of the vertices in A (i.e., $|X \cap A| \geq 6$). Then $(X \setminus A) \cup B$ is a cluster deletion set and $|(X \setminus A) \cup B| \leq |X| - 6 + 3 < |X|$.

For case (ii), let r, r', r'' be the three sides of the clause gadget and let $X_{r''}$ be the set of vertices in X that lie in side r'' . Assume that X does not contain all vertices from two sides r, r' . Since X must contain all but one vertex from each side, we may assume without loss of generality that $X_r = \{u_{r,1,i}, \dots, u_{r,6,i}\}$, $|X_{r'}| = \{u_{r',1,i}, \dots, u_{r',6,i}\}$, and $|X_{r''}| \geq 6$ (note that it may contain all vertices in the third side r''). Let $\alpha_{r''} := 1$, if $L_i^{r''} = x_j$, and $\alpha_{r''} := 2$, if $L_i^{r''} = \neg x_j$ and let

$$X' := (X \setminus X_{r''}) \cup \{u_{r,7,i}, u_{r',7,i}\} \cup \{v_{\alpha_{r''},s,j} \mid s \in \{1,2,3\}\}.$$

Then, $|X'| \leq |X| - 6 + 2 + 3 = |X| - 1$. Moreover, X' is a cluster deletion set in G , because $X \setminus X' \subseteq \{u_{r'',s,i} \mid s \in \{1, \dots, 7\}\}$, but all of these vertices are isolated and, therefore, not part of any P_3 in $G - X'$. Hence, X is not minimum. ◀

A similar statement is also true for vertex gadgets:

► **Lemma 11.** *Let G be the graph output by Construction 9 and $X \subseteq V(G)$ be a minimum cluster deletion set. Then, in each variable gadget of G , X contains all vertices in one of the two sides of the gadget.*

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Proof. If X does not contain all vertices in either side of a vertex gadget corresponding to the variable x_j , it must contain all but one vertex from each side. Let C_{i_1}, \dots, C_{i_t} be the clauses that contain the literal x_j and let r_1, \dots, r_t be the sides of each of the corresponding clause gadgets whose vertices are adjacent to the side in the vertex gadget of x_j . Then, X must contain all but one vertex in the side $r_{t'}$ of the clause gadget for $C_{i_{t'}}$, for each $t' \in \{1, \dots, t\}$. By Lemma 10, it follows that X contains all vertices in each of those sides. Hence, removing from X all vertices in the $r = 1$ side of x_j 's vertex gadget and adding the remaining vertex in the $r = 2$ side yields a smaller cluster deletion set. \blacktriangleleft

► **Lemma 12.** *Let G be the graph output by Construction 9 and $C \subseteq V(G)$ be a minimum cluster deletion set. Then, C is a vertex cover of G . Hence, $\text{cd}(G) = \text{vc}(G)$.*

Proof. Clearly $\text{cd}(G) \leq \text{vc}(G)$. We show that $\text{cd}(G) \geq \text{vc}(G)$. By Lemmas 10 and 11, C covers all edges within each clause and each variable gadget. It remains to show that edges between these gadgets are covered. The only such edges are between sides of a clause gadget and sides of a variable gadget when these two sides correspond to the same literal. Then, C must contain all vertices in one of these two sides, since, otherwise, $G - C$ would contain an induced P_3 . Hence, C also covers all edges between those two sides. \blacktriangleleft

► **Lemma 13.** *Let φ be a formula in 3-CNF and G the graph output by Construction 9 on input φ . Then, φ is satisfiable if and only if $\text{vc}(G) \leq \ell$.*

Proof. First, suppose that φ is satisfiable and that $\alpha: \{x_1, \dots, x_n\} \rightarrow \{0, 1\}$ is a satisfying assignment. We extend α to literals on this variable set in the natural way. Since α satisfies every clause in φ , there is an $\alpha_i \in \{1, 2, 3\}$ for every $i \in \{1, \dots, m\}$ such that $\alpha(L_i^{\alpha_i}) = 1$. Let

$$\begin{aligned} C := & \{u_{r,s,i} \mid r \in \{1, 2, 3\} \setminus \{\alpha_i\}, s \in \{1, \dots, 7\}, i \in \{1, \dots, m\}\} \\ & \cup \{v_{1,s,j} \mid s \in \{1, 2, 3\}, j \in \{1, \dots, n\}, \alpha(x_j) = 1\} \\ & \cup \{v_{2,s,j} \mid s \in \{1, 2, 3\}, j \in \{1, \dots, n\}, \alpha(x_j) = 0\}. \end{aligned}$$

First, note that $|C| = 14m + 3n = k$. Secondly, we claim that C is a vertex cover of G . Clearly, all edges within clause gadgets and all edges within vertex gadgets are covered, because C contains all but one side in each of those gadgets. Edges between a vertex gadget and a clause gadget are covered because C contains vertices in sides of vertex gadgets, unless the literal this side corresponds to is not satisfied by α , but if this is the case, then C contains all sides of clause gadgets that correspond to this literal.

Now, suppose that $C \subseteq V(G)$, $|C| \leq k$, is a vertex cover of G . We may assume that C is minimum. Hence, by Lemmas 10–12, it contains all vertices in at two sides of every clause gadget and all vertices in exactly one side of every variable gadget. Let $\alpha: \{x_1, \dots, x_n\} \rightarrow \{0, 1\}$ with:

$$\alpha(x_j) := \begin{cases} 1, & \text{if } \{v_{1,s,j} \mid s \in \{1, 2, 3\}\} \subseteq C, \\ 0, & \text{if } \{v_{2,s,j} \mid s \in \{1, 2, 3\}\} \subseteq C. \end{cases}$$

We claim that α satisfies φ . Let C_i be a clause in φ . One of the three sides of the gadget representing C_i is not contained in C . This side corresponds to the literal $L_i^r \in \{x_j, \neg x_j\}$. If $L_i^r = x_j$, then all vertices in $\{u_{r,s,i} \mid s \in \{1, \dots, 7\}\}$ are adjacent to all vertices in $\{v_{1,s,j} \mid s \in \{1, 2, 3\}\}$. Since $\{u_{r,s,i} \mid s \in \{1, \dots, 7\}\} \not\subseteq C$, it follows that $\{v_{1,s,j} \mid s \in \{1, 2, 3\}\} \subseteq C$ and, therefore, $\alpha(x_j) = 1$. Hence, α satisfies the clause C_i . The case where $L_i^r = \neg x_j$ is analogous. \blacktriangleleft

Theorem 8 follows from the preceding lemmas.

Proof of Theorem 8. Clearly, Construction 9 can be computed in polynomial time. The claim follows by Lemmas 12 and 13. ◀

We remark that the NP-hardness of VERTEX COVER ABOVE CLUSTER DELETION NUMBER holds even if we are given a minimum cluster deletion set as part of the input. To show this, we slightly adapt Construction 9. Let (G, k) be an instance given in Construction 9. We further introduce $7m/3 + n$ complete graphs on three vertices (we may assume that m is divisible by 3, otherwise we add dummy clauses). Denote these vertices by T and observe that $|T| = 7m + 3n$. We add an edge between each vertex in $V(G)$ and each vertex in T . Let H denote the resulting graph. Observe that $V(G)$ is a cluster vertex deletion set of size $21m + 6n$ of H . Moreover, $\text{cd}(H) \geq \text{cd}(G) + |T| \geq 21m + 6n$. It is not difficult to show that any vertex cover of size at most $21m + 6n$ of G contains every vertex of T . Thus, H has a vertex cover of size at most $21m + 6n$ if and only if G has a vertex cover of size $21m + 6n - |T| = k$. Lemma 13 establishes the correctness of the reduction.

9 Feedback Vertex Set below Vertex Cover

FEEDBACK VERTEX SET BELOW n is generally known as the MAXIMUM INDUCED FOREST problem and is known to be $W[1]$ -hard with respect to the solution size [27]. In the following, we consider FEEDBACK VERTEX SET BELOW VERTEX COVER, essentially the same problem but with a slightly smaller parameter. We show that this change is sufficient to make the problem NP-hard even if the parameter is fixed at two.

► **Theorem 14.** FEEDBACK VERTEX SET BELOW VERTEX COVER is NP-hard even if $\ell = \text{vc}(G) - \text{fvs}(G) = 2$.

Proof. We reduce from FEEDBACK VERTEX SET. Let $I = (G, k)$ be an instance of the latter problem. We assume without loss of generality that G has at least one edge. Let $\lambda = |V(G)| - k - 2$. We construct a graph H with $\text{vc}(H) = |V(G)|$ that contains a feedback vertex set of size at most $k' = |V(G)| - 2$ if and only if G contains a feedback vertex set of size at most k . Note that then $\ell = \text{vc}(H) - k' = 2$.

The construction of H is as follows: Add a copy of G to H . Attach a leaf to every vertex $v \in V(G)$, that is, add a new vertex u_v and the edge $\{v, u_v\}$ to H . Add a set V^* of λ vertices to $V(H)$ and make each $u \in V^*$ adjacent to every vertex in $V(G)$.

Suppose G contains a feedback vertex set S of size at most k . Then the set $S \cup V^*$ is a feedback vertex set of size at most $k + \lambda = |V(G)| - 2 = k'$ for H , as $H - (S \cup V^*)$ is isomorphic to the forest $G - S$ with a leaf attached to every vertex.

Conversely, suppose that H contains a feedback vertex set S' of size at most k' . We claim that there exists a feedback vertex set of size at most k' in H that contains none of the leaves u_v and all vertices in V^* . Clearly, if S' contains a leaf u_v attached to some $v \in V(G)$, then $(S' \setminus \{u_v\}) \cup \{v\}$ is also a feedback vertex set of size at most $|V(G)| - 2$ in H . Hence, we may assume that $V^* \subseteq S' \subseteq V^* \cup V(G)$.

Next, suppose that $V^* \setminus S' \neq \emptyset$. If $|V^* \setminus S'| \geq 2$, then S' contains at least $|V(G)| - 1$ vertices of $V(G)$ since otherwise two vertices in $V^* \setminus S'$ and two vertices in $V(G)$ form a cycle of length four. Note, however, that $|S'| \leq k' = |V(G)| - 2$. Thus, we may assume that $|V^* \setminus S'| \leq 1$. Towards showing that $V^* \setminus S' = \emptyset$, suppose that there is a vertex $w \in V^* \setminus S'$. Then, for every edge $\{u, v\} \in E(G)$, we have $u \in S'$ or $v \in S'$, as otherwise u, v , and w induce a cycle in $H - S'$. Thus, $S' \setminus V^*$ is a vertex cover of G . Let $x \in S' \cap V(G)$ be

arbitrary. Such a vertex exists by the assumption that G has at least one edge. We claim that $S^* = (S' \setminus \{x\}) \cup \{w\}$ is a feedback vertex set for H of size at most k' . If it is not, then $H - S^*$ has a cycle that contains x . Let y and z be two neighbors of x in this cycle. Note that $y, z \in V(G) \setminus S'$. Since $S' \cap V(G)$ is a vertex cover of G , it follows that the neighborhoods of y and z in $H - S^*$ are $\{x, u_y\}$ and $\{x, u_z\}$, respectively. The vertices u_y and u_z have degree one, and thus we have a contradiction to the existence of the aforementioned cycle. Thus, the claim follows. Lastly, having a solution S' with $V^* \subseteq S' \subseteq V^* \cup V(G)$ of size at most k' implies that $S = S' \setminus V^*$ is a feedback vertex set of size at most k for G .

Finally, to show that $\ell = \text{fvs}(H) - \text{vc}(H) = 2$, we need to show that $\text{vc}(H) = \text{fvs}(H) + 2 = |V(G)|$. As $V(G)$ is a vertex cover of H , we have $\text{vc}(H) \leq |V(G)|$. Since $\{\{v, u_v\} \mid v \in V(G)\}$ is a matching of size $|V(G)|$ in H , we have $\text{vc}(H) \geq |V(G)|$. \blacktriangleleft

10 Conclusion

The goal of this work is to extend the above guarantee paradigm in parameterized complexity beyond the previously considered lower bounds on vertex cover, namely the maximum matching size and the optimal LP relaxation solution. We approached this issue by considering various structural graph parameters that are upper-bounded by the vertex cover number. This work sketches a rough contour of the parameterized complexity landscape of these kinds of parameterizations of both VERTEX COVER and FEEDBACK VERTEX SET. It raises a number of immediate open questions, of which we highlight four:

- (i) Is VERTEX COVER ABOVE TREEWIDTH also fixed-parameter tractable on arbitrary graphs? We also leave this question open for FEEDBACK VERTEX SET.
- (ii) In Section 7, we showed that VERTEX COVER ABOVE FEEDBACK VERTEX NUMBER is $W[1]$ -hard. A natural question to ask is whether this problem is NP-hard for a constant parameter value or whether it is in XP, that is, whether it can be decided by an algorithm with running time $\mathcal{O}(n^{f(\ell)})$ for an arbitrary computable function f .
- (iii) One can naturally generalize graph parameters like feedback vertex number or cluster deletion number by fixing a graph class \mathcal{F} and defining the \mathcal{F} -free deletion number of any graph G as the size of a smallest set $X \subseteq V(G)$ such that $G - X$ does not contain any $H \in \mathcal{F}$ as an induced subgraph. If \mathcal{F} contains a graph that is not edgeless, then vertex cover number upper-bounds the \mathcal{F} -free deletion number. It would be interesting to find a graph class \mathcal{F} such that VERTEX COVER ABOVE \mathcal{F} -FREE DELETION is FPT or to rule out the existence of such a class. We have only answered this question (always in the negative) if \mathcal{F} is any of the following classes: all cycles, $\{P_3\}$, all complete graphs.
- (iv) Moving beyond parameterized complexity, can graphs in which the difference parameters we have considered are small be characterized in an elegant way? For instance, one can easily prove that $\text{vc}(G) = \text{fvs}(G)$ if and only if G is edgeless. We are not aware of any simple characterization of graphs where $\text{vc}(G) - \text{fvs}(G) = 1$ or $\text{vc}(G) - \text{fvs}(G) \leq c$ for a larger constant c . Such a characterization could be useful for answering (ii).

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