

Degree-Restricted Strength Decompositions and Algebraic Branching Programs

Fulvio Gesmundo  

Saarland University, Saarbrücken, Germany

Purnata Ghosal 

University of Warwick, UK

Christian Ikenmeyer 

University of Warwick, UK

Vladimir Lysikov  

QMATH, Department of Mathematical Sciences, University of Copenhagen, Denmark

Abstract

We analyze Kumar’s recent quadratic algebraic branching program size lower bound proof method (CCC 2017) for the power sum polynomial. We present a refinement of this method that gives better bounds in some cases.

The lower bound relies on Noether-Lefschetz type conditions on the hypersurface defined by the homogeneous polynomial. In the explicit example that we provide, the lower bound is proved resorting to classical intersection theory.

Furthermore, we use similar methods to improve the known lower bound methods for slice rank of polynomials. We consider a sequence of polynomials that have been studied before by Shioda and show that for these polynomials the improved lower bound matches the known upper bound.

2012 ACM Subject Classification Theory of computation → Algebraic complexity theory

Keywords and phrases Lower bounds, Slice rank, Strength of polynomials, Algebraic branching programs

Digital Object Identifier 10.4230/LIPIcs.FSTTCS.2022.20

Funding *Purnata Ghosal*: DFG IK 116/2-1 and EPSRC EP/W014882/1. The work was done while this author was at the University of Liverpool.

Christian Ikenmeyer: DFG IK 116/2-1 and EPSRC EP/W014882/1. The work was done while this author was at the University of Liverpool.

Vladimir Lysikov: ERC 818761 and VILLUM FONDEN via the QMATH Centre of Excellence (Grant No. 10059).

Acknowledgements We would like to thank Daniele Agostini for many helpful discussions during the development of this work. We also thank Edoardo Ballico, Luca Chiantini, Giorgio Ottaviani and Kristian Ranestad for pointing out useful references on the Noether-Lefschetz property and related results. We thank the anonymous referees for their helpful comments.

1 Introduction

Homogeneous algebraic branching programs are a fundamental machine model for the computation of homogeneous polynomials. Their noncommutative version is completely understood (even in terms of border complexity) since Nisan’s 1991 paper [21]; they are the model of choice in [7] for succinct presentation of a system of homogeneous polynomial equations; and they can be used to phrase Valiant’s famous determinant versus permanent question [26] in a homogeneous way: Does the minimal size of the required homogeneous algebraic branching program for the permanent polynomial grow superpolynomially? Phrasing Valiant’s question in this way removes the “padding” problem in geometric complexity theory, so this is a way of circumventing the GCT no-go results in [18, 8].



© Fulvio Gesmundo, Purnata Ghosal, Christian Ikenmeyer, and Vladimir Lysikov; licensed under Creative Commons License CC-BY 4.0

42nd IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS 2022).

Editors: Anuj Dawar and Venkatesan Guruswami; Article No. 20; pp. 20:1–20:15



Leibniz International Proceedings in Informatics

Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Even though Valiant’s problem is the flagship problem in algebraic complexity theory, only very weak size bounds on homogeneous algebraic branching programs are known. The best lower bound so far was recently proved by Kumar [19] for the power sum polynomial. His proof and recent lower bounds proofs in other algebraic computation models (Chatterjee et al. [10] for general algebraic branching programs, and Kumar and Volk [20] for determinantal complexity) employ decompositions of the form

$$F = \sum_{k=1}^r G_k H_k + R, \tag{1}$$

where F is the polynomial for which the lower bound is proven, and G_k, H_k, R are polynomials such that $\deg R < \deg F$ and $G_k(0) = H_k(0) = 0$. Kumar’s recent $(d - 1)\lceil \frac{n}{2} \rceil$ bound on the size of homogeneous algebraic branching programs can be proven by considering simpler decompositions

$$F = \sum_{k=1}^r G_k H_k, \tag{2}$$

where F is a homogeneous polynomial and G_k, H_k are homogeneous polynomials of degree strictly smaller than F . Decompositions of this form have been investigated in algebraic geometry as well: for instance, in [11, 23], they were used to study rational points on certain algebraic varieties; they appear in [9] to characterize complete intersections contained in a given hypersurface; recently, they were used in [1] to give a proof of Stillman’s conjecture. Following [1], we say that (2) is a *strength decomposition* of the polynomial F ; the minimum r for which a strength decomposition exists is called the *strength* of F . In the literature, the strength of F is also called *Schmidt rank* or *h-invariant*.

In this paper we analyze Kumar’s lower bound proof method. The proof in [19] is tailored to the power sum polynomial, but the method is applicable to every polynomial. We present the general version of Kumar’s technique in Section 3.1. We refine this technique introducing the notion of k -restricted strength of a polynomial and we provide a lower bound for this notion based on geometric properties of the hypersurface defined by the polynomial. These properties are based on the non-existence of subvarieties of low codimension and low degree in the associated hypersurface and can be interpreted as higher codimension versions of a Noether-Lefschetz type condition [16].

We apply the refined method to give a lower bound on the size of a homogeneous algebraic branching program for an explicit family of polynomials

$$P_{n,d}(x_0, \dots, x_{2n}) = x_0^d + \sum_{k=1}^n x_{2k-1} x_{2k}^{d-1};$$

$P_{n,d}$ is a homogeneous polynomial of degree d in $N = 2n + 1$ variables. Kumar’s technique directly applied to this polynomial gives a lower bound $\lceil \frac{N}{4} \rceil (d - 1)$. For $d < 2^{N/4}$ we improve the lower bound by an additive term of approximately $N/2$. If the degree is exponential in N , we get a further additive improvement of order $\frac{N}{2} d^{O(1/N)}$, see Corollary 17(c).

In Section 3.4, we further study the notion of slice rank of homogeneous polynomials, which is a special case of k -restricted strength when $k = 1$. Theorem 18 gives a method to prove lower bounds on the slice rank. Using this result, in Theorem 20 we compute the slice rank of polynomials

$$S_{n,d}(x_0, \dots, x_{n+1}) = \sum_{i=0}^{n-1} x_i x_{i+1}^{d-1} + x_n x_0^{d-1} + x_{n+1}^d.$$

that have been studied by Shioda [24, 25]. For $n = 4$ and 6 we find that the slice rank is equal to $\frac{N}{2} + 1$, where $N = n + 2$ is the number of variables. To the authors' knowledge, this is the first lower bound for the slice rank better than $\lceil N/2 \rceil$. This translates to a lower bound $\frac{N}{2}(d-1) + 2$ on the homogeneous ABP complexity. Again, this is the first lower bound better than Kumar's $\lceil \frac{N}{2} \rceil(d-1)$. We conjecture that these bounds continue to hold for $S_{n,d}$ with arbitrary even n , see Conjecture 22.

2 Preliminaries

We work over the field of complex numbers \mathbb{C} . A homogeneous linear polynomial is called a *linear form*. The ideal generated by polynomials F_1, \dots, F_m is denoted by $\langle F_1, \dots, F_m \rangle$. An ideal is homogeneous if it admits a set of homogeneous generators. Every (homogeneous) ideal in a polynomial ring admits a finite set of (homogeneous) generators [13, Theorem 1.2].

2.1 Projective geometry

Since we work with homogeneous polynomials, it is convenient to work in projective space \mathbb{P}^n , which is defined as $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^\times$, that is, points in \mathbb{P}^n correspond to lines through the origin in \mathbb{C}^{n+1} . Given a nonzero vector $v = (v_0, \dots, v_n) \in \mathbb{C}^{n+1}$, write $[v] = (v_0 : \dots : v_n)$ for the corresponding point in \mathbb{P}^n . We refer to [17] for basics of projective geometry and we only record some basic facts. Given a homogeneous ideal $I = \langle F_1, \dots, F_m \rangle$, write $Z(I) = Z(F_1, \dots, F_m) = \{[v] \in \mathbb{P}^n : F_j(v) = 0 \text{ for every } j\}$; a subset $X \subset \mathbb{P}^n$ is a variety if $X = Z(I)$ for some homogeneous ideal I . A variety is irreducible if it is not the proper union of two varieties; every variety X can be written uniquely as a union of finitely many irreducible subvarieties $X = \bigcup_1^r X_j$; X_1, \dots, X_r are called irreducible components of X . We refer to [17, Ch. 11 and Ch. 18] for the definitions and the basic properties of dimension and degree of a variety X , denoted respectively $\dim X$ and $\deg X$. The dimension of \mathbb{P}^n is n . The codimension of $X \subseteq \mathbb{P}^n$ is $\text{codim } X = n - \dim X$; if $X = \emptyset$, $\text{codim } X = n + 1$. A variety X is called *hypersurface* if all its irreducible components have codimension 1; in this case $X = Z(F)$ is defined by a principal ideal $\langle F \rangle$.

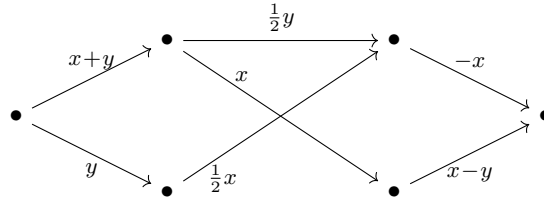
A (*projective*) *linear subspace* in \mathbb{P}^n is a variety defined by linear forms. The codimension of $Z(L_1, \dots, L_r)$, for some linear forms L_1, \dots, L_r , equals the number of linearly independent elements among L_1, \dots, L_r ; in particular $\text{codim } Z(L_1, \dots, L_r) \leq r$. A *line* is a linear subspace of dimension 1. The line spanned by two distinct points $[x], [y] \in \mathbb{P}^n$ is the unique line containing them. This line consists of all points of the form $[\alpha x + \beta y]$ where $(\alpha, \beta) \neq (0, 0)$.

Let $X \subset \mathbb{P}^n$ be a variety. The *projective cone* over X with vertex $p \notin X$ is the union of all lines connecting p with a point in X .

Given a hypersurface $Z(F) \subseteq \mathbb{P}^n$, write $\text{Sing}(F) = Z(\frac{\partial F}{\partial x_i} : i = 0, \dots, n)$, which is a subvariety of $Z(F)$. For example, if $F = x_0^d + x_1^d + x_2^d$, then $\text{Sing}(F) = Z(x_0^{d-1}, x_1^{d-1}, x_2^{d-1}) = \emptyset \subseteq \mathbb{P}^2$; if $F = x_0^d + x_1 x_2^{d-1}$ then $\text{Sing}(F) = Z(x_0^{d-1}, x_2^{d-1}, x_1 x_2^{d-2}) = Z(x_0, x_2) = \{[0 : 1 : 0]\} \subseteq \mathbb{P}^2$. If the factorization of F into irreducible polynomials does not have repeated factors, then $\text{Sing}(F)$ coincides with the *singular locus* of the hypersurface $Z(F)$, see, e.g., in [17, Ch. 14].

We mention the following two fundamental results in algebraic geometry.

► **Theorem 1** (Krull height theorem for polynomial rings [2, Cor. 11.17]). *Let $X \subset \mathbb{P}^n$ be a variety, with $X = Z(F_1, \dots, F_c)$. Then all irreducible components of X have codimension at most c .*



■ **Figure 1** A homogeneous algebraic branching program of size 4, computing $(x + y) \cdot \frac{1}{2}y \cdot (-x) + (x + y) \cdot x \cdot (x - y) + y \cdot \frac{1}{2}x \cdot (-x) = x^3 - x^2y - \frac{3}{2}xy^2$.

► **Theorem 2** (Bezout inequality, see [6, Thm. 8.28]). *If a projective variety $X \subset \mathbb{P}^n$ is cut out by c polynomials of degrees d_1, \dots, d_c , the sum of degrees of its irreducible components is at most $\prod_{i=1}^c d_i$.*

Further, in the proof of Lemma 16, we require some basics of intersection theory. We introduce the required notions and references in Section 3.3.

2.2 Algebraic branching programs

The computational model of algebraic branching programs was first formally defined by Nisan [21] in the context of noncommutative computation, but essentially the same model was used by Valiant in his famous proof of universality of determinant [26]. The computational power of algebraic branching programs is intermediate between the one of general arithmetic circuits and the one of arithmetic formulas. It is a convenient model for algebraic methods, because its power can be captured by restrictions of determinants or iterated matrix multiplication polynomials, which allows for the use of well developed tools from algebra and algebraic geometry. In this paper we only consider homogeneous algebraic branching programs.

► **Definition 3.** *A layered directed graph is a directed graph in which the set of vertices is partitioned into layers indexed by integers so that each edge connects vertices in consecutive layers.*

► **Definition 4.** *A homogeneous algebraic branching program (ABP) in variables x_1, \dots, x_n is a layered directed graph with one source and one sink, and with edges labeled by linear forms in x_1, \dots, x_n . The weight of a path in an ABP is the product of labels on the edges of the path. The polynomial computed between vertices u and v is the sum of the weights of all paths from u to v . The polynomial computed by an ABP is the polynomial computed between the source and the sink.*

The size of an ABP is the number of its inner vertices, namely all vertices except the source and the sink. For a homogeneous polynomial F , its homogeneous ABP complexity $B_{\text{hom}}(F)$ is the minimal size of a homogeneous ABP computing F .

See Figure 1 for an example of an ABP.

2.3 Strength and slice rank of polynomial

► **Definition 5.** *Let F be a homogeneous polynomial of degree d . A strength decomposition of F is a decomposition of the form*

$$F = \sum_{k=1}^r G_k H_k$$

where G_k and H_k are homogeneous polynomials of degree less than d . The strength of F is

$$\text{str}(F) = \min\{r : F \text{ has a strength decomposition with } r \text{ summands}\}.$$

The following is a basic lower bound for the strength of a polynomial. It appears in the introduction of [1] and [3, Remark 4.3]; in [19] it is mentioned with a reference to a personal communication with Saptharishi.

► **Proposition 6.** $\text{str}(F) \geq \lceil \frac{1}{2} \text{codim Sing}(F) \rceil$.

Proof. If $F = \sum_{k=1}^r G_k H_k$, then $\frac{\partial}{\partial x_i} F = \sum_{k=1}^r G_k \frac{\partial}{\partial x_i} H_k + \sum_{k=1}^r H_k \frac{\partial}{\partial x_i} G_k$. Thus all the partial derivatives of F lie in the ideal $\langle G_1, \dots, G_r, H_1, \dots, H_r \rangle$ and therefore the zero set $Z(G_1, \dots, G_r, H_1, \dots, H_r)$ is contained in $\text{Sing}(F)$.

Therefore, applying Theorem 1, we deduce

$$\text{codim Sing}(F) \leq \text{codim } Z(G_1, \dots, G_r, H_1, \dots, H_r) \leq 2r$$

and the required lower bound follows. ◀

► **Remark 7.** The bound of Proposition 6 gives essentially the only known lower bound method for strength which can be applied to explicit polynomials. Different methods are applied to polynomials of a specific form satisfying an unspecified genericity condition: for instance, in [3], it is shown that a polynomial of the form $F = x_1^2 f_1 + x_2^2 f_2 + x_3^2 f_3 + x_4^2 f_4$ with generic f_1, \dots, f_4 has strength 4; this is however achieved using indirect methods [12, 5].

► **Definition 8.** Let F be a homogeneous polynomial of degree d . A slice rank decomposition of F is a decomposition of the form

$$F = \sum_{k=1}^r L_k H_k$$

where L_k are linear forms. The minimal number of summands in a strength decomposition of F is called the slice rank of F and is denoted by $\text{sr}(F)$.

We point out that the notion of slice rank of tensors [22] is related but geometrically very different from the slice rank of homogeneous polynomials defined above.

Clearly, slice rank decompositions are a special class of strength decompositions and thus $\text{sr}(F) \geq \text{str}(F)$. It is known that for generic polynomials the optimal strength decomposition is a slice rank decomposition [4]; in particular $\text{str}(F) = \text{sr}(F)$ for generic F .

Slice rank decompositions of a polynomial F have a clear geometric interpretation in terms of linear subspaces contained in the hypersurface $Z(F)$.

► **Proposition 9.** Let F be a homogeneous polynomial. We have $\text{sr}(F) \leq r$ if and only if $Z(F)$ contains a linear subspace of codimension r .

Proof. The polynomial F admits a decomposition $F = \sum_{k=1}^r L_k H_k$ for linear forms L_1, \dots, L_k if and only if $F \in \langle L_1, \dots, L_k \rangle$. The ideal $\langle L_1, \dots, L_k \rangle$ is radical, in the sense of [13, Sec. 1.6]. Therefore, by the classic Nullstellensatz [13, Thm. 1.6], the condition $F \in \langle L_1, \dots, L_k \rangle$ is equivalent to the condition $Z(F) \supset Z(L_1, \dots, L_r)$. ◀

3 Degree-restricted strength decompositions and lower bounds on ABP size

3.1 Basic properties and connection to ABPs

In this section we introduce the degree-restricted strength decompositions and present a streamlined proof of Kumar's lower bound generalized to arbitrary polynomials based on Proposition 6.

► **Definition 10.** Let F be a homogeneous polynomial of degree d . A strength decomposition $F = \sum_{k=1}^r G_k H_k$ is called j -restricted if $\deg G_k = j$ for all k . The j -restricted strength $\text{str}_j(F)$ is the minimal number of summands in a j -restricted strength decomposition of F .

The following basic properties are clear from the definition.

► **Proposition 11.** Let F be a homogeneous polynomial of degree d and let j be an integer such that $1 \leq j < d$. The following statements hold.

- (a) $\text{str}_j(F) \geq \text{str}(F)$;
- (b) $\text{str}_j(F) = \text{str}_{d-j}(F)$;
- (c) $\text{str}_1(F) = \text{sr}(F)$;

► **Theorem 12.** For every homogeneous polynomial F of degree d

$$B_{\text{hom}}(F) \geq \sum_{j=1}^{d-1} \text{str}_j(F).$$

Proof. Let A be a homogeneous ABP computing F with source s and sink t . Denote by $A[v, w]$ the polynomial computed between vertices v and w . Let V_j be the set of vertices in the j -th layer. Since each path from the source to the sink contains exactly one vertex from each layer, we have

$$F = A[s, t] = \sum_{v \in V_j} A[s, v]A[v, t].$$

If v lies in the j -th layer then $\deg A[s, v] = j$, because each path from s to v contains j edges. Thus F has a j -restricted strength decomposition with $|V_j|$ summands, showing $|V_j| \geq \text{str}_j(F)$. Summing over all layers, we obtain the desired lower bound. ◀

Theorem 12 is similar to Nisan's result for noncommutative ABPs [21, Thm. 1], but in the noncommutative setting the analogue of strength can be easily described as the rank of the partial derivative matrix. In the commutative setting there is no similar characterization of strength.

From Theorem 12, Proposition 11(a) and Proposition 6, we immediately obtain the following B_{hom} lower bounds technique.

► **Corollary 13** (Kumar's singular locus lower bounds technique). For every homogeneous polynomial $F \in \mathbb{C}[x_0, \dots, x_n]$ of degree d

$$B_{\text{hom}}(F) \geq (d-1) \lceil \frac{1}{2} \text{codim Sing}(F) \rceil.$$

In particular, if $\text{Sing}(F) = \emptyset$, then

$$B_{\text{hom}}(F) \geq (d-1) \lceil \frac{n+1}{2} \rceil.$$

The power sum $F = x_0^d + \dots + x_n^d$ of degree d in $n+1$ variables satisfies $\text{Sing}(F) = \emptyset$ and hence a direct application of Corollary 13 gives $B_{\text{hom}}(x_0^d + \dots + x_n^d) \geq (d-1) \lceil \frac{n+1}{2} \rceil$, which recovers Kumar's result [19].

3.2 Lower bound on degree-restricted strength

In this section we prove a lower bound on the degree-restricted strength of polynomials. It is based on the connection between strength decompositions and low degree subvarieties in $Z(F)$, which generalizes Proposition 9. We provide an explicit sequence of polynomials for which we obtain a lower bound that is slightly stronger than the one from Proposition 6.

► **Theorem 14.** *Let $F \in \mathbb{C}[x_0, \dots, x_n]_d$ be a homogeneous polynomial. Suppose that the zero set $Z(F)$ does not contain irreducible subvarieties X with $\text{codim } X \leq c$ and $\text{deg } X < s$ for some $s \geq 2$. Then*

$$\text{sr}(F) \geq c + 1$$

and

$$\text{str}_k(F) \geq \min\{c + 1, \lceil \log_k s \rceil\}$$

Proof. The statement for the slice rank follows from Proposition 9 as $Z(F)$ does not contain linear subspaces, that is, irreducible subvarieties of degree 1, of codimension c .

Assume F has a k -restricted decomposition

$$F = \sum_{j=1}^r G_j H_j.$$

Consider the variety $Y = Z(G_1, \dots, G_r)$. Since F lies in the ideal $\langle G_1, \dots, G_r \rangle$, Y is a subvariety of $Z(F)$. Since Y is defined by r polynomials of degree k , by Theorem 1, the codimension of every irreducible component of Y is at most r ; moreover, by Theorem 2 the sum of degrees of its irreducible components is at most k^r , hence the same holds for each component.

Suppose $r \leq c$. Since $Z(F)$ does not contain subvarieties X with $\text{codim } X \leq c$ and $\text{deg } X < s$, for each irreducible component X of Y we have $k^r \geq \text{deg } X \geq s$, so $r \geq \log_k s$. Since r is an integer, we obtain the lower bound $r \geq \lceil \log_k s \rceil$. Hence, either $r \geq c + 1$ or $r \geq \lceil \log_k s \rceil$, and we conclude $r \geq \min\{c + 1, \lceil \log_k s \rceil\}$ as desired. ◀

► **Remark 15.** In Theorem 14 it suffices to require that $Z(F)$ does not contain subvarieties of codimension exactly c and degree smaller than s . This is a consequence of Bertini's Theorem, see [17, Sec. 18]. Indeed, if X is an irreducible subvariety of $Z(F)$ with $\text{codim } X < c$, let X' be the intersection of X with $c - \text{codim } X$ generic hyperplanes; then $\text{codim } X' = c$ and $\text{deg } X = \text{deg } X'$.

Consider the family of polynomials

$$P_{n,d}(x_0, x_1, \dots, x_{2n}) = x_0^d + \sum_{k=1}^n x_{2k-1} x_{2k}^{d-1}$$

with $d \geq 3$. Note that it is clear from the definition of $P_{n,d}$ that $\text{str}_k(P_{n,d}) \leq n + 1$. Also note that $\text{Sing}(P_{n,d})$ is the linear subspace given by $x_0 = x_2 = x_4 = \dots = x_{2n} = 0$. Its codimension is $n + 1$, so the singular locus lower bound on $B_{\text{hom}}(P_{n,d})$ from Corollary 13 is

$$B_{\text{hom}}(P_{n,d}) \geq (d - 1) \lceil \frac{n+1}{2} \rceil.$$

Corollary 17(c) below will provide an improvement of this lower bound, based on Theorem 14. In order to apply Theorem 14, we need a lower bound on the degree of subvarieties of $Z(P_{n,d})$ of low codimension. This is obtained resorting to an intersection theoretic argument, which is explained in the next section.

3.3 Intersection theory of $Z(P_{n,d})$

This section requires some background in intersection theory, for which we refer to [14] and [15]. We use some facts about Chow groups of a variety, which is one of the fundamental objects studied in intersection theory.

Let X be a variety. Given an integer $a \geq 0$, the Chow group $\text{CH}_a(X)$ is an abelian group associated to the variety X . Every irreducible subvariety $Y \subset X$ of dimension a corresponds to an element $[Y] \in \text{CH}_a(X)$ and these elements generate $\text{CH}_a(X)$. Thus, $\text{CH}_a(X)$ consists of integer linear combinations of irreducible a -dimensional subvarieties of X modulo a certain equivalence relation called *rational equivalence*, the definition of which we do not reproduce here. The elements of $\text{CH}_a(X)$ are called *algebraic cycle classes* of dimension a on X .

If $Y \subset X$ is a subvariety, then every subvariety $Z \subset Y$ is also a subvariety of X . This gives rise to a homomorphism $\iota_*: \text{CH}_a(Y) \rightarrow \text{CH}_a(X)$ sending a cycle class of a subvariety Z of Y to the cycle class of the same variety as a subvariety of X . This homomorphism is called the *pushforward* induced by the inclusion $\iota: Y \hookrightarrow X$. It is a special case of the proper pushforward [15, §1.4].

In addition, we can consider the open set $U = X \setminus Y$ as a variety and its subvarieties. The map j^* sending a cycle class $[Z]$ on X to a cycle class $[Z \cap U]$ on U is well defined. Again, this is a special case of a more general construction of flat pullback [15, §1.7].

Thus for $Y \subset X$ we have $\iota_*: \text{CH}_a(Y) \rightarrow \text{CH}_a(X)$ and $j^*: \text{CH}_a(X) \rightarrow \text{CH}_a(X \setminus Y)$. It is an important fact that these homomorphisms compose to give an exact sequence

$$\text{CH}_a(Y) \rightarrow \text{CH}_a(X) \rightarrow \text{CH}_a(X \setminus Y) \rightarrow 0;$$

in other words, $\ker j^* = \text{im } \iota_*$. This exact sequence is called the *excision exact sequence*, see [15, Prop. 1.8].

For simple varieties the Chow groups can be constructed explicitly. For the projective space \mathbb{P}^n the class in $\text{CH}_a(\mathbb{P}^n)$ of a variety is determined by its degree. More precisely, we have $\text{CH}_a(\mathbb{P}^n) \cong \mathbb{Z}$ where the isomorphism is given by the map $\text{deg}: \text{CH}_a(\mathbb{P}^n) \rightarrow \mathbb{Z}$ sending the class of a subvariety Z to $\text{deg } Z$ (see [14, Thm. 2.1] or [15, Ex. 1.9.3]). A consequence of this is that the degree is well defined for cycle classes on projective varieties. That is, if $X \subseteq \mathbb{P}^n$ is a projective variety and Z is a subvariety of X , then the degree of Z can be read from the cycle class $[Z] \in \text{CH}_a(X)$ by applying the pushforward $\iota_*: \text{CH}_a(X) \rightarrow \text{CH}_a(\mathbb{P}^n)$.

On the other hand, for the affine space the Chow groups $\text{CH}_a(\mathbb{A}^n) = 0$ are trivial for every $a < n$ (see [14, Thm. 1.13] or [15, §1.9]).

Finally, we need a statement which relates the Chow groups of a projective cone X over Y to the Chow groups of Y . Recall that X is a cone over Y (with vertex $p \notin Y$) if X is the union of lines $\langle y, p \rangle$ for $y \in Y$. In this case $\text{CH}_a(Y) \cong \text{CH}_{a+1}(X)$ [15, Ex. 2.6.2]. The isomorphism is given by a map $\alpha: \text{CH}_a(Y) \rightarrow \text{CH}_{a+1}(X)$ which sends a cycle class of a subvariety $Z \subset Y$ to the cycle class of the cone over Z .

► **Lemma 16.** *Let $F \in \mathbb{C}[x_0, \dots, x_n]_d$ be a homogeneous polynomial of degree $d \geq 2$. Set $G = F + x_{n+1}x_{n+2}^{d-1}$. Suppose that the degree of every subvariety $Y \subseteq Z(F) \subseteq \mathbb{P}^n$ with $\dim Y = a$ is divisible by s . Then the degree of every subvariety $Y' \subseteq Z(G) \subseteq \mathbb{P}^{n+2}$ with $\dim Y' = a + 1$ is divisible by s .*

Proof. Let $X = Z(G) \subset \mathbb{P}^{n+2}$ and $Z = Z(F) \subset \mathbb{P}^n$. Let $Y \subset \mathbb{P}^{n+1}$ be the variety given by the equation $F(x_0, \dots, x_n) = 0$ in \mathbb{P}^{n+1} . It consists of all points of the form $[\alpha x + \beta e_{n+1}]$ with $F(x) = 0$, so it is the projective cone over Z with the vertex $[e_{n+1}]$.

The intersection theory statements about projective cones discussed above says that the Chow group $\text{CH}_{a+1}(Y)$ is isomorphic to the Chow group $\text{CH}_a(Z)$. The isomorphism $\alpha: \text{CH}_a(Z) \rightarrow \text{CH}_{a+1}(Y)$ takes the class of a subvariety in Z to the class of the cone over this subvariety.

Let H be the hyperplane given by $x_{n+2} = 0$ in \mathbb{P}^{n+2} . Note that $X \cap H \subset H$ is isomorphic to $Y \subset \mathbb{P}^{n+1}$, so we can identify $X \cap H$ with Y . Let U be the open subset $X \setminus Y$. This subset is an affine variety in $\mathbb{A}^{n+2} = \mathbb{P}^{n+2} \setminus H$ given by the equation $F + x_{n+1} = 0$. Thus $U \cong \text{graph } F \cong \mathbb{A}^{n+1}$. It follows that the Chow group $\text{CH}_{a+1}(U)$ is trivial.

The exactness of the excision exact sequence

$$\text{CH}_{a+1}(Y) \rightarrow \text{CH}_{a+1}(X) \rightarrow \text{CH}_{a+1}(U) \rightarrow 0$$

implies that the inclusion pushforward $\iota^* : \text{CH}_{a+1}(Y) \rightarrow \text{CH}_{a+1}(X)$ is surjective.

Both the cone map $\alpha : \text{CH}_a(Z) \rightarrow \text{CH}_{a+1}(Y)$ and $\iota^* : \text{CH}_{a+1}(Y) \rightarrow \text{CH}_{a+1}(X)$ preserve the degree. Composing them, we obtain a degree-preserving surjective homomorphism from $\text{CH}_a(Z)$ to $\text{CH}_{a+1}(X)$. Since the degree of every dimension a subvariety of Z is divisible by s , the same is true for cycle classes in $\text{CH}_a(Z)$ and therefore, for cycle classes in $\text{CH}_{a+1}(X)$, including dimension $a + 1$ subvarieties of X . ◀

► **Corollary 17.** *If $n \geq 1$ and $d \geq 2$, then*

- (a) $\text{sr}(P_{n,d}) = n + 1$;
 - (b) $\text{str}_k(P_{n,d}) \geq \min\{n + 1, \lceil \log_k d \rceil\}$;
 - (c) $\text{B}_{\text{hom}}(P_{n,d}) \geq (d - 1)\lceil \frac{n+1}{2} \rceil + 2\lfloor \frac{n+1}{2} \rfloor$ for $d \leq 2^{\frac{n+1}{2}}$;
- $$\text{B}_{\text{hom}}(P_{n,d}) \geq (d - 1)\lceil \frac{n+1}{2} \rceil + 2\lfloor \frac{n+1}{2} \rfloor \lfloor d^{\frac{1}{n+1}} \rfloor + \sum_{j=\lfloor d^{\frac{1}{n+1}} \rfloor + 1}^{\lfloor d^{\frac{2}{n+1}} \rfloor} (\lceil \log_j d \rceil - \lceil \frac{n+1}{2} \rceil)$$
- for all
- d
- .

Proof. The polynomial $P_{1,d} = x_0^d + x_1 x_2^{d-1}$ is irreducible, e.g., by the Eisenstein criterion [13, Ex. 18.11] applied to it as an element of $\mathbb{C}[x_1, x_2][x_0]$ with the prime ideal $\langle x_1 \rangle$.

It follows that $Z(P_{1,d})$ is an irreducible variety of degree d in \mathbb{P}^2 , so it does not contain subvarieties of codimension 1 other than itself; in particular, every subvariety of codimension 1 has degree divisible by d . Using Lemma 16 inductively, we obtain that every subvariety X of $Z(P_{n,d})$ with $\text{codim } X = n$ has degree divisible by d . The lower bounds for the slice rank and the restricted strength follow by Theorem 14.

The lower bound for the homogeneous ABP complexity follows by Proposition 11(b) and Theorem 12. If $d \leq 2^{\frac{n+1}{2}}$, we use $\text{str}_1(P_{n,d}) = \text{str}_{d-1}(P_{n,d}) = n + 1$ and $\text{str}_j(P_{n,d}) \geq \lceil \frac{n+1}{2} \rceil$ from Proposition 6 for other j to get

$$\text{B}_{\text{hom}}(P_{n,d}) \geq \sum_{j=1}^{d-1} \text{str}_j(P_{n,d}) \geq 2(n + 1) + (d - 3)\lceil \frac{n + 1}{2} \rceil = (d - 1)\lceil \frac{n + 1}{2} \rceil + 2\lfloor \frac{n + 1}{2} \rfloor$$

For the second bound in (c), we separate the sum $\sum_{j=1}^{d-1} \text{str}_j(P_{n,d})$ into three parts.

For $j \leq d^{\frac{1}{n+1}}$ we have $\text{str}_j(P_{n,d}) = \text{str}_{d-j}(P_{n,d}) \geq n + 1$, for $d^{\frac{1}{n+1}} < j \leq d^{\frac{2}{n+1}}$ we use the lower bound $\text{str}_j(P_{n,d}) = \text{str}_{d-j}(P_{n,d}) \geq \lceil \log_j d \rceil$, and for $d^{\frac{2}{n+1}} < j < d - d^{\frac{2}{n+1}}$ Proposition 6 gives $\text{str}_j(P_{n,d}) \geq \lceil \frac{n+1}{2} \rceil$. ◀

We point out that determining explicit hypersurfaces which do not contain low codimension subvarieties of low degree is an extremely hard problem. It is related to the Noether-Lefschetz Theorem, a classical result which, in particular, implies that if F is a general homogeneous polynomial of degree d , then $Z(F)$ does not contain subvarieties X with $\text{codim } X = 2$ and $\text{deg } X \leq d$. A consequence of [27] is that if F is a general homogeneous polynomial of degree $d \geq 6$ in five variables, then $Z(F) \subseteq \mathbb{P}^4$ does not contain subvarieties X with $\text{codim } X \leq 3$ and $\text{deg}(X) \leq d$. Stronger cohomological results hold for very general hypersurfaces; a series of conjectures and open problems is proposed in [16].

3.4 Slice rank lower bound

In this section, we give examples of polynomials for which we prove a slice rank lower bound stronger than the one induced by the strength lower bound of Proposition 6. In one instance, we show an improved lower bound for a polynomial with $\text{Sing}(F) = \emptyset$.

We define the *Shioda polynomials* of degree $d \geq 3$ in $n + 2$ variables:

$$S_{n,d}(x_0, \dots, x_{n+1}) = \sum_{i=0}^{n-1} x_i x_{i+1}^{d-1} + x_n x_0^{d-1} + x_{n+1}^d.$$

The polynomials $S_{n,d}$ were investigated by Shioda [24, 25] as explicit examples for a cohomological version of the Noether-Lefschetz Theorem in middle dimension.

For even n the decomposition

$$S_{n,d} = \sum_{k=0}^{n/2-1} x_{2k+1} (x_{2k+2}^{d-1} + x_{2k} x_{2k+1}^{d-2}) + x_n \cdot x_0^{d-1} + x_{n+1} \cdot x_{n+1}^{d-1} \quad (3)$$

shows that $\text{sr}(S_{n,d}) \leq \frac{n}{2} + 2$. We prove matching lower bounds for $n = 2$ and $n = 4$; we conjecture that the bound holds for all even values of n .

First consider a modified polynomial

$$\hat{S}_{n,d}(x_1, \dots, x_{n+1}) = \sum_{i=1}^{n-1} x_i x_{i+1}^{d-1} + x_{n+1}^d,$$

which is obtained from $S_{n,d}$ by setting $x_0 = 0$. It is easy to verify that $\text{Sing}(\hat{S}_{n,d})$ coincides with the point $[e_{n-1}] = (0 : \dots : 1 : 0 : 0)$; here we use homogeneous coordinates $(x_1 : \dots : x_{n+1})$ on \mathbb{P}^n .

If n is even, then $\text{codim Sing}(\hat{S}_{n,d}) = n$ and Proposition 6 gives a lower bound $\text{sr}(\hat{S}_{n,d}) \geq \frac{n}{2}$. An analog of (3) gives the upper bound $\frac{n}{2} + 1$. We provide a matching lower bound, relying on the following result which improves the lower bound of Proposition 6.

► **Theorem 18.** *Let $F \in \mathbb{C}[x_0, \dots, x_n]$ be a homogeneous polynomial with $\text{codim Sing}(F) = s$ even. Then $\text{sr}(F) = \frac{s}{2}$ if and only if there is a linear space $Q \subset \mathbb{Z}(F)$ of codimension $\frac{s}{2}$ containing one of the irreducible components of $\text{Sing}(F)$.*

Proof. If $Q \subset \mathbb{Z}(F)$ is a linear space with $\text{codim } Q = s/2$, then $\text{sr}(F) = s/2$ by Proposition 9.

Conversely, suppose $\text{sr}(F) = s/2$ and let $F = \sum_{k=1}^{s/2} L_k H_k$ be a minimal slice rank decomposition of F . Let $Q = \mathbb{Z}(L_1, \dots, L_{s/2})$, and let $X = \mathbb{Z}(L_1, \dots, L_{s/2}, H_1, \dots, H_{s/2})$. By the minimality of the decomposition, Q is a linear space of codimension $s/2$. Moreover, $X \subseteq Q \subseteq \mathbb{Z}(F)$.

Since X is defined by s polynomials, by Theorem 1 all irreducible components of X have codimension at most s . As in Proposition 6, X is contained in $\text{Sing}(F)$. Therefore, for every irreducible component X' of X , we have $s = \text{codim Sing}(F) \leq \text{codim } X \leq \text{codim } X' \leq s$. This shows that X' and $\text{Sing}(F)$ have the same dimension, therefore X' is an irreducible component of $\text{Sing}(F)$. Since $X' \subseteq X \subseteq Q$, we conclude. ◀

► **Lemma 19.** *If $d \geq 3$ and n is even, then $\text{sr}(\hat{S}_{n,d}) = \frac{n}{2} + 1$*

Proof. If n is even, Proposition 6 gives the lower bound $\text{sr}(\hat{S}_{n,d}) \geq \frac{n}{2}$. We use induction on n to prove that $\text{sr}(\hat{S}_{n,d}) \neq \frac{n}{2}$. If $n = 0$ the statement is clear.

Suppose by contradiction $\text{sr}(\hat{S}_{n,d}) = \frac{n}{2}$. By Theorem 18 there exists a projective linear subspace $Q \subset Z(\hat{S}_{n,d})$ of dimension $\frac{n}{2}$ containing the singular point $[e_{n-1}]$. Let $H = Z(x_{n-1})$ and let $Q' = Q \cap H$. Since Q contains the point $[e_{n-1}]$, which is not in H , we have $\dim Q' = \frac{n}{2} - 1$.

Observe that for every point $(v_1 : \dots : v_n) \in Q'$, we have $v_n = 0$. To see this, fix $[v] \in Q'$ and consider the line spanned by $[v] \in Q'$ and $[e_{n-1}]$; this line is contained in Q , hence in $Z(\hat{S}_{n,d})$. In particular, for every $\alpha, \beta \in \mathbb{C}$, we have

$$0 = \hat{S}_{n,d}(\alpha v + \beta e_{n-1}) = \alpha^d \left(\sum_{k=1}^{n-3} v_k v_{k+1}^{d-1} + v_{n+1}^d \right) + \alpha^{d-1} \beta v_n^{d-1}.$$

Therefore, this expression must be 0 as a polynomial in α, β and in particular $v_n = 0$ because v_n^{d-1} is the coefficient of $\alpha^{d-1} \beta$.

This shows that the existence of a subspace $Q \subset Z(\hat{S}_{n,d})$ of dimension $\frac{n}{2}$ containing $[e_{n-1}]$ implies the existence of a subspace $Q' \subset Z(\hat{S}_{n,d}) \cap Z(x_{n-1}, x_n)$ of dimension $\frac{n}{2} - 1$. Note that substituting $x_{n-1} = x_n = 0$ into $\hat{S}_{n,d}$, one obtains, up to renaming the variables, the polynomial $\hat{S}_{n-2,d}$. Therefore, the existence of Q' implies $\text{sr}(\hat{S}_{n-2,d}) = \frac{n}{2} - 1 = \frac{n-2}{2}$, in contradiction with the induction hypothesis. This concludes the proof. \blacktriangleleft

► **Theorem 20.** *If $d \geq 5$, then $\text{sr}(S_{4,d}) = 4$.*

Proof. Let $\rho: \mathbb{P}^5 \rightarrow \mathbb{P}^5$ be the map defined by $\rho(x_0 : x_1 : \dots : x_5) = (x_4 : x_0 : x_1 : x_2 : x_3 : x_5)$ which cyclically permutes the first 5 coordinates of a point $(x_0 : x_1 : \dots : x_5)$. Note that the hypersurface $Z(S_{4,d})$ is mapped to itself by ρ .

The lower bound given by Proposition 6 is $\text{sr}(S_{4,d}) \geq 3$ and assume by contradiction that equality holds. Then there exists a linear space $Q \subset Z(S_{4,d}) \subset \mathbb{P}^5$ of codimension 3.

First, we prove a series of claims about the plane Q culminating with the claim that Q contains one of the five points $[e_k] = \rho^k[e_0]$, where $e_0 = (1, 0, \dots, 0)$. Then we derive a contradiction with the lower bound for $\hat{S}_{n,d}$.

Let A_0 be the line $Z(x_1, x_3, x_4, x_5) \subseteq \mathbb{P}^5$ and $A_k = \rho^k A_0$. In other words, A_0 is the set of points of the form $(x_0 : 0 : x_2 : 0 : 0 : 0)$ and A_k is obtained by cyclically shifting the first five coordinates.

► **Claim 20.1.** Q intersects $A_0 \cup A_1 \cup A_2$.

Proof. Since $\dim Q = 2$, by [17, Prop. 11.4] it intersects any codimension 2 linear subspace. Let $p = (p_0 : 0 : p_2 : 0 : p_4 : p_5) \in Q \cap Z(x_1, x_3)$ and $q = (q_0 : q_1 : 0 : q_3 : 0 : q_5) \in Q \cap Z(x_2, x_4)$ be two points which lie in the respective intersections.

If $p = q$, then $p = q = [e_0] \in A_0$ and the Claim is verified.

Suppose $p \neq q$. The line joining p and q lies in Q and, therefore, in $Z(S_{4,d})$. Let $\hat{p}, \hat{q} \in \mathbb{C}^6$ be representatives of p and q respectively. We have $S_{4,d}(\alpha \hat{p} + \beta \hat{q}) = 0$ for all values of α and β . Expand this expression as a polynomial in α, β , and consider the coefficients of $\alpha^d, \alpha^{d-2} \beta^2, \alpha^{d-3} \beta^3$; these must be 0, hence

$$\begin{aligned} p_4 p_0^{d-1} + p_5^d &= 0, \\ \binom{d-1}{2} p_4 p_0^{d-3} q_0^2 + \binom{d}{2} p_5^{d-2} q_5^2 &= 0, \\ \binom{d-1}{3} p_4 p_0^{d-4} q_0^3 + \binom{d}{3} p_5^{d-3} q_5^3 &= 0. \end{aligned}$$

20:12 Degree-Restricted Strength Decompositions and Algebraic Branching Programs

Rewrite these equations as

$$\begin{aligned} p_4 p_0^{d-1} &= -p_5^d, \\ (d-2)p_4 p_0^{d-3} q_0^2 &= -d p_5^{d-2} q_5^2, \\ (d-3)p_4 p_0^{d-4} q_0^3 &= -d p_5^{d-3} q_5^3. \end{aligned}$$

If $p_5 \neq 0$, then dividing the equations by $p_4 p_0^{d-1} = -p_5^d$, we obtain

$$\begin{aligned} (d-2) \left(\frac{q_0}{p_0} \right)^2 &= d \left(\frac{q_5}{p_5} \right)^2 \\ (d-3) \left(\frac{q_0}{p_0} \right)^3 &= d \left(\frac{q_5}{p_5} \right)^3 \end{aligned}$$

from which we have $q_0 = q_5 = 0$, which implies $q \in A_1$. If, on the other hand, $p_5 = 0$, then either $p_0 = 0$ and $p \in A_2$, or $p_4 = 0$ and $p \in A_0$. \triangleleft

► **Claim 20.2.** If $Q \cap A_0 \neq \emptyset$ and $Q \cap A_2 \neq \emptyset$, then Q contains $[e_0]$, $[e_2]$, or $[e_4]$.

Proof. Let p be a point in $Q \cap A_0$ and $q \in Q \cap A_2$, so $p = (p_0 : 0 : p_2 : 0 : 0 : 0)$ and $q = (0 : 0 : q_2 : 0 : q_4 : 0)$. If $p = q$, then they are equal to $[e_2]$. If $p \neq q$, then there is a point on the line that they span which has zero second coordinate. Let this linear combination be $r = (r_0 : 0 : 0 : 0 : r_4 : 0)$. Since $r \in Q \subset Z(S_{4,n})$, we have $S_{4,n}(r) = r_4 r_0^{d-1} = 0$, so r is either $[e_0]$ or $[e_4]$. \triangleleft

► **Claim 20.3.** Q contains one of the five points $[e_k]$.

Proof. Let $S = \{k \mid Q \cap \rho^k A_0 \neq \emptyset\}$. Since $\rho^5 = \text{id}$, we can see S as a subset of $\mathbb{Z}/5\mathbb{Z}$. Claim 20.1 implies that S contains at least one element of $\{0, 1, 2\}$. Because of the cyclic symmetry, an analogous statement is true for every three consecutive values in $\mathbb{Z}/5\mathbb{Z}$. Similarly, cyclically shifted versions of Claim 20.2 imply that if S contains k and $k+2$, then Q contains one of the five basis points.

Without loss of generality, $0 \in S$. If $2 \in S$ or $3 \in S$, then Claim 20.2 guarantees Q contains one of the five basis points. If both $2, 3 \notin S$, then Claim 20.1 applied to the consecutive triples $\{1, 2, 3\}$ and $\{2, 3, 4\}$ implies that $1, 4 \in S$. Since they differ by two, Claim 20.2 applies and we conclude. \triangleleft

By shifting Q cyclically we can assume that $[e_1] \in Q$.

► **Claim 20.4.** If $[e_1] \in Q$, then Q lies in the hyperplane $Z(x_0)$.

Proof. Note that if the line spanned by two points $[v], [w]$ lies in a hypersurface $Z(F)$, then $\sum \frac{\partial F}{\partial x_i}(v)w_j = 0$. Indeed, the function $f(t) = F(\alpha + tw)$ is identically 0 and so is its derivative in t at $t = 0$. By the chain rule, we obtain the desired equality. In particular, for every two points $p, q \in Q$, we obtain $\sum \frac{\partial S_{4,n}}{\partial x_i}(p)q_i = 0$. Fixing $p = [e_1]$, this guarantees $q_0 = 0$ for every $q \in Q$, showing $Q \subseteq Z(x_0)$. \triangleleft

We deduce that Q is contained in $Z(S_{4,n}) \cap Z(x_0) = Z(S_{4,n}, x_0) = Z(\hat{S}_{4,n}, x_0)$. Therefore $Q \subset Z(\hat{S}_{4,n})$ when regarded as a hypersurface in $Z(x_0) = \mathbb{P}^n$. By Proposition 9, this implies $\text{sr}(\hat{S}_{4,n}) \leq 2$, in contradiction with Lemma 19. This contradiction completes the proof. \blacktriangleleft

As a corollary we obtain a lower bound for homogeneous ABP size for Shioda's polynomials in six variables, which improves on Kumar's lower bound for a polynomial with the same number of variables.

► **Corollary 21.** $B_{\text{hom}}(S_{4,d}) \geq 3(d-1) + 2$.

Proof. We have $\text{str}_k(S_{4,n}) \geq 3$ from Proposition 6 and $\text{str}_{d-1} = \text{str}_1(S_{4,n}) = \text{sr}(S_{4,n}) = 4$ from Theorem 20. Using Theorem 12 we get the required lower bound. ◀

A similar (but easier) argument can be used to prove $\text{sr}(S_{2,d}) = 3$. It follows that $B_{\text{hom}}(S_{2,d}) \geq 2(d-1) + 2$. We conjecture that this can be generalized to all Shioda polynomials.

► **Conjecture 22.** For n even we have $\text{sr}(S_{n,d}) = \frac{n}{2} + 2$ and, consequently,

$$B_{\text{hom}}(S_{n,d}) \geq \frac{n+2}{2}(d-1) + 2.$$

4 Geometry of algebraic branching programs

We have seen that ABPs are related to degree-restricted strength decompositions, and degree-restricted strength decompositions are closely related to subvarieties of the hypersurface defined by the computed polynomial. One can also connect existence of ABPs to subvarieties directly.

► **Theorem 23.** Let F be a homogeneous polynomial of degree d . F is computed by a homogeneous ABP with w_k vertices in layer k if and only if there exists a chain of ideals

$$I_1 \supset I_2 \supset \cdots \supset I_{d-1} \supset I_d = \langle F \rangle$$

such that I_k is generated by w_k homogeneous polynomials of degree k .

Proof. Suppose a homogeneous ABP A computes the polynomial F . Recall that we denote the polynomial computed between vertices v and w by $A[v, w]$. Let s and t be the source and the sink of A , and let V_k be the set of vertices in the k -th layer.

Define I_k to be the ideal generated by polynomials $A[s, v]$ for all $v \in V_k$. These polynomials are homogeneous degree k polynomials, because every path from s to $v \in V_k$ has length k . If $w \in V_{k+1}$, then

$$A[s, w] = \sum_{v \in V_k} A[s, v]A[v, w], \tag{4}$$

so all generators of I_{k+1} lie in I_k and thus $I_k \supset I_{k+1}$. The last layer of A contains only the sink, and the corresponding ideal is $\langle F \rangle$.

On the other hand, given a sequence of ideals $I_1 \supset I_2 \supset \cdots \supset I_{d-1} \supset I_d = \langle F \rangle$ such that I_k is generated by homogeneous polynomials of degree k . Let G_{k1}, \dots, G_{kw_k} be the generators of I_k . Since $I_j \supset I_{j+1}$, we have

$$G_{k+1,j} = \sum_{i=1}^{w_k} G_{ki}L_{kij} \tag{5}$$

for some linear forms L_{kij} . Let A be an ABP with w_k vertices in layer k such that the edge from the i -th vertex in the k -th layer to the j -th vertex in the $(k+1)$ -th layer is labeled by L_{kij} . Then the equations (4) coincide with (5) and thus the ABP computes the generator F of the last ideal. ◀

Geometrically, this implies that $Z(F)$ contains a chain of subvarieties $X_1 \subset X_2 \subset \cdots \subset X_{d-1} \subset Z(F)$ where each X_k is cut out by w_k polynomials of degree k .

References

- 1 T. Ananyan and M. Hochster. Small subalgebras of polynomial rings and Stillman’s conjecture. *J. Amer. Math. Soc.*, 33(1):291–309, 2020.
- 2 M. F. Atiyah and I. G. Macdonald. *Introduction to Commutative Algebra*. Addison Wesley, 1969.
- 3 E. Ballico, A. Bik, A. Oneto, and E. Ventura. The set of forms with bounded strength is not closed. *arXiv*, 2020. [arXiv:2012.01237](https://arxiv.org/abs/2012.01237).
- 4 E. Ballico, A. Bik, A. Oneto, and E. Ventura. Strength and slice rank of forms are generically equal, 2021. [arXiv:2102.11549](https://arxiv.org/abs/2102.11549).
- 5 A. Bik. *Strength and noetherianity for infinite tensors*. PhD thesis, Universität Bern, 2020.
- 6 P. Bürgisser, M. Clausen, and M. A. Shokrollahi. *Algebraic complexity theory*, volume 315 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1997.
- 7 P. Bürgisser, F. Cucker, and P. Lairez. Rigid continuation paths II. Structured polynomial systems. *arXiv*, 2020. [arXiv:2010.10997](https://arxiv.org/abs/2010.10997).
- 8 P. Bürgisser, C. Ikenmeyer, and G. Panova. No occurrence obstructions in geometric complexity theory. *J. Amer. Math. Soc.*, 32(1):163–193, 2019.
- 9 E. Carlini, L. Chiantini, and A. V. Geramita. Complete intersections on general hypersurfaces. *Michigan Math. J.*, 57:121–136, 2008.
- 10 P. Chatterjee, M. Kumar, A. She, and B. L. Volk. A Quadratic Lower Bound for Algebraic Branching Programs. In *35th Computational Complexity Conference (CCC 2020)*, volume 169 of *LIPICs*, pages 2:1–2:21. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020.
- 11 H. Davenport and D. J. Lewis. Non-homogeneous cubic equations. *J. London Math. Soc.*, 39:657–671, 1964.
- 12 Jan Draisma. Topological Noetherianity of polynomial functors. *Journal of the American Mathematical Society*, 32(3):691–707, 2019.
- 13 D. Eisenbud. *Commutative Algebra: with a view toward algebraic geometry*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.
- 14 D. Eisenbud and J. Harris. *3264 and All That - A Second Course in Algebraic Geometry*. Cambridge University Press, Cambridge, 2016.
- 15 W. Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.
- 16 P. Griffiths and J. Harris. On the Noether-Lefschetz theorem and some remarks on codimension-two cycles. *Math. Ann.*, 271(1):31–51, 1985.
- 17 J. Harris. *Algebraic geometry. A first course*, volume 133 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1992.
- 18 C. Ikenmeyer and G. Panova. Rectangular Kronecker coefficients and plethysms in geometric complexity theory. *Adv. Math.*, 319:40–66, 2017.
- 19 M. Kumar. A quadratic lower bound for homogeneous algebraic branching programs. *Comput. Complex.*, 28(3):409–435, 2019.
- 20 M. Kumar and B. L. Volk. A Lower Bound on Determinantal Complexity. In *36th Computational Complexity Conference, CCC 2021, July 20-23, 2021, Toronto, Ontario, Canada (Virtual Conference)*, volume 200 of *LIPICs*, pages 4:1–4:12. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021.
- 21 N. Nisan. Lower bounds for non-commutative computation. In *Proceedings of the 23rd ACM Symp. on Th. of Comp.*, STOC’91, pages 410–418, New York, NY, USA, 1991. ACM.
- 22 W. F. Sawin and T. Tao. Notes on the “slice rank” of tensors, 2016. available at <https://terrytao.wordpress.com/2016/08/24/notes-on-the-slice-rank-of-tensors/>.
- 23 W. M. Schmidt. The density of integer points on homogeneous varieties. *Acta Math.*, 154(3-4):243–296, 1985.

- 24 T. Shioda. On the Picard number of a complex projective variety. *Ann. Sci. École Norm. Sup. (4)*, 14(3):303–321, 1981.
- 25 T. Shioda. Algebraic cycles on a certain hypersurface. In *Algebraic geometry (Tokyo/Kyoto, 1982)*, volume 1016 of *Lecture Notes in Math.*, pages 271–294. Springer, Berlin, 1983.
- 26 Leslie G. Valiant. Completeness Classes in Algebra. In *Proceedings of the 11th Annual ACM Symposium on Theory of Computing, April 30 - May 2, 1979, Atlanta, Georgia, USA*, pages 249–261. ACM, 1979.
- 27 X. Wu. On a conjecture of Griffiths-Harris generalizing the Noether-Lefschetz theorem. *Duke Math. J.*, 60(2):465–472, 1990.