Playing (Almost-)Optimally in Concurrent Büchi and Co-Büchi Games

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Abstract

We study two-player concurrent stochastic games on finite graphs, with Büchi and co-Büchi objectives. The goal of the first player is to maximize the probability of satisfying the given objective. Following Martin's determinacy theorem for Blackwell games, we know that such games have a value. Natural questions are then: does there exist an optimal strategy, that is, a strategy achieving the value of the game? what is the memory required for playing (almost-)optimally?

The situation is rather simple to describe for turn-based games, where positional pure strategies suffice to play optimally in games with parity objectives. Concurrency makes the situation intricate and heterogeneous. For most ω-regular objectives, there do indeed not exist optimal strategies in general. For some objectives (that we will mention), infinite memory might also be required for playing optimally or almost-optimally.

We also provide characterizations of local interactions of the players to ensure positionality of (almost-)optimal strategies for Büchi and co-Büchi objectives. This characterization relies on properties of game forms underpinning the formalism for defining local interactions of the two players. These well-behaved game forms are like elementary bricks which, when they behave well in isolation, can be assembled in graph games and ensure the good property for the whole game.

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1 Introduction

Stochastic concurrent games. Games on graphs are an intensively studied mathematical tool, with wide applicability in verification and in particular for the controller synthesis problem, see for instance [16, 1]. We consider two-player stochastic concurrent games played on finite graphs. For simplicity (but this is with no restriction), such a game is played over a finite bipartite graph called an arena: some states belong to Nature while others belong to the players. Nature is stochastic, and therefore assigns a probabilistic distribution over the players' states. In each players' state, a local interaction between the two players (called Player A and Player B) happens, specified by a two-dimensional table. Such an interaction is resolved as follows: Player A selects a probability distribution over the rows while Player B selects a probability distribution over the columns of the table; this results into a distribution over the cells of the table, each one pointing to a Nature state of the graph. An example of game arena (with no Nature states – we could add dummy deterministic Nature states) is given in Figure 1 (this example comes from [10]). At state q_0, the interaction between the two players is given by the table, and each player has two actions: if Player A plays the second row and Player B the first column, then the game proceeds to state $\top$: in that case, the game always goes back to q_0.

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Globally, the game proceeds as follows: starting at an initial state q_0, the two players play in the local interaction of the current state, and the joint choice determines (stochastically) the next Nature state of the game, itself moving randomly to players' states; the game then proceeds subsequently from the new players' state. The way players make choices is given by strategies, which, given the sequence of states visited so far (the so-called history), assign local strategies for the local interaction of the state the game is in. For application in controller synthesis, strategies will correspond to controllers, hence it is desirable to have strategies simple to implement. We will be in particular interested in strategies which are *positional*, i.e. strategies which only depend on the current state of the game, not on the whole history. When each player has fixed a strategy (say $\mathsf{s_A}$ for Player A and $\mathsf{s_B}$ for Player B), this defines a probability distribution $\mathbb{P}^{q_0}_{\mathsf{s_A},\mathsf{s_B}}$ over infinite sequences of states of the game. The objectives of the two players are opposite (we assume a zero-sum setting): together with the game, a measurable set W of infinite sequences of states is fixed; the objective of Player A is to maximize the probability of W while the objective of Player B is to minimize it.

Back to the example of Figure 1. If Player A (resp. B) plays the first row (resp. column) with probability p_A (resp. p_B), then the probability to move to $\bot$ in one step is $(1-p_\mathsf{A})\cdot(1-p_\mathsf{B})$. If Player A repeatedly plays the same strategy at q_0 with $p_\mathsf{A} < 1$, by playing $p_\mathsf{B} = 0$, Player B will enforce $\bot$ almost-surely; however, if Player A plays $p_\mathsf{A} = 1$, then by playing $p_\mathsf{B} = 1$, Player B enforces staying in q_0, hence visiting $\top$ with probability 0. On the contrary Player A can ensure visiting $\top$ infinitely often with probability $1 - \varepsilon$ for every $\varepsilon > 0$, by playing iteratively at q_0 the first row of the table with probability $1 - \varepsilon_k$ (k the number of visits to q_0) and the second row with probability ε_k, where the sequence $(\varepsilon_k)_k$ decreases fast to zero (see Appendix A).

Values and (almost-)optimal strategies. As mentioned above, Player A wants to maximize the probability of W, while Player B wants to minimize this probability. Formally, given a strategy $\mathsf{s_A}$ for Player A, its value is measured by $\inf_{\mathsf{s_B}} \mathbb{P}^{q_0}_{\mathsf{s_A},\mathsf{s_B}}(W)$, and Player A wants to maximize that value. Dually, given a strategy $\mathsf{s_B}$ for Player B, its value is measured by $\sup_{\mathsf{s_A}} \mathbb{P}^{q_0}_{\mathsf{s_A},\mathsf{s_B}}(W)$, and Player B wants to minimize that value. Following Martin's determinacy theorem for Blackwell games [14], it actually holds that the game has a *value* given by

$$\chi_{q_0} = \sup_{\mathsf{s_A}} \inf_{\mathsf{s_B}} \mathbb{P}^{q_0}_{\mathsf{s_A},\mathsf{s_B}}(W) = \inf_{\mathsf{s_B}} \sup_{\mathsf{s_A}} \mathbb{P}^{q_0}_{\mathsf{s_A},\mathsf{s_B}}(W)$$

While this ensures the existence of almost-optimal strategies (that is, ε-optimal strategies for every $\varepsilon > 0$) for both players, it says nothing about the existence of optimal strategies, which are strategies achieving χ_{q_0}. In general, except for safety objectives, optimal strategies may not exist, as witnessed by the example of Figure 1, which uses a Büchi condition. Also, it says nothing about the complexity of optimal strategies (when they exist) and ε-optimal strategies. Complexity of a strategy is measured in terms of memory that is used by the strategy: while general strategies may depend on the whole history of the game, a *positional* strategy only depends on the current state of the game; a *finite-memory* strategy records a finite amount of information using a finite automaton; the most complex ones, the *infinite-memory* strategies require more than a finite automaton to record information necessary to take decisions.

Back to the game of Figure 1, assuming the Büchi condition "visit $\top$ infinitely often", the game is such that $\chi_{q_0} = 1$. However Player A has no optimal strategy, and can only achieve $1 - \varepsilon$ for every $\varepsilon > 0$ with an infinite-memory strategy, and any positional strategy has value 0.

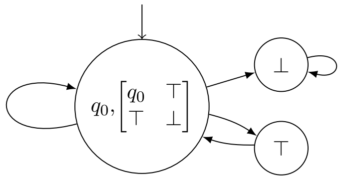

Figure 1 A concurrent game with objective Büchi($\{\top\}$).

$$\mathcal{F}_{q_0} = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$$

Figure 2 The local interaction at state q_0.

$$\langle \mathcal{F}_{q_0}, \mu_v \rangle = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Figure 3 The local interaction at state q_0 valued by the vector μ_v giving the value of states.

The contributions of this work. We are interested in memory requirements for optimal and ε-optimal strategies in concurrent games with parity objectives. The situation is rather heterogeneous: while safety objectives enjoy very robust properties (existence of positional optimal strategies in all cases, see for instance [11, Thm. 1]), the situation appears as much more complex for parity objectives (already with three colors): there may not exist optimal strategies, and when they exist, optimal strategies as well as ε-optimal strategies require in general infinite memory (the case of optimal strategies was proven in [10] while the case of ε-optimal strategies is a consequence of the Büchi case, studied in [11, Thm. 2]). The case of reachability objectives was studied with details in [2]: optimal strategies may not exist, but there exists a positional strategy that is 1) optimal from each state from where there exists an optimal strategy, and 2) ε-optimal from the other states.

In this paper, we focus on Büchi and co-Büchi objectives. Few things were known for those games: specifically, it was shown in [11, Thm. 2] that Büchi and co-Büchi concurrent games may have no optimal strategies, and that ε-optimal strategies may require infinite memory for Büchi objectives. We show in addition that, when optimal strategies exist everywhere, optimal strategies can be chosen positional for Büchi objectives but may require infinite memory for co-Büchi objectives. We more importantly characterize “well-behaved” local interactions (i.e. interactions of the two players at each state, which are given by tables) for ensuring positionality of (ε-)optimal strategies in the various settings where they do not exist in the general case. We follow the approach used in [2] for reachability objectives and abstract those local interactions into game forms, where cells of the table are now seen as variables (some of them being equal). For instance, the game form associated with state q_0 in the running example has three outcomes: x, y and z, and it is given in Figure 2. Game forms can be seen as elementary bricks that can be used to build games on graphs. Given a property we want to hold on concurrent games (e.g. the existence of positional optimal strategies in Büchi games), we characterize those bricks that are safe w.r.t. that property, that is the game forms that behave well when used individually, the ones ensuring that, when they are the only non-trivial local interaction in a concurrent game, the property holds. Then, we realize that they also behave well when used collectively: if all local interactions are safe in a concurrent game, then the whole game ensures the property of interest. We obtain a clear-cut separation: if all local interactions are safe in a concurrent game, then the property is necessarily ensured; on the other hand, if a game form is not safe, one can build a game where it is the only non-trivial local interaction that does ensure the property. Our contributions can be summarized as follows:

1. In the general setting of prefix-independent objectives, we characterize positional uniformly optimal strategies using locally optimal strategies (i.e. strategies which are optimal at local interactions) and constraints on values of end-components generated by the strategies (Lemma 16).

2. We study Büchi concurrent games. We first show that there is a positional uniformly optimal strategy in a Büchi game as soon as it is known that optimal strategies exist from every state (Proposition 17). To benefit from this result, we give a (sufficient and necessary) condition on (game forms describing) local interactions to ensure the existence of optimal strategies. In particular, if all local interactions are well-behaved (w.r.t. the condition), then it will be the case that positional uniformly optimal strategies exist (Theorem 19). We also give a weaker necessary and sufficient condition on local interactions to ensure the existence of positional ε-optimal strategies in Büchi games, since in general, infinite memory might be required (Theorem 22).
3. We study co-Büchi concurrent games. We first show that optimal strategies might require infinite memory in co-Büchi games,in contrast with Büchi games (Subsection 6.1). We also characterize local interactions that ensure the existence of positional optimal strategies in co-Büchi games (Theorem 25). It is not useful to do the same for positional ε-optimal strategies since it is always the case that such strategies exist [5].

Additional details and proofs can be found in the arXiv version of this paper [3].

All these results show the contrast between concurrent games and turn-based games: indeed, in the latter (which have attracted more attention these last years), pure (that is, deterministic) positional optimal strategies always exist for parity objectives (hence in particular, for Büchi and co-Büchi objectives) [15, 7, 18]. The results presented in this paper hence show the complexity inherent to concurrent interactions in games. Those have nevertheless retained some attention over the last twenty years [10, 5, 9, 6, 12], and are relevant in applications [13].

2 Game Forms

A *discrete probabilisty distribution* over a non-empty finite set Q is a function $\mu : Q \to [0, 1]$ such that $\sum_{x \in Q} \mu(x) = 1$. The *support* $\mathsf{Supp}(\mu)$ of a probabilistic distribution $\mu : Q \to [0, 1]$ corresponds to the set of non-zeros of the distribution: $\mathsf{Supp}(\mu) = \{q \in Q \mid \mu(q) \in]0, 1]\}$. The set of all distributions over the set Q is denoted $\mathcal{D}(Q)$.

Informally, game forms are two-dimensional tables with variables while games in normal forms are game forms whose outcomes are real values in $[0, 1]$. Formally:

▶ **Definition 1** (Game form and game in normal form). *A* game form *(GF for short) is a tuple $\mathcal{F} = \langle \mathsf{St_A}, \mathsf{St_B}, \mathsf{O}, \varrho \rangle$ where $\mathsf{St_A}$ (resp. $\mathsf{St_B}$) is a non-empty finite set of actions available to Player A (resp. B), O is a non-empty set of outcomes, and $\varrho : \mathsf{St_A} \times \mathsf{St_B} \to \mathsf{O}$ is a function that associates an outcome to each pair of actions. When the set of outcomes O is equal to $[0, 1]$, we say that $\mathcal{F}$ is a* game in normal form*. For a valuation $v \in [0, 1]^{\mathsf{O}}$ of the outcomes, the notation $\langle \mathcal{F}, v \rangle$ refers to the game in normal form $\langle \mathsf{St_A}, \mathsf{St_B}, [0, 1], v \circ \varrho \rangle$.*

An example of game form (resp. game in normal form) is given in Figure 2 (resp. 3), where $\mathsf{St_A}$ (resp. $\mathsf{St_B}$) are rows (resp. columns) of the table and x, y, z (resp. $0, 1$) are the possible outcomes of the game form (resp. game in normal form). We use game forms to represent interactions between two players. The strategies available to Player A (resp. B) are convex combinations of actions given as the rows (resp. columns) of the table. In a game in normal form, Player A tries to maximize the outcome, whereas Player B tries to minimize it.

▶ **Definition 2** (Outcome of a game in normal form). *Let $\mathcal{F} = \langle \mathsf{St_A}, \mathsf{St_B}, [0, 1], \varrho \rangle$ be a game in normal form. The set $\mathcal{D}(\mathsf{St_A})$ (resp. $\mathcal{D}(\mathsf{St_B})$) is the set of (mixed) strategies available to Player A (resp. B). For a pair of strategies $(\sigma_\mathsf{A}, \sigma_\mathsf{B}) \in \mathcal{D}(\mathsf{St_A}) \times \mathcal{D}(\mathsf{St_B})$, the outcome $\mathsf{out}_\mathcal{F}(\sigma_\mathsf{A}, \sigma_\mathsf{B})$ in $\mathcal{F}$ of the strategies $(\sigma_\mathsf{A}, \sigma_\mathsf{B})$ is $\mathsf{out}_\mathcal{F}(\sigma_\mathsf{A}, \sigma_\mathsf{B}) := \sum_{a \in \mathsf{St_A}} \sum_{b \in \mathsf{St_B}} \sigma_\mathsf{A}(a) \cdot \sigma_\mathsf{B}(b) \cdot \varrho(a, b) \in [0, 1]$.*

▶ **Definition 3** (Value of a game in normal form and optimal strategies). *Let* $\mathcal{F} = \langle \mathsf{St}_\mathsf{A}, \mathsf{St}_\mathsf{B}, [0,1], \varrho \rangle$ *be a game in normal form, and* $\sigma_\mathsf{A} \in \mathcal{D}(\mathsf{St}_\mathsf{A})$ *be a strategy for Player* A*. The* value *of strategy* σ_A *is* $\mathsf{val}_\mathcal{F}(\sigma_\mathsf{A}) := \inf_{\sigma_\mathsf{B} \in \mathcal{D}(\mathsf{St}_\mathsf{B})} \mathsf{out}_\mathcal{F}(\sigma_\mathsf{A}, \sigma_\mathsf{B})$*, and analogously for Player* B*, with a* sup *instead of an* inf*. When* $\sup_{\sigma_\mathsf{A} \in \mathcal{D}(\mathsf{St}_\mathsf{A})} \mathsf{val}_\mathcal{F}(\sigma_\mathsf{A}) = \inf_{\sigma_\mathsf{B} \in \mathcal{D}(\mathsf{St}_\mathsf{B})} \mathsf{val}_\mathcal{F}(\sigma_\mathsf{B})$*, it defines the* value *of the game* $\mathcal{F}$*, denoted* $\mathsf{val}_\mathcal{F}$*.*

A strategy $\sigma_\mathsf{A} \in \mathcal{D}(\mathsf{St}_\mathsf{A})$ *ensuring* $\mathsf{val}_\mathcal{F} = \mathsf{val}_\mathcal{F}(\sigma_\mathsf{A})$ *is said to be* optimal*. The set of all optimal strategies for Player* A *is denoted* $\mathsf{Opt}_\mathsf{A}(\mathcal{F}) \subseteq \mathcal{D}(\mathsf{St}_\mathsf{A})$*, and analogously for Player* B*. Von Neumann's minimax theorem [17] ensures the existence of optimal strategies (for both players).*

In the following, strategies in games in normal forms will be called GF-strategies, in order not to confuse them with strategies in concurrent (graph) games.

3 Concurrent Stochastic Games

We introduce the definition of a concurrent arena played on a finite graph, and of a concurrent game by adding a winning condition.

▶ **Definition 4** (Finite stochastic concurrent arena and game). *A* concurrent arena $\mathcal{C}$ *is a tuple* $\langle Q, A, B, \mathsf{D}, \delta, \mathsf{dist} \rangle$ *where* Q *is a non-empty set of states,* A *(resp.* B*) is the non-empty finite set of actions available to Player* A *(resp.* B*),* D *is the set of Nature states,* $\delta : Q \times A \times B \to \mathsf{D}$ *is the transition function and* $\mathsf{dist} : \mathsf{D} \to \mathcal{D}(Q)$ *is the distribution function.*

A concurrent game *is a pair* $\langle \mathcal{C}, W \rangle$ *where* $\mathcal{C}$ *is a concurrent arena and* $W \subseteq Q^\omega$ *is Borel. The set* W *is called the* (winning) objective *and it corresponds to the set of paths winning for Player* A *and losing for Player* B*.*

For the rest of the section, we fix a concurrent game $\mathcal{G} = \langle \mathcal{C}, W \rangle$. An important class of objectives are the *prefix-independent* objectives, that is objectives W such that an infinite path is in W if and only if one of its suffixes is in W: for all $\rho \in Q^\omega$ and $\pi \in Q^+$, $\rho \in W \Leftrightarrow \pi \cdot \rho \in W$. In this paper, we more specifically focus on Büchi and co-Büchi objectives, which informally correspond to the infinite paths seeing infinitely often, or finitely often, a given set of states. We also recall the definitions of the reachability and safety objectives.

▶ **Definition 5** (Büchi and co-Büchi, Reachability and Safety objectives). *Consider a target set of states* $T \subseteq Q$*. We define the following objectives:*

- $\mathsf{Büchi}(T) := \{\rho \in Q^\omega \mid \forall i \in \mathbb{N}, \exists j \geq i,\ \rho_j \in T\}$; $\mathsf{Reach}(T) := \{\rho \in Q^\omega \mid \exists i \in \mathbb{N}, \rho_i \in T\}$;
- $\mathsf{coBüchi}(T) := \{\rho \in Q^\omega \mid \exists i \in \mathbb{N},\ \forall j \geq i,\ \rho_j \notin T\}$; $\mathsf{Safe}(T) := \{\rho \in Q^\omega \mid \forall i \in \mathbb{N}, \rho_i \notin T\}$.

In concurrent games, game forms appear at each state and specify how the interaction of the Players determines the next (Nature) state. In fact, this corresponds to the local interactions of the game.

▶ **Definition 6** (Local interactions). *The* local interaction *at state* $q \in Q$ *is the game form* $\mathcal{F}_q = \langle A, B, \mathsf{D}, \delta(q, \cdot, \cdot) \rangle$*. That is, the* GF*-strategies available for Player* A *(resp.* B*) are the actions in* A *(resp.* B*) and the outcomes are the Nature states.*

As an example, the local interaction at state q_0 in Figure 1 is represented in Figure 2, up to a renaming of the outcomes.

A strategy of a given Player then associates to every history (i.e. every finite sequence of states) a GF-strategy in the local interaction at the current state, in other words it associates a distribution on the actions available at the local interaction to the given Player.

▶ **Definition 7** (Strategies). *A* strategy *for Player* A *is a function* $\mathsf{s_A} : Q^+ \to \mathcal{D}(A)$ *such that, for all* $\rho = q_0 \cdots q_n \in Q^+$, $\mathsf{s_A}(\rho) \in \mathcal{D}(A)$ *is a* GF*-strategy for Player* A *in the game form* $\mathcal{F}_{q_n}$*. A strategy* $\mathsf{s_A} : Q^+ \to \mathcal{D}(A)$ *for Player* A *is* positional *if, for all* $\pi = \rho \cdot q \in Q^+$ *and* $\pi' = \rho' \cdot q' \in Q^+$*, if* $q = q'$*, then* $\mathsf{s_A}(\pi) = \mathsf{s_A}(\pi')$ *(that is, the strategy only depends on the current state of the game). We denote by* $\mathsf{S}^{\mathsf{A}}_{\mathcal{C}}$ *and* $\mathsf{PS}^{\mathsf{A}}_{\mathcal{C}}$ *the set of all strategies and positional strategies in arena* $\mathcal{C}$ *for Player* A*. The definitions are analogous for Player* B*.*

Before defining the outcome of the game given a strategy for a Player, we define the probability to go from state q to state q', given two GF-strategies in the game form $\mathcal{F}_q$.

▶ **Definition 8** (Probability Transition). *Let* $q \in Q$ *be a state and* $(\sigma_{\mathsf{A}}, \sigma_{\mathsf{B}}) \in \mathcal{D}(A) \times \mathcal{D}(B)$ *be two* GF*-strategies in the game form* $\mathcal{F}_q$*. For a state* $q' \in Q$*, the probability to go from* q *to* q' *if the players play* σ_{A} *and* σ_{B} *in* q *is equal to* $\mathbb{P}_{\sigma_{\mathsf{A}},\sigma_{\mathsf{B}}}(q, q') := \mathsf{out}_{\langle \mathcal{F}_q, \mathsf{dist}(\cdot)(q') \rangle}(\sigma_{\mathsf{A}}, \sigma_{\mathsf{B}})$*.*

From this, given two strategies, we deduce the probability of any cylinder supported by a finite path, and consequently of any Borel set in Q^ω. From a state $q_0 \in Q$, given two strategies $\mathsf{s_A}$ and $\mathsf{s_B}$, this probability distribution is denoted $\mathbb{P}^{\mathcal{C},q_0}_{\mathsf{s_A},\mathsf{s_B}} : \mathsf{Borel}(Q) \to [0, 1]$. Let us now define the value of a strategy and of the game.

▶ **Definition 9** (Value of strategies and of the game). *Let* $\mathsf{s_A} \in \mathsf{S}^{\mathsf{A}}_{\mathcal{C}}$ *be a Player* A *strategy. The* value *of* $\mathsf{s_A}$ *is the function* $\chi_{\mathcal{G}}[\mathsf{s_A}] : Q \to [0, 1]$ *such that for every* $q \in Q$, $\chi_{\mathcal{G}}[\mathsf{s_A}](q) := \inf_{\mathsf{s_B} \in \mathsf{S}^{\mathsf{B}}_{\mathcal{C}}} \mathbb{P}^{\mathcal{C},q}_{\mathsf{s_A},\mathsf{s_B}}[W]$*.*

The value *for Player* A *is the function* $\chi_{\mathcal{G}}[\mathsf{A}] : Q \to [0, 1]$ *such that for all* $q \in Q$*, we have* $\chi_{\mathcal{G}}[\mathsf{A}](q) := \sup_{\mathsf{s_A} \in \mathsf{S}^{\mathsf{A}}_{\mathcal{C}}} \chi_{\mathcal{G}}[\mathsf{s_A}](q)$*. The value for Player* B *is defined similarly by reversing the supremum and infimum.*

By Martin's result on the determinacy of Blackwell games [14], for all concurrent games $\mathcal{G} = \langle \mathcal{C}, W \rangle$*, the values for both Players are equal, which defines the* value *of the game:* $\chi_{\mathcal{G}} := \chi_{\mathcal{G}}[\mathsf{A}] = \chi_{\mathcal{G}}[\mathsf{B}]$*. Furthermore, a strategy* $\mathsf{s_A} \in \mathsf{S}^{\mathsf{A}}_{\mathcal{C}}$ *which ensures* $\chi_{\mathcal{G}}[\mathsf{s_A}](q) = \chi_{\mathcal{G}}(q)$ *from some state* $q \in Q$*, is said to be* optimal *from* q*. If, in addition it ensures* $\chi_{\mathcal{G}}[\mathsf{s_A}] = \chi_{\mathcal{G}}$*, it is said to be* uniformly optimal*.*

We mention a very useful result of [4]. Informally, it shows that if there is a state with a positive value, then there is a state with value 1.

▶ **Theorem 10** (Theorem 1 in [4]). *Let* $\mathcal{G}$ *be a concurrent game with a prefix-independent objective. If there is a state* $q \in Q$ *such that* $\chi_{\mathcal{G}}(q) > 0$ *(resp.* $\chi_{\mathcal{G}}(q) < 1$*), then there is a state* $q' \in Q$ *such that* $\chi_{\mathcal{G}}(q') = 1$ *(resp.* $\chi_{\mathcal{G}}(q') = 0$*).*

Finally, we define the Markov decision process which is induced by a positional strategy, and its end-components.

▶ **Definition 11** (Induced Markov decision process). *Let* $\mathsf{s_A} \in \mathsf{PS}^{\mathsf{A}}_{\mathcal{C}}$ *be a positional strategy. The* Markov decision process Γ *(MDP for short) induced by the strategy* $\mathsf{s_A}$ *is the triplet* $\Gamma := \langle Q, B, \iota \rangle$ *where* Q *is the set of states,* B *is the set of actions and* $\iota : Q \times B \to \mathcal{D}(Q)$ *is a map associating to a state and an action a distribution over the states. For all* $q \in Q$, $b \in B$ *and* $q' \in Q$*, we set* $\iota(q, b)(q') := \mathbb{P}^{\mathsf{s_A}(q),b}_{q,q'}$*.*

The induced MDP Γ is a special case of a concurrent game where Player A does not play (A could be a singleton) and the set of Player B strategies is the same as in $\mathcal{C}$. The useful objects in MDPs are the end components [8], i.e. sub-MDPs that are strongly connected.

▶ **Definition 12** (End component and sub-game). *Let* $\mathsf{s_A} \in \mathsf{PS}^{\mathsf{A}}_{\mathcal{C}}$ *be a positional strategy, and* Γ *its induced MDP. An* end component *(EC for short)* H *in* Γ *is a pair* (Q_H, β) *such that* $Q_H \subseteq Q$ *is a subset of states and* $\beta : Q_H \to \mathcal{P}(B) \setminus \emptyset$ *associates to each state a non-empty set of actions compatible with the EC* H *such that:*

- *for all $q \in Q_H$ and $b \in \beta(q)$, $\mathsf{Supp}(\iota(q,b)) \subseteq Q_H$;*
- *the underlying graph (Q_H, E) is strongly connected, where $(q,q') \in E$ if and only if there is $b \in \beta(q)$ such that $q' \in \mathsf{Supp}(\iota(q,b))$.*

We denote by $\mathsf{D}_H \subseteq \mathsf{D}$ the set of Nature states compatible with the EC H: $\mathsf{D}_H = \{d \in \mathsf{D} \mid \mathsf{Supp}(d) \subseteq Q_H\}$. Note that, for all $q \in Q_H$ and $b \in \beta(q)$, we have $\delta(q, \mathsf{Supp}(\mathsf{s_A}(q)), b) \subseteq \mathsf{D}_H$.

The end component H can be seen as a concurrent arena. In that case, it is denoted $\mathcal{C}_H^{\mathsf{s_A}}$.

4 Uniform Optimality with Positional Strategies

In this section, we present a necessary and sufficient condition for a positional strategy to be uniformly optimal. Note that this result holds for any prefix-independent objective for which, in any finite MDP with the complement objective $Q^\omega \setminus W$, there is a positional optimal strategy. This is in particular the case for the Büchi and co-Büchi objectives (in fact, it holds for all parity objectives).

We assume a concurrent game $\mathcal{G} = \langle \mathcal{C}, W \rangle$ is given for this section, that W is Borel and prefix-independent, and that $v := \chi_{\mathcal{G}} \in [0,1]^Q$ is the value (function) of the game. We first define the crucial notion of *local optimality*: informally, a Player A positional strategy is locally optimal if, at each local interaction, it is optimal in the game in normal form induced by the values of the game. As the outcomes of a local interaction are the Nature states, we have to be able to lift the valuation v of the states into a valuation of the Nature states. This is done via a convex combination in the definition below.

▶ **Definition 13** (Lifting a valuation of the states). *Let $w : Q \to [0,1]$ be an arbitrary valuation of the states. We define $\mu_w : \mathsf{D} \to [0,1]$, the* lift *of the valuation w to Nature states in the following way: $\mu_w(d) := \sum_{q \in Q} \mathsf{dist}(d)(q) \cdot w(q)$ for all $d \in \mathsf{D}$.*

We can now define the local optimality of a positional strategy.

▶ **Definition 14** (Local optimality). *A Player A positional strategy $\mathsf{s_A} \in \mathsf{PS}_{\mathcal{C}}^{\mathsf{A}}$ is* locally optimal *if for all $q \in Q$: $\mathsf{val}_{\langle \mathcal{F}_q, \mu_v \rangle}(\mathsf{s_A}(q)) = v(q)$ (i.e. the GF-strategy $\mathsf{s_A}(q)$ is optimal in $\langle \mathcal{F}_q, \mu_v \rangle$).*

Interestingly, in the MDP induced by a locally optimal strategy, all the states in a given EC have the same value (w.r.t. the valuation v). This is stated in the proposition below.

▶ **Proposition 15** (Proposition 18 in [2]). *For every locally optimal Player A positional strategy $\mathsf{s_A} \in \mathsf{PS}_{\mathcal{C}}^{\mathsf{A}}$, for all EC $H = (Q_H, \beta)$ in the MDP induced by the strategy $\mathsf{s_A}$, there exists a value $v_H \in [0,1]$ such that, for all $q \in Q_H$, we have $v(q) = v_H$.*

We now discuss how local optimality relates to uniform optimality. We first observe that local optimality is necessary for uniform optimality, and we illustrate this on the example of Figure 4: in this (partly depicted) game, Nature states are omitted, and values w.r.t. the valuation v are written in red close to the states. For instance, if Player A plays the top row and Player B the left column, the state q_0 is reached. The local interaction $\mathcal{F}_{q_0}$ at state q_0, valued with the lift μ_v is then depicted in Figure 5. Assume $\mathsf{s_A}$ is a Player A positional strategy that is not locally optimal at q_0: the convex combination of the values in the second column (i.e. the values of the states q_1, q_2) is less than $1/2$, i.e. $p \cdot 1/4 + (1-p) \cdot 3/4 = 1/2 - \varepsilon < 1/2$, where $p \in [0,1]$ is the probability chosen by $\mathsf{s_A}(q_0)$ to play the top row. A Player B strategy $\mathsf{s_B}$ whose values at states q_1, q_2 are $(\varepsilon/2)$-close to $v(q_1), v(q_2)$ and that plays the second row with probability 1 at q_0, ensures that the value of the game w.r.t. $\mathsf{s_A}, \mathsf{s_B}$ is at most $p \cdot (1/4 + \varepsilon/2) + (1-p) \cdot (3/4 + \varepsilon/2) \le 1/2 - \varepsilon/2 < 1/2 = v(q_0)$. Thus, the strategy $\mathsf{s_A}$ is not optimal from q_0.

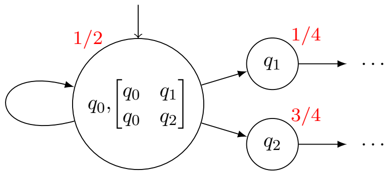

Figure 4 A concurrent game where the values of the states are depicted near them in red.

$$\langle \mathcal{F}_{q_0}, \mu_v \rangle = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{bmatrix}$$

Figure 5 The local interaction at state q_0 valued by the function v giving the value of states.

However, local optimality is not a sufficient condition for uniform optimality, as it can be seen in Figure 1. A game with a Büchi objective $\mathsf{Büchi}(\{\top\})$ is depicted (Player A wins if the state $\top$ is seen infinitely often). In this game, the valued local interaction at state q_0 is also depicted in Figure 3. A Player A locally optimal strategy plays the top row with probability 1 at state q_0. If such a strategy is played, Player B can ensure the game never to leave the state q_0 (by playing the left column), thus ensuring value 0 from q_0 (with $v(q_0) = 1$). In fact, in this game, there is no Player A optimal strategy.

Overall, from a state $q \in Q$, the local optimality of a Player A positional strategy $\mathsf{s_A}$ ensures that, for every Player B strategy, the convex combination of the values v_H of the ECs H, weighted by the probability to reach them from q, is at least the value of the state q. However, this does not guarantee anything concerning the $\chi_\mathcal{G}[\mathsf{s_A}]$-value of the game if it never leaves a specific EC H, it may be 0 whereas $v_H > 0$ (that is what happens in the game of Figure 1). For the strategy $\mathsf{s_A}$ to be uniformly optimal, it must ensure that the value under $\mathsf{s_A}$ of all states q in the EC H is at least v_H (i.e. for all Player B strategies $\mathsf{s_B}$ which ensure staying in H, the value under $\mathsf{s_A}$ and $\mathsf{s_B}$ is at least v_H). Furthermore, by Theorem 10 applied to H, as soon as at least one state has a value smaller than 1, there is one state with value 0. It follows that, for the strategy $\mathsf{s_A}$ to be uniformly optimal, it must be the case that in every EC H such that $v_H > 0$, in the game restricted to H, the value of the game is 1. Note that this exactly corresponds to the condition the authors of [2] have stated in the case of a reachability objective. Uniform optimality can finally be characterized as follows.

▶ **Lemma 16.** *Assume that W is Borel and prefix-independent, and that in all finite MDPs with objective $Q^\omega \setminus W$, there is a positional optimal strategy. Let $\mathsf{s_A} \in \mathsf{PS}^{\mathsf{A}}_{\mathcal{C}}$ be a positional Player A strategy. It is uniformly optimal if and only if:*

- *it is locally optimal;*
- *for all ECs H in the MDP induced by the strategy $\mathsf{s_A}$, if $v_H > 0$, then for all $q \in Q_H$, we have $\chi_{\mathcal{C}_H^{\mathsf{s_A}}}(q) = 1$ (i.e. in the sub-game formed by the EC H, the value of state q is 1).*

Note that what is proved in the [3] is slightly more general than Lemma 16 as it deals with an arbitrary valuation of the states, not only the one giving the value of the game. This generalization will be used in particular to prove that a positional strategy is ε-optimal from all states in Subsection 5.3.

5 Playing Optimally with Positional Strategies in Büchi Games

In this section, we focus on Büchi objectives. We will distinguish several frameworks. First we consider the case when optimal strategies exist from every state and show that in that case, there is a positional strategy that is uniformly optimal. Furthermore, we show that this always occurs as soon as all local interactions at states not in the target are *reach-maximizable*, the condition ensuring the existence of optimal strategies in reachability games [2]. Finally, we study the game forms necessary and sufficient to ensure the existence of positional almost-optimal strategies (i.e. ε-optimal strategies, for all $\varepsilon > 0$).

5.1 Playing optimally when optimal strategies exist from every state

First we observe that the value of a Büchi game is closely related to the value of a well-chosen reachability game. We let $\mathcal{G} = \langle \mathcal{C}, \mathsf{Büchi}(T) \rangle$ be a concurrent Büchi game. We modify $\mathcal{G}$ by replacing every state q in the target T, whose value in the Büchi game is $u := \chi_{\mathcal{G}}(q)$ by a state that has probability u to go to $\top$ (the new target in the reachability game) and probability $1 - u$ to go to a sink non-target state $\bot$. In that case, the value of the original Büchi game is the same as the value of the obtained reachability game, which we denote $\mathcal{G}_{\mathsf{reach}} := \langle \mathcal{C}_{\mathsf{reach}}, \mathsf{Reach}(\{\top\}) \rangle$. It is straightforward to show that the values of both games are the same from all states, and that optimal strategies in the Büchi game will be also optimal in the reachability game. Vice-versa, optimal strategies in the reachability game can be lifted to the Büchi game by augmenting it with a locally optimal strategy at target states.

▶ **Proposition 17.** *For all Büchi games in which an optimal strategy exists from every state, there exists a Player* A *positional strategy that is uniformly optimal.*

5.2 Game forms ensuring the existence of optimal strategies

Proposition 17 assumes the existence of optimal strategies from every state. However, there exist Büchi games with no optimal strategies, and even for which almost-optimal strategies require infinite memory. An example of such a game is depicted in Figure 1 (some details will be given in Subsection 5.3). This justifies the interest of having Büchi games in which optimal strategies exist from every state "by design".

In [2], the authors have proven a necessary and sufficient condition on game forms to ensure the existence of optimal strategies in all finite reachability games using these game forms as local interactions. This condition, called *reach-maximizable* game forms (RM for short) is not detailed here but is available in the [3]. This formalizes as follows:

▶ **Theorem 18** (Lem 33 and Thm 36 in [2])**.** *In all reachability games* $\mathcal{G} = \langle \mathcal{C}, \mathsf{Reach}(T) \rangle$ *where all interactions at states in* $Q \setminus T$ *are RM, there exist positional uniformly optimal strategies (for Player* A*). Furthermore, if a game form* $\mathcal{F}$ *is not RM, one can build a reachability game where* $\mathcal{F}$ *is the only non-trivial interaction, in which there is no optimal strategy for Player* A.

In the previous subsection, we have described how to translate a Büchi game $\mathcal{G} = \langle \mathcal{C}, \mathsf{Büchi}(T) \rangle$ into a reachability game $\mathcal{G}_{\mathsf{reach}} := \langle \mathcal{C}_{\mathsf{reach}}, \mathsf{Reach}(\{\top\}) \rangle$ while keeping the same value and the same local interactions in states outside the target T; furthermore, the local interactions in all states in T in $\mathcal{G}_{\mathsf{reach}}$ become trivial (i.e. the outcome is independent of the actions of the players), which are RM. Hence, if all local interactions at states in $Q \setminus T$ in the original game $\mathcal{G}$ are RM, then all game forms in $\mathcal{G}_{\mathsf{reach}}$ are RM, thus there is a uniformly optimal positional strategy for Player A in that game by Theorem 18. Such a strategy can then be translated back into the Büchi game to get a positional Player A uniformly optimal strategy. We obtain the following result.

▶ **Theorem 19.** *In all Büchi games* $\mathcal{G} = \langle \mathcal{C}, \mathsf{Büchi}(T) \rangle$ *where all interactions at states in* $Q \setminus T$ *are RM, there exist positional uniformly optimal strategies (for Player* A*). Furthermore, if a game form* $\mathcal{F}$ *is not RM, one can build a Büchi game where* $\mathcal{F}$ *is the only non-trivial interaction, in which there is no optimal strategy for Player* A.

$$\mathcal{F}_{q_0} = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$$

Figure 6 The game form used as local interaction at q_0.

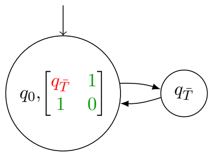

Figure 7 The Büchi game $\mathcal{G}_1$.

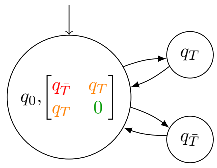

Figure 8 The Büchi game $\mathcal{G}_2$.

5.3 GFs ensuring the existence of positional almost-optimal strategies

While positional strategies are sufficient to play optimally in Büchi games where optimal strategies do actually exist, the case is really bad for those Büchi games where optimal strategies do not exist. Indeed, it can be the case that infinite memory is necessary to play almost-optimally, as we illustrate in the next paragraph. Then we characterize game forms that ensure the existence of positional almost-optimal strategies.

An example of a Büchi game where playing almost-optimally requires infinite memory. Consider the Büchi game $\mathcal{G} = \langle \mathcal{C}, \mathsf{Büchi}(\top) \rangle$ in Figure 1. As argued earlier (in Section 4), there are no optimal strategies in that game. First, notice that the value of the game at state q_0 is 1. However, any finite-memory strategy (i.e. a strategy that can be described with a finite automaton) has value 0. Indeed, consider such a Player A strategy $\mathsf{s_A}$. There is some probability $p > 0$ such that if $\mathsf{s_A}$ plays an action with a positive probability, this probability is at least p. If the strategy $\mathsf{s_A}$ plays, at state q_0 the bottom row with positive probability and if Player B plays the right column with probability 1, then state $\perp$ is reached with probability at least p. If this happens infinitely often, then the state $\perp$ is eventually reached almost-surely: the value of the strategy is then 0. Hence, Player A has to play, from some time on, the top row with probability 1. From that time on, Player B can play the left column with probability 1, leading to avoid state $\top$ almost-surely. Hence the value of strategy $\mathsf{s_A}$ is 0 as well. In fact, all ε-optimal strategies (for every $\varepsilon > 0$) cannot be finite-memory strategies. A Player A strategy with value at least $1 - \varepsilon$ (with $0 < \varepsilon < 1$) will have to play the bottom row with positive probability infinitely often, but that probability will have to decrease arbitrarily close to 0. More details are given in [3].

Game forms ensuring the existence of positional ε-optimal strategies. forms which ensure the existence of positional almost-optimal strategies in Büchi games. The approach is inspired by the one developed in [2] for reachability games.

We start by discussing an example, and then generalize the approach. Let us consider the game form $\mathcal{F}$ depicted in Figure 6, where $\mathsf{O} = \{x, y, z\}$. We embed this game form into two different environments, depicted in Figures 7 and 8. These define two Büchi games using the following interpretation: (a) values 0 and 1 in green represent output values giving the probability to satisfy the Büchi condition when these outputs are selected; (b) other outputs lead to either orange state q_T (a target for the Büchi condition) or red state $q_{\bar{T}}$ (not a target). In particular, the game of Figure 8 is another representation of the game of Figure 1.

Let us compare these two games. First notice that there are no optimal strategies in both cases, as already argued for the game in Figure 8; the arguments are similar for the game of Figure 7. Interestingly, in the game $\mathcal{G}_1$ in Figure 7, there are positional almost-optimal strategies (it is in fact a reachability game) whereas there are none in the game $\mathcal{G}_2$ in Figure 8 (as already discussed above); despite the fact that the local interaction at q_0 valued with μ_v, for v the value vector of the game, is that of Figure 3 in both cases.

We analyze the two settings to better understand the differences. A positional ε-optimal strategy $\mathsf{s_A}$ at q_0 in both games has to be an ε-optimal GF-strategy $\mathsf{s_A}(q_0) \in \mathcal{D}(\mathsf{A})$ in the game in normal form $\langle \mathcal{F}_{q_0}, \mu_v \rangle$ (similarly to how a uniformly optimal positional strategy needs to be locally optimal (Lemma 16)). Consider for instance the Player A positional strategy $\mathsf{s_A}$ that plays (at q_0) the top row with probability $1-\varepsilon$ (which is ε-optimal in $\langle \mathcal{F}_{q_0}, \mu_v \rangle$). This strategy has value $1-\varepsilon$ in $\mathcal{G}_1$, but has value 0 in $\mathcal{G}_2$. In both cases, if Player B plays the left column, the target is seen infinitely often almost-surely (since the bottom row is played with positive probability). The difference arises if Player B plays the right column with probability 1: in $\mathcal{G}_1$, the target is reached and never left with probability $1-\varepsilon$; however, in $\mathcal{G}_2$, with probability $1-\varepsilon$, the game visits the target but loops back to q_0, hence playing for ever this strategy leads with probability 1 to the green value 0. Actually, $\mathsf{s_A}$ is ε-optimal if for each column, either there is no green value but at least one orange q_T, or there are green values and their average is at least $1-\varepsilon$.

This intuitive explanation can be generalized to any game form $\mathcal{F}$ as follows, which we will embed in several environments. To define an environment, we fix (i) a partition $\mathsf{O} = \mathsf{O_{Lp}} \uplus \mathsf{O_{Ex}}$ of the outcomes (Lp stands for loop – i.e. orange and red outcomes – and Ex stands for exit – i.e. the green outcomes), (ii) a partial valuation of the outcomes $\alpha : \mathsf{O_{Ex}} \to [0,1]$ and a probability $p_T : \mathsf{O_{Lp}} \to [0,1]$ to visit the target T in the next step and loop back (above, the probability was either 1 (represented by orange state q_T) or 0 (represented by red state $q_{\bar{T}}$)). We then consider the Büchi game $\mathcal{G}^{\mathsf{Büchi}}_{\mathcal{F},\alpha,p_T}$ which embeds game form $\mathcal{F}$ in the environment given by α and p_T as follows: an outcome $o \in \mathsf{O_{Lp}}$ leads to q_T with probability $p_T(o)$ and $q_{\bar{T}}$ otherwise; from q_T or $q_{\bar{T}}$, surely we go back to q_0; an outcome $o \in \mathsf{O_{Ex}}$ leads to the target T with probability $\alpha(o)$ and outside T otherwise (in both cases, it stays there forever); this is formally defined in [3]. We want to specify that there are positional ε-optimal strategies in the game $\mathcal{G}^{\mathsf{Büchi}}_{\mathcal{F},\alpha,p_T}$ if and only if there are ε-optimal GF-strategies in the game form $\mathcal{F}$ ensuring the properties described above. However, to express what is an ε-optimal strategy, we need to know the value of the game $\mathcal{G}^{\mathsf{Büchi}}_{\mathcal{F},\alpha,p_T}$ at state q_0: to do so, we use [11], in which the value of concurrent games with parity objectives (generalizations of Büchi and co-Büchi objectives) is computed using μ-calculus. In our case, this can be expressed with nested fixed point operations, as described in the Appendix of [3]. We denote this value $u^{\mathsf{Büchi}}_{\mathcal{F},\alpha,p_T}$ or simply u. We can now define almost-Büchi maximizable (aBM for short) game form.

▶ **Definition 20** (Almost-Büchi maximizable game forms)**.** *Consider a game form $\mathcal{F}$, a partition of the outcomes $\mathsf{O} = \mathsf{O_{Lp}} \uplus \mathsf{O_{Ex}}$, a partial valuation of the outcomes $\alpha : \mathsf{O_{Ex}} \to [0,1]$ and a probability $p_T : \mathsf{O_{Lp}} \to [0,1]$ to visit the target T. The game form $\mathcal{F}$ is* almost-Büchi maximizable *(aBM for short) w.r.t. α and p_T if for all $0 < \varepsilon < u$, there exists a GF-strategy $\sigma_\mathsf{A} \in \mathcal{D}(\mathsf{St_A})$ such that, for all $b \in \mathsf{St_B}$, letting $A_b := \{a \in \mathsf{Supp}(\sigma_\mathsf{A}) \mid \varrho(a,b) \in \mathsf{O_{Ex}}\}$, either:*

- *$A_b = \emptyset$, and there exists $a \in \mathsf{Supp}(\sigma_\mathsf{A})$ such that $p_T(\varrho(a,b)) > 0$ (i.e. if all outcomes loop back to q_0, there is a positive probability to visit T: left column in the game $\mathcal{G}_2$);*
- *$A_b \neq \emptyset$, and $\sum_{a \in A_b} \sigma_\mathsf{A}(a) \cdot \alpha(\varrho(a,b)) \geq (u-\varepsilon) \cdot \sigma_\mathsf{A}(A_b)$ (i.e. for the action b, the value of σ_A restricted to the outcomes in $\mathsf{O_{Ex}}$ is at least $u - \varepsilon$: right column in the game $\mathcal{G}_1$).*

The game form $\mathcal{F}$ is almost-Büchi maximizable (aBM for short) *if for all partitions $\mathsf{O} = \mathsf{O_{Lp}} \uplus \mathsf{O_{Ex}}$, for all partial valuations $\alpha : \mathsf{O_{Ex}} \to [0,1]$ and probability function $p_T : \mathsf{O_{Lp}} \to [0,1]$, it is aBM w.r.t. α and p_T.*

This definition relates to the existence of positional almost-optimal strategies in $\mathcal{G}^{\mathsf{Büchi}}_{\mathcal{F},\alpha,p_T}$:

▶ **Lemma 21.** *The game form $\mathcal{F}$ is aBM w.r.t. α and p_T if and only if there are positional almost-optimal strategies from state q_0 in the game $\mathcal{G}^{\mathsf{Büchi}}_{\mathcal{F},\alpha,p_T}$.*

More interestingly, if a concurrent Büchi game has all its game forms aBM (possibly except at target states), then there exist positional almost-optimal strategies.

▶ **Theorem 22.** *Consider a Büchi game $\mathcal{G} = \langle \mathcal{C}, \mathsf{Büchi}(T) \rangle$ and assume that all local interactions at states in $Q \setminus T$ are aBM. Then, for every $\varepsilon > 0$, there is a positional strategy that is ε-optimal from every state $q \in Q$.*

Proof sketch. Let $\varepsilon > 0$. We build a positional Player A strategy $\mathsf{s_A}$ and then apply (a slight generalization of) Lemma 16 to show that it is ε-optimal. Let $v := \chi_{\mathcal{G}}$ be the value vector of the game and $u \in v[Q] \setminus \{0\}$ be some positive value of the game. Consider the set $Q_u := v^{-1}[\{u\}] \subseteq Q$ of states whose values w.r.t. v is u. We define the strategy $\mathsf{s_A}$ on each Q_u for $u \in v[Q]$ and then glue the portions of $\mathsf{s_A}$ together. This is possible because the Player A strategy we build enforces end-components in which the value given by v of all states is the same (similarly to Proposition 15 for locally optimal strategies).

In Figure 9, the set Q_u corresponds to the white area. Target states (T) are in orange, while non-target states are in red (q_1, q_2, q_3 in the figure). From every state of Q_u, there may be several split arrows, which correspond to choices by Player B (actions $b \in \mathsf{St_B}$); black split edges stay within Q_u while blue edges partly lead outside Q_u; once a split edge is chosen by Player B, Player A may ensure any leaving edge with some positive probability.

In a state $q \in Q_u \cap T$, the strategy $\mathsf{s_A}$ only needs to be locally optimal, i.e. such that $\mathsf{val}_{\langle \mathcal{F}_q, \mu_v \rangle}(\mathsf{s_A}(q)) = v(q)$. Then, the states in $Q_u \setminus T$ will be considered one by one. Let $q \in Q_u \setminus T$, we consider the Büchi game $\mathcal{G}^{\mathsf{Büchi}}_{\mathcal{F}_q,\alpha,p_T}$ built from the aBM game form $\mathcal{F}_q$ and the immediate environment of q as follows: the partial valuation α of the outcomes (i.e. the Nature states) is defined on those with a positive probability to reach $Q \setminus Q_u$ (i.e. the green area – green outcomes in Figures 7, 8), and p_T maps a Nature state d staying in Q_u to the probability $\mathsf{dist}(d)[Q_u \cap T]$ to reach the target T (i.e. the probability to reach the orange states in Figure 9 – they correspond to the orange and red outcomes in Figures 7, 8).

In Figure 9, states in red are non-winning yet (non-target in the first stage) and therefore if Player B can choose a black split-edge leading only to red states, then the value of game $\mathcal{G}^{\mathsf{Büchi}}_{\mathcal{F}_q,\alpha,p_T}$ is 0 (this is the case of q_2). On the other hand, if all split-edges are either blue or black with at least one orange end, then the value of game $\mathcal{G}^{\mathsf{Büchi}}_{\mathcal{F}_q,\alpha,p_T}$ is positive (this is the case of states q_1 and q_3). Then, we realize that there must be at least one state $q \in Q_u \setminus T$ such that $u^{\mathsf{Büchi}}_{\mathcal{F}_q,\alpha,p_T} \geq u$ (this is a key argument, and it is due to the definition of $u^{\mathsf{Büchi}}_{\mathcal{F}_q,\alpha,p_T}$, which relates to how the value in Büchi games is computed via fixed-point operations). In Figure 9, there are actually two such states, q_1 and q_3: the value at q_1 is 1 while the value at q_3 is an average of the values at the two (green) ends of the (blue) split-edge. We can then use the facts that $\mathcal{F}_{q_1}$ and $\mathcal{F}_{q_3}$ are aBM and apply the definition with a well-chosen $0 < \varepsilon_u \leq \varepsilon$ to obtain the GF-strategies played by the strategy $\mathsf{s_A}$ at states q_1 and q_3.

We then iterate the process by going to the second stage by considering that the previously dealt states (q_1 and q_3 in our example) are now orange, as in Figure 10: they are now considered as targets. The property that $u^{\mathsf{Büchi}}_{\mathcal{F}_q,\alpha,p_T} \geq u$ (with a new p_T taking into account the larger set of targets) then propagates throughout the game to all states in the white area. The strategy $\mathsf{s_A}$ is now fully defined on Q_u, and we can check that, under any Player B strategy, if the game eventually stays within an EC within Q_u, it will reach the target T infinitely often almost-surely. Indeed, from each state, there is either a positive probability to leave the EC (i.e. see a green outcome) or a positive probability to get closer to the target (i.e. see an orange outcome). ◀

With Lemma 21 and Theorem 22, we obtain a corollary analogous to Theorem 19 for aBM game forms and the existence of almost-optimal strategies.

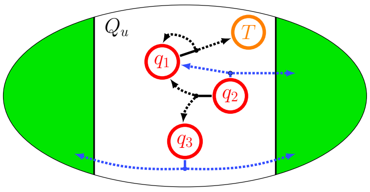

Figure 9 First stage of the construction of strategy $\mathsf{s_A}$ within Q_u.

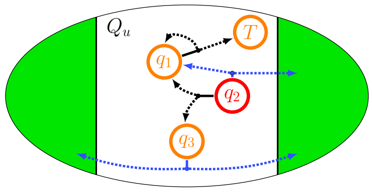

Figure 10 Second stage of the construction of strategy $\mathsf{s_A}$ within Q_u.

▶ **Corollary 23.** *In all Büchi games $\mathcal{G} = \langle \mathcal{C}, \mathsf{Büchi}(T) \rangle$ where all interactions at states in $Q \setminus T$ are aBM, there exist positional uniformly almost-optimal strategies (for Player* A*). Furthermore, if a game form $\mathcal{F}$ is not aBM, one can build a Büchi game where $\mathcal{F}$ is the only non-trivial interaction, in which there is no positional almost-optimal strategy for Player* A*.*

Note that game forms that are not aBM may behave well in some environments, even though they do not behave well in some other environments. Hence, it might be the case that in a specific Büchi game with local interactions that are not aBM, there are some positional almost-optimal strategies. This observation stands for all the type of game forms we have characterized: RM, aBM and also coBM in the next section.

Finally, note that it is decidable whether a game form is aBM.

▶ **Proposition 24.** *It is decidable if a game form is aBM. Moreover, all RM game forms are aBM game forms and there is a game form that is aBM but not RM.*

6 Playing Optimally in co-Büchi Games

Although they may seem quite close, Büchi and co-Büchi objectives do not enjoy the same properties in the setting of concurrent games. For instance, we have seen that there are Büchi games in which a state has value 1 but any finite-memory strategy has value 0. This cannot happen in co-Büchi games, since strategies with values arbitrarily close to 1 can be found among positional strategies [5]. We have also seen in Section 5 that in all concurrent Büchi games, if there is an optimal strategy from all states, then there is a uniformly optimal positional strategy. As we will see in the next subsection, this does not hold for co-Büchi games. This shows that concurrency in games complicates a lot the model: the results of this paper has to be put in regards with the model of turn-based games where pure positional strategies are sufficient to play optimally, for parity objectives [15].

6.1 Optimal strategies may require infinite memory in co-Büchi games

Consider the game depicted in Figure 11, which uses the same convention as in the previous section. The objective is $W = \mathsf{coBüchi}(\{q_T, q'_T, \top\})$ where the state $\top$ is not represented, but implicitly present via the green values (for instance, green value $1/2$ leads to $\top$ with probability $1/2$ and to $\bot$ with probability $1/2$). Let $A := \{a_1, a_2, a_3\}$ with a_1 the top row and a_3 the bottom row and $B := \{b_1, b_2, b_3\}$ with b_1 the leftmost column and b_3 the rightmost column. If a green value is not reached, Player A wins if and only if eventually, the red state is not seen anymore. The values of the states $q_0, q_T, q_{T'}$ and $q_{\bar{T}}$ are the same and are at most $1/2$. Indeed, if Player B almost-surely plays b_3 at q_0, she ensures that the value of the game from q_0 is at most $1/2$. Let us argue that any Player A positional strategy has value less than $1/2$ and exhibit an infinite-memory Player A strategy whose value is $1/2$.

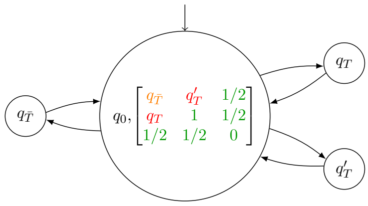

Figure 11 The co-Büchi game $\mathcal{G}_1$.

$$\mathcal{F} = \begin{bmatrix} x & z & t \\ y & r & t \\ t & t & s \end{bmatrix}$$

Figure 12 The local interaction at state q_0.

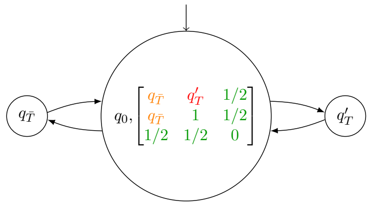

Figure 13 Another co-Büchi game ($\mathcal{G}_2$) built on $\mathcal{F}$.

Consider a Player A positional strategy $\mathsf{s_A}$. We define a Player B strategy $\mathsf{s_B}$ as follows: if $\mathsf{s_A}(q_0)(a_3) = \varepsilon > 0$, then we set $\mathsf{s_B}(q_0)(b_3) := 1$ and the value of the game w.r.t. $\mathsf{s_A}, \mathsf{s_B}$ is at most $1/2 - \varepsilon < 1/2$. If $\mathsf{s_A}(q_0)(a_1) = 1$, we set $\mathsf{s_B}(q_0)(b_2) := 1$ and the state $q'_T \in T$ is visited infinitely often almost-surely. Otherwise, $\mathsf{s_A}(q_0)(a_2) > 0$ and $\mathsf{s_A}(q_0)(a_3) = 0$, hence choosing $\mathsf{s_B}(q_0)(b_1) := 1$ ensures that the state $q_T \in T$ is visited infinitely often almost-surely. In the last two cases, $\mathsf{s_A}$ has value 0. Overall, any positional strategy $\mathsf{s_A}$ has value less than $1/2$.

We briefly describe a Player A optimal strategy $\mathsf{s_A}$ (whose value is $1/2$). The idea is the following: along histories that have not visited q'_T yet (this happens when Player B has not played b_2), $\mathsf{s_A}$ plays a_1 with very high probability $1 - \varepsilon_k < 1$ and a_2 with probability $\varepsilon_k > 0$, where k denotes the number of steps. The values $(\varepsilon_k)_{k \in \mathbb{N}}$ are chosen so that, if Player B only plays b_1 with probability 1, then the state $q_T \in T$ is seen finitely often almost-surely. Some details are given in Appendix B. After the first visit to q'_T, Player $\mathsf{s_A}$ switches to a positional strategy of value $\frac{1}{2} - \varepsilon'_k$, for $\varepsilon'_k > 0$ and k is the number of steps after the first visit to q'_T. A first visit to q'_T occurs when b_2 is played by Player B. The value of $\mathsf{s_A}$ after that point is then $(1 - \varepsilon_k) \cdot (\frac{1}{2} - \varepsilon'_k) + \varepsilon_k \cdot 1$. It suffices to choose ε'_k small enough so that the above value is at least $1/2$. Such a Player A strategy is optimal from q_0. It follows that, contrary to the Büchi case, requiring that positional optimal strategies exists from all states in a co-Büchi game is stronger than requiring that optimal strategies exists from all states.

6.2 GFs in co-Büchi games which ensure positional optimal strategies

Contrary to the Büchi objectives, RM game forms do not suffice to ensure the existence of positional uniformly optimal strategies in co-Büchi games, see Proposition 26. In this subsection, we characterize the game forms ensuring this property.

We proceed as in Subsection 5.3, and we explain the approach on an example. Consider the game form $\mathcal{F}$ depicted in Figure 12, that is the local interaction at state q_0 of the game in Figure 11. Let $\mathsf{O} = \mathsf{O_{Ex}} \uplus \mathsf{O_{Lp}}$ for $\mathsf{O_{Ex}} := \{t, r, s\}$, $\mathsf{O_{Lp}} := \{x, y, z\}$ and consider the partial valuation $\alpha : \mathsf{O_{Ex}} \to [0, 1]$ such that $\alpha(t) := 1/2$, $\alpha(r) := 1$ and $\alpha(s) := 0$. We then consider two probability functions $p_T^1, p_T^2 : \mathsf{O_{Lp}} \to [0, 1]$ such that $p_T^1(x), p_T^2(x) := 0$, $p_T^1(z), p_T^2(z) := 1$, $p_T^1(y) := 1$ whereas $p_T^2(y) := 0$. The games $\mathcal{G}_1 := \mathcal{G}^{\mathsf{coBüchi}}_{\mathcal{F},\alpha,p_T^1}$ (Figure 11) and $\mathcal{G}_2 := \mathcal{G}^{\mathsf{coBüchi}}_{\mathcal{F},\alpha,p_T^2}$ (Figure 13) are defined just like their Büchi counterparts, except for the objective which is $W = \mathsf{coBüchi}(T)$ with $q_T, q'_T \in T$ and $q_{\bar{T}} \notin T$. We have already argued that there is no positional optimal strategy in the game $\mathcal{G}_1$ (Figure 11). The issue is the following: when considering a locally optimal strategy (recall Lemma 16), there is a column where, in the support of the strategy, there is a red outcome (i.e. positive probability to reach T) and no green outcome. Then, if Player B plays, with probability 1, the corresponding action, she ensures that the set T is visited infinitely often almost-surely. Now, in the game $\mathcal{G}_2$ of Figure 12, any Player A positional strategy playing a_3 with probability 0 and a_2 with positive probability is uniformly optimal. Indeed, in that case, (i) if b_1 is played, then the set T is not seen; (ii) if b_2 or b_3 are played, then the game will end in a green outcome almost-surely.

Table 1 Game forms necessary and sufficient for the existence of positional strategies for (almost-)optimality.

	Positional Optimal Strategy		Positional ε-Optimal Strategy	
	T	$Q \setminus T$	T	$Q \setminus T$
$\mathsf{Safe}(T)$	**All**	**All**	**All**	**All**
$\mathsf{Reach}(T)$	**All**	**RM**	**All**	**All**
$\mathsf{Büchi}(T)$	**All**	**RM**	**All**	**aBM**
$\mathsf{coBüchi}(T)$	**RM**	**coBM**	**All**	**All**

In the general case, as for Büchi games, the value of co-Büchi games can be computed with nested fixed points (see [3]). Using this value, we can define the notion of co-Büchi maximizable game forms (coBM for short), which, while requiring a slightly more complex setting, formalizes the intuition given in the previous example, see [3]. It ensures a lemma analogous to Lemma 21 in the context of co-Büchi games.

Furthermore, coBM game forms also ensure that, when they are used in a co-Büchi game, there always exists a positional uniformly optimal strategy.

▶ **Theorem 25.** *In all co-Büchi games $\mathcal{G} = \langle \mathcal{C}, \mathsf{coBüchi}(T) \rangle$ where all local interactions at states in T are RM and all local interactions at states in $Q \setminus T$ are coBM, there exist positional uniformly optimal strategies (for Player A). Furthermore, if a game form $\mathcal{F}$ is not coBM, one can build a co-Büchi game where $\mathcal{F}$ is the only non-trivial interaction where there is no positional optimal strategy for Player A.*

▶ **Proposition 26.** *It is decidable if a game form is coBM. All coBM game forms are RM game forms. There exists a game form that is RM but not coBM.*

7 Conclusion

We have studied game forms and defined various conditions such that these game forms in isolation behave properly w.r.t. some fixed property (like existence of optimal strategies for Büchi objectives), and we have proven that they can be used collectively in graph games while preserving this property. These conditions, summarized in Table 1, give the unique way to construct games which will satisfy good memory properties by construction for playing (almost-)optimally.

Let us explicit how to read a specific row of this table, say the third one for the Büchi objective. A Büchi objective is defined along with a target $T \subseteq Q$. The game forms necessary and sufficient to ensure the existence of positional optimal strategies are given in the leftmost part of the table, and to ensure the existence of positional almost-optimal strategies, in the rightmost part of the table. For instance, for the leftmost part, if a game form is not RM, there is a Büchi game built from it – where it is the only non-trivial local interaction – in which there is no positional optimal strategy. Conversely, if in a Büchi game, all local interactions at states outside the target T are RM game forms (no further assumption is made on the game forms appearing at states in T), then there is a positional optimal strategy. The rightmost part of the table can be read similarly.

Finally, we would like to mention that all two-variable game forms $\mathcal{F} = \langle \mathsf{St}_\mathsf{A}, \mathsf{St}_\mathsf{B}, \mathsf{O}, \varrho \rangle$ (i.e. such that $|\mathsf{O}| \leq 2$) are coBM (this is a direct consequence of the definition), hence RM. From this, we obtain as a corollary of our results that in all finite Büchi and co-Büchi games where all local interactions are two-variable game forms, both players have positional uniformly optimal strategies.

▶ **Corollary 27.** *In a Büchi or co-Büchi game $\mathcal{G}$ such that for all $q \in Q$, $|\delta(q, A, B)| \leq 2$, both players have a positional uniformly optimal strategy.*

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A Infinite memory to play ε-optimal strategies in Büchi games

Consider the game of Figure 1. This game is explained in [10]. Assume that the sets of actions are $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$ with a_1 corresponding to the top row and b_1 to the left column.

Consider some positive probability $p > 0$ and consider a Player A strategy $\mathsf{s_A}$ such that the probability to play a_2 is either 0 or at least p. Let us show that the value of such a strategy is 0. Specifically, consider a Player B strategy $\mathsf{s_B}$ such that, for all $\rho \cdot q_0 \in Q^+$, we have:

$$\mathsf{s_B}(\rho \cdot q_0) := \begin{cases} b_1 & \text{if } \mathsf{s_A}(\rho \cdot q_0)(a_2) = 0 \\ b_2 & \text{if } otherwise \end{cases}$$

Each time $\mathsf{s_A}(\rho \cdot q_0)(a_2) \geq p$, then the probability to reach the state $\bot$ is reached with probability at least p. Hence, if this happens infinitely often, then the state $\bot$ is seen with probability 1. Otherwise, the state q_0 is never left after some moment on. In both cases, the set T is seen only finitely often. It follows that the value of the game with strategies $\mathsf{s_A}, \mathsf{s_B}$ is 0.

Now, consider some $\varepsilon > 0$ let us exhibit a Player A strategy of value at least $1-\varepsilon$. The idea is the following. Consider a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ of positive values such that $\lim_{n\to\infty} \Pi_{i=0}^n (1 - \varepsilon_i) \geq 1 - \varepsilon$. Then, consider a Player A strategy such that $\mathsf{s_A}(\rho)(a_1) := 1 - \varepsilon_k$ where $k \in \mathbb{N}$ denotes the number of times the state $\top$ is visited in ρ. Then, the state q_0 is seen indefinitely with probability 0. If the state $\top$ has been seen already k times, then the probability to stay at state q_0 for n steps is at most $(1 - \varepsilon_k)^n \to_{n\to\infty} 0$. Furthermore, the probability to ever reach the state $\bot$ is at most $\lim_{n\to\infty} \Pi_{i=0}^n (1 - \varepsilon_i) \geq 1 - \varepsilon$. Overall, regardless of Player B's strategy, the probability to visit the set T infinitely often is at least $1 - \varepsilon$. Note that such a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ could be equal, for instance, to $\varepsilon_k := 1 - (1 - \varepsilon)^{\frac{1}{2^{k+1}}}$. Then, for $n \in \mathbb{N}$:

$$\Pi_{i=0}^n (1 - \varepsilon_i) = (1 - \varepsilon)^{\sum_{i=0}^n \frac{1}{2^{i+1}}} = (1 - \varepsilon)^{(1 - \frac{1}{2^{n+1}})} \to_{n\to\infty} 1 - \varepsilon$$

B Playing optimally in the game of Figure 11

We are given a probabilistic process such that, at each step either event T or $\neg T$ occur. Furthermore, at step $n \in \mathbb{N}$, the probability that the event T occurs is equal to ε_n. We will denote by $\mathbb{P}$ the probability measure of the corresponding measurable sets. In the following, we will use the following notations:

- for all $n \in \mathbb{N}$, $\mathsf{X}_n T$ refers to the event: the event T occurs at step n.
- for all $n \in \mathbb{N}$: $\Diamond_n T := \cup_{k\geq n} (\mathsf{X}_k T)$ refers to the event: the event T occurs at some point after step n.
- $\Box\Diamond T := \cap_{n\in\mathbb{N}} (\Diamond_n T)$ refers to the event: the event T occurs infinitely often.

An infinite path (which can be seen as an infinite sequence of elementary events) is winning for the Büchi objective if it corresponds to the event $\Box\Diamond T$. We want to define a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ such that, for all $k \in \mathbb{N}$ we have $\varepsilon_k > 0$ and $\mathbb{P}(\Box\Diamond T) = 0$. In fact, it suffices to consider a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ whose sum converges (i.e. such that $\sum_{n=0}^{\infty} \varepsilon_n < \infty$), for instance $\varepsilon_n := \frac{1}{2^{n+1}}$ for all $n \in \mathbb{N}$. Indeed, for all $n \in \mathbb{N}$, we have:

$$\mathbb{P}(X_n T) = \varepsilon_n$$

Furthermore:

$$\mathbb{P}(\Diamond_n T) = \mathbb{P}(\cup_{k \geq n} \mathsf{X}_k T) \leq \sum_{k=n}^{\infty} \mathbb{P}(\mathsf{X}_k T) = \sum_{k=n}^{\infty} \varepsilon_k$$

It follows that:

$$\mathbb{P}(\Box \Diamond T) = \mathbb{P}(\cap_{n \in \mathbb{N}} \Diamond_n T) = \lim_{n \to \infty} \mathbb{P}(\Diamond_n T) \leq \lim_{n \to \infty} \sum_{k=n}^{\infty} \varepsilon_k = 0$$