Phase Semantics for Linear Logic with Least and Greatest Fixed Points

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Abstract

The truth semantics of linear logic (*i.e.* phase semantics) is often overlooked despite having a wide range of applications and deep connections with several denotational semantics. In phase semantics, one is concerned about the provability of formulas rather than the contents of their proofs (or refutations). Linear logic equipped with the least and greatest fixpoint operators (μMALL) has been an active field of research for the past one and a half decades. Various proof systems are known *viz.* finitary and non-wellfounded, based on explicit and implicit (co)induction respectively.

In this paper, we extend the phase semantics of multiplicative additive linear logic (*a.k.a.* MALL) to μMALL with explicit (co)induction (*i.e.* $\mu\text{MALL}^{\text{ind}}$). We introduce a Tait-style system for μMALL called μMALL_ω where proofs are wellfounded but potentially infinitely branching. We study its phase semantics and prove that it does not have the finite model property.

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1 Introduction

Fixpoint logics: from truth to proofs. Fixpoint logics were first introduced in the study of inductive definability [1] in recursion theory which predates its first application in computer science as an expressive database query language [3]. In order to define the language of a fixpoint logic, one introduces explicit fixpoint construct(s) and closes it under these construct(s) thus obtaining a richer language. First order logic extended with various fixpoint operators have been extensively explored in model theory [38]. In the propositional case, the (multi)modal μ-calculus (the extension of basic modal logic K with least and greatest fixpoint operators) is probably the most well-studied. Introduced by Scott and Bakker in an unpublished manuscript, the logic has been historically studied in the formal methods and verification community [16]. More recently, there has been a growing interest in its structural proof-theory. The most important result in this direction is the completeness of Hilbert-style axiomatisations for the logic, which has turned out be notoriously difficult [34, 51, 50].

Various proof systems for fixpoint logics. The setting of our paper is μMALL, the extension of multiplicative additive linear logic by fixpoints *viz.* the propositional fragment of the logic introduced by Baelde and Miller in [7] who proposed the sequent calculus system $\mu\mathsf{MALL}^{\mathsf{ind}}$ for the logic. All these formalisations *viz.* the Hilbert-style axiomatisations of μ-calculus and $\mu\mathsf{MALL}^{\mathsf{ind}}$ employ inference rules that express an *explicit* (co)induction scheme *i.e.* the induction hypothesis must be provided explicitly. However, sequent calculi with explicit (co)induction do not have the subformula property in spite of having analyticity. In fact, it is generally accepted that we do not have *true* cut elimination for any logic equipped with a theory of inductive definitions [40]. However, one can consider an alternative formalisation of inductive reasoning *viz. implicit* induction, which avoids the need for explicitly specifying (co)induction invariants. This formalism generally recovers true cut elimination but at the price of infinitary axiomatisation of the fixpoints. There are two approaches to this.

The first approach is to consider a Tait-style system *i.e.* infinitary wellfounded derivations which use a so-called ω-rule with infinitely many premises of finite approximations of a fixpoint. Such rules arise in various areas of logic, notably as Carnap's rule [15] in arithmetic. A complete Tait-style system has been proposed for fixpoint logics *viz.* for the μ-calculus [35] and star-continuous action lattices [43] (where the ω-rule construes the Kleene star as an ω-iteration of finite concatenations).

The second approach is to define a non-wellfounded and/or a circular proof system with finitely branching inferences. Such systems have been extensively studied in the setting of μMALL [44, 28, 6, 23]. Circular proofs have deep roots in the history of logic and mathematical reasoning: starting with Euclid's [27] heuristic of infinite descent through the more rigorous studies of Fermat [22]. A systematic investigation of the connection between circular proofs and reasoning by infinite descent has been carried out by Brotherston and Simpson [11, 12, 13].

Proof systems difficult to compare. Brotherston and Simpson conjectured that (in the setting of Martin-Löf's inductive definitions) circular proofs derive the same statements as finitary proofs with explicit induction. The so-called *Brotherston-Simpson conjecture* remained open for about a decade until Berardi and Tatsuta [8, 10] answered it negatively for the general case. On the other hand, if the logic contains arithmetic, the conjecture is known to be true; proved independently by Simpson [47], and Berardi and Tatsuta [9].

Note that the Brotherston-Simpson conjecture is heavily dependent on the base logic since the availability of structural rules or modal constructs induce subtle differences. For instance, the modal μ-calculus coincides on all systems. On the other hand, in Kleene Algebras, which is a substructural logic, the wellfounded, circular, and Tait-style systems are indeed different. In the setting of linear logic, in a recent work [20], the circular and the non-wellfouded system has been shown to be separate. However, the exact relation between finitary and circular system is still open. There is good reason for the difficulty. The very restricted use of structural rules in the linear setting induces a much more refined provability relation. Therefore, in this paper, we study the provability semantics *a.k.a.* the *phase semantics* of these logics as a first step of tackling the Brotherston-Simpson conjecture. Categorical semantics of circular proofs of the additive fragment have been studied in [44]. More recently, coherence space semantics have been studied for $\mu\mathsf{MALL}^{\mathsf{ind}}$ [24] as well as preliminary results on $\mu\mathsf{MALL}_\omega$ denotational semantics [26]. These semantics interpret formulas as well as their proofs thereby preserving their computational content. On the other hand, phase semantics is a coarser interpretation that allows for expressing strong invariant of linear logic provability and has been notably used to prove decidability results [36, 19] and cut admissibility results [42].

Contributions. In this paper, we shall develop a phase semantics for various proof systems for μMALL. Our first contribution is the notion of μ-phase models which are bespoke mathematical objects that are shown to be sound and complete interpretation for $\mu\mathsf{MALL}^{\mathsf{ind}}$. The completeness proof is intricate via Tait-Girard reducibility candidates and is inspired from [42]. Subsequently, we obtain the cut-admissibility of $\mu\mathsf{MALL}^{\mathsf{ind}}$ by semantic means. Furthermore, we design a new Tait-style proof system for μMALL *viz.* $\mu\mathsf{MALL}_\omega$ and obtain its phase semantics and cut-admissibility. In this case, the crux of the completeness proof is the obtention of a wellfounded notion of the rank of a formula. Cut-admissibility lets us show that $\mu\mathsf{MALL}_\omega$ and $\mu\mathsf{MALL}^{\mathsf{ind}}$ are different systems in terms of provability. Finally, we show that $\mu\mathsf{MALL}_\omega$ does not have a finite model property using an idea by Lafont [36].

Organisation of the paper. The paper is organised as follows. In Section 2, we give a brief exposition of linear logic and its phase semantics, relevant fixpoint theorems and describe the syntax and relevant properties of μMALL. In Section 3, we develop the phase semantics of μMALL *wrt.* the wellfounded proof system $\mu\mathsf{MALL}^{\mathsf{ind}}$. In Section 4, we introduce the Tait-style proof system $\mu\mathsf{MALL}_\omega$ and study its semantics, cut-admissibility and finite model property. Finally, we conclude in Section 5 discussing directions of future work. Two appendices complement the paper: a table summarizing the proof systems used in this paper is provided in Appendix A, proof details are provided in Appendix B.

Notation. Let F and G be formulas. $F(G/x)$ denotes that every occurrence of x in F is replaced by G. If x is clear from the context, we simply write it as $F(G)$. A special case is $F^n(x)$ which denotes $\overbrace{F(F(\cdots(F}^{n}(x))\cdots)$. We write a^n as a macro for $\overbrace{a \parr \ldots \parr a}^{n}$. Let Γ be any sequent. Then, $\mathcal{L} \vdash \Gamma$ ($\mathcal{L} \vdash_{cf} \Gamma$ respectively) denotes that there is a proof (cut-free proof respectively) of Γ in the system $\mathcal{L}$. Finally, for any finite set S, its cardinality is $|S|$.

2 Background

2.1 Linear logic and phase semantics

Substructural logics are logics lacking at least one of the usual structural rules. In *linear logic* [30], one of the most well-studied substructural logics, sequents are effectively multisets and the use of contraction and weakening is carefully controlled. Conjunction and disjunction each have two versions in linear logic: *multiplicative* and *additive*. Consequently the units have multiplicative and additive versions as well.

	conjunction	disjunction	true	false
multiplicative	$\otimes$	$\parr$	$\mathbf{1}$	$\bot$
additive	$\&$	$\oplus$	$\top$	$\mathbf{0}$

The logical system thus obtained is called multiplicative-additive linear logic (MALL) and its inference rules are depicted in Figure 1 (sequents being construed as finite multisets). Full linear logic extends MALL by incorporating certain "exponential" modalities, written $?F$ and, dually, $!F$. Because of the absence of contraction and weakening, linear logic is resource-conscious *i.e.* one is concerned over the number of times that a given sentence is used in the proof of another sentence. To get a flavour of *phase semantics*, the provability semantics of linear logic [30], it is informative to characterise the set of lists Γ of formulas that make a formula F provable.

Identity rules	$\dfrac{}{\vdash F, F^{\perp}}$ (id)	$\dfrac{\vdash \Gamma_1, F \quad \vdash \Gamma_2, F^{\perp}}{\vdash \Gamma_1, \Gamma_2}$ (cut)		
Logical rules **multiplicative connectives**	$\dfrac{\vdash \Gamma, F_1, F_2}{\vdash \Gamma, F_1 ⅋ F_2}$ (⅋)	$\dfrac{\vdash \Gamma_1, F_1 \quad \vdash \Gamma_2, F_2}{\vdash \Gamma_1, \Gamma_2, F_1 \otimes F_2}$ ($\otimes$)	$\dfrac{}{\vdash \mathbf{1}}$ ($\mathbf{1}$)	$\dfrac{\vdash \Gamma}{\vdash \Gamma, \perp}$ ($\perp$)
additive connectives	$\dfrac{\vdash \Gamma, F_i}{\vdash \Gamma, F_1 \oplus F_2}$ ($\oplus_i$)	$\dfrac{\vdash \Gamma, F_1 \quad \vdash \Gamma, F_2}{\vdash \Gamma, F_1 \& F_2}$ ($\&$)	$\dfrac{}{\vdash \Gamma, \top}$ ($\top$)	No rule for $\mathbf{0}$

Figure 1 Inference rules for MALL, where $i \in \{1, 2\}$.

▶ **Definition 1.** *For a formula F, define* $\mathsf{Pr}(F) = \{\Gamma \mid \mathsf{MALL} \vdash \Gamma, F\}$ *and* $\mathsf{Pr}_{cf}(F) = \{\Gamma \mid \mathsf{MALL} \vdash_{cf} \Gamma, F\}$.

Let us examine some properties of $\mathsf{Pr}(F)$. First, notice that the axiom rule ensures that for any F, $F^{\perp} \in \mathsf{Pr}(F)$. Invertibility of the ($\perp$) rule gives us $\mathsf{Pr}(\perp)$ is the set of all provable sequents. Similar observations on the invertibility of the ($\&$) rule inform that $\mathsf{Pr}(G \& G) = \mathsf{Pr}(F) \cap \mathsf{Pr}(G)$. For the (non-invertible) connectives $\otimes$ and $\oplus$, we only have $\mathsf{Pr}(F \otimes G) \supseteq \mathsf{Pr}(F) \cdot \mathsf{Pr}(G) = \{\Gamma, \Delta \mid \Gamma \in \mathsf{Pr}(F), \Delta \in \mathsf{Pr}(G)\}$ and $\mathsf{Pr}(F \oplus G) \supseteq \mathsf{Pr}(F) \cup \mathsf{Pr}(G)$. This suggests that the algebraic model for linear logic should simultenously be a monoid and lattice *i.e.* a *residuated lattice.*

$\mathsf{Pr}(\perp)$ plays an major role in this approach, especially when considering it together with the cut inference. Indeed, for any F, one has that $\mathsf{Pr}(F^{\perp}) = \{\Gamma \mid \forall \Delta \in \mathsf{Pr}(F), \Gamma, \Delta \in \mathsf{Pr}(\perp)\}$. This naturally suggests to consider the operation $S^{\perp} = \{\Gamma \mid \forall \Delta \in S, \Gamma \cdot \Delta \in \mathsf{Pr}(\perp)\}$ which induces a closure operator $(\bullet)^{\perp\perp}$ on the set of multisets of linear formulas. As we will soon see, $\mathsf{Pr}(F)$ is closed under the double negation operation for any F.

These are the basic design principles of phase semantics: interpreting linear formulas as closed subsets of a monoid for the closure operation induced by the orthogonality relation *w.r.t.* a specific subset $\perp\!\!\!\perp$ of the monoid which is an *abstraction of the provable sequents.*

▶ **Definition 2.** *A* **phase space** *is a 4-tuple* $\mathcal{M} = (M, 1, \cdot, \perp\!\!\!\perp)$ *where* $(M, 1, \cdot)$ *is a commutative monoid and* $\perp\!\!\!\perp \subseteq M$. *For* $X, Y \subseteq M$, *define the following operations:* $XY := \{x \cdot y \mid x \in X, y \in Y\}$ *and* $X^{\perp} := \{y \mid \forall x \in X, x \cdot y \in \perp\!\!\!\perp\}$.

Facts *are those* $X \subseteq M$ *such that* $X = X^{\perp\perp}$. *(Equivalently,* $X = Y^{\perp}$ *for some* $Y \subseteq M$.*)*

▶ **Example 3.** Consider the additive monoid $(\mathbb{Z}, 0, +)$ and let $\perp\!\!\!\perp = \{0\}$. For any set $S \subseteq \mathbb{Z}$, $S^{\perp} = \{y \mid \forall x \in S. x + y = 0\}$. Therefore, if S is not singleton then $S^{\perp} = \emptyset$; so, $S^{\perp\perp} = \mathbb{Z}$. On the other hand, $\{x\}^{\perp} = \{-x\}$. The facts of this phase space are $\emptyset$, singleton sets, and $\mathbb{Z}$.

▶ **Proposition 4.** *Let* $X, Y \subseteq M$. *Then the following properties hold.*

1. $X \subseteq Y^{\perp} \iff XY \subseteq \perp\!\!\!\perp$
2. $XX^{\perp} \subseteq \perp\!\!\!\perp$
3. $X \subseteq Y \implies Y^{\perp} \subseteq X^{\perp}$
4. $X \subseteq X^{\perp\perp}$
5. $X^{\perp\perp\perp} = X^{\perp}$
6. $(X \cup Y)^{\perp} = X^{\perp} \cap Y^{\perp}$

Let X and Y be facts. We define the following operations on facts.

$$X \otimes Y := (XY)^{\perp\perp} \qquad X ⅋ Y := (X^{\perp}Y^{\perp})^{\perp}$$

$$X \& Y := X \cap Y \qquad X \oplus Y := (X \cup Y)^{\perp\perp}$$

▶ **Proposition 5.** *Let* X, Y *be facts. Then,* $1 \in X ⅋ Y \iff X^{\perp} \subseteq Y$.

Fix a phase space $\mathcal{M}$ and let $\mathcal{X}$ be its set of facts. Fix $V : \mathcal{A} \to \mathcal{X}$ where $\mathcal{A}$ is the set of atoms. A phase space along with such a valuation V is called a ***phase model***. The semantics $[\![F]\!]$ of a MALL formula F is parameterised by a valuation (suppose, V) which we will denote by $[\![F]\!]^V$. We are now ready to define the semantics which is defined inductively as follows:

$$
\begin{array}{lll}
[\![a]\!]^V = V(a) & [\![a^\perp]\!]^V = [\![a]\!]^{V\perp} & a \in \mathcal{A} \\
[\![\mathbf{1}]\!]^V = \{1\}^{\perp\perp} & [\![\perp]\!]^V = \perp\!\!\!\perp & \\
[\![\mathbf{0}]\!]^V = \{\emptyset\}^{\perp\perp} & [\![\top]\!]^V = M & \\
[\![F_1 \odot F_2]\!]^V = [\![F_1]\!]^V \odot [\![F_2]\!]^V & & \odot \in \{\otimes, \parr, \&, \oplus\}
\end{array}
$$

When V is clear from the context, we shall drop it, simply writing $[\![F]\!]$. Finally, we generalise the interpretation to sequents of the form $\Gamma = F_1, \ldots, F_n$ as $[\![\Gamma]\!] = [\![F_1 \parr F_2 \parr \ldots \parr F_n]\!]$.

▶ **Theorem 6** (MALL Soundness [30]). *If* MALL $\vdash \Gamma$ *then for all phase models* $(\mathcal{M}, V)$*,* $1 \in [\![\Gamma]\!]^V$*.*

▶ **Example 7.** To illustrate the utility of the phase semantics, we show that in any provable multiplicative formula F (*i.e.* a MALL formula with only multiplicative connectives), an atom occurs exactly as many times as its negation. Fix an arbitrary atom a occurring in F. Let $[\![F]\!]^V$ be the interpretation of F in the phase space in Example 3 *w.r.t.* the valuation V that maps the atom a to $\{1\}$ and every other atom to $\{0\}$. In this phase space, it is easy to see that $X \otimes Y = X \parr Y = \{x + y \mid x \in X, y \in Y\}$. By Theorem 6, if F is provable, $0 \in [\![F]\!]^V$ hence number of occurrences of a in F is equal to the number of occurrences of $a^\perp$.

Note that a syntactic proof would require the heavy tool of MALL cut-admissibility.

▶ **Definition 8.** *The* **syntactic model** $(\mathsf{MALL}^\bullet, \emptyset, \cdot, \perp\!\!\!\perp, V)$ *is a phase model such that:*

- $(\mathsf{MALL}^\bullet, \emptyset, \cdot)$*, called* **syntactic monoid***, is the free commutative monoid generated by all formulas. In other words,* $\mathsf{MALL}^\bullet$ *is the set of all sequents construed as finite multisets, the empty multiset* $\emptyset$ *is the monoid identity, and multiset union is the monoid operation.*
- $\perp\!\!\!\perp = \mathsf{Pr}(\perp)$ i.e. $\perp\!\!\!\perp$ *is set of all provable sequents.*
- $V(a) = \mathsf{Pr}(a)$ *for all atoms* $a \in \mathcal{A}$*.*

▶ Remark 9. Note that for the syntactic model to be well-defined one needs to show that $\mathsf{Pr}(a)$ is a fact in the phase space $(\mathsf{MALL}^\bullet, \emptyset, \cdot, \perp\!\!\!\perp)$.

▶ **Lemma 10** (Adequation Lemma for MALL)**.** *For all formulas* F*,* $[\![F]\!]^V \subseteq \mathsf{Pr}(F)$*.*

▶ **Theorem 11** (MALL Completeness [30])**.** *If for all phase models* $(\mathcal{M}, V)$*,* $1 \in [\![\Gamma]\!]^V$ *then* MALL $\vdash \Gamma$.

Proof sketch. Suppose for any phase model $(\mathcal{M}, V)$, $1 \in [\![\Gamma]\!]^V$. In particular, this holds for the syntactic model. By Lemma 10, $[\![\Gamma]\!]^V \subseteq \mathsf{Pr}(\Gamma)$ (construing Γ as a parr formula). Therefore, $\emptyset \in \mathsf{Pr}(\Gamma)$ (recall $\emptyset$ is the unit of syntactic monoid). Hence, $\vdash \Gamma$. ◀

Okada [42] observed that one can obtain the cut-admissibility of MALL for free by slightly modifying the definition of the syntactic monoid. Now define $\perp\!\!\!\perp = \mathsf{Pr}_{cf}(\perp)$ and $V(a) = \mathsf{Pr}_{cf}(a)$. The refined adequation lemma $[\![F]\!] \subseteq \mathsf{Pr}_{cf}(F)$ follows exactly as in Lemma 10.

▶ **Theorem 12** (MALL cut-free completeness [42])**.** *If for any phase model* $(\mathcal{M}, V)$*,* $1 \in [\![\Gamma]\!]^V$ *then* MALL $\vdash_{cf} \Gamma$.

▶ **Corollary 13.** MALL *admits cuts.*

Proof. Suppose MALL $\vdash \Gamma$. By Theorem 6, $\emptyset \in [\![\Gamma]\!]^V$ for the syntactic model of MALL. By Theorem 12, MALL $\vdash_{cf} \Gamma$. ◀

2.2 Fixpoint theory and fixpoint logic

In this section we will recall some background on the fundamental fixpoint theorems of lattice theory. Not only will we use them several times in our technical proofs, but also, they will provide the intuition about the design of proof systems with fixpoint rules and their corresponding semantics. For the rest of this subsection, let $(S, \leqslant_S)$ be a complete lattice with least element $\bot$ and greatest element $\top$.

▶ **Theorem 14** ([33, 49]). *Let $f : S \to S$ be a monotonic function. The set of fixpoints of f is non-empty and equipped with $\leqslant_S$ forms a complete lattice.*

▶ **Definition 15.** *Let $f : S \to S$ be a monotonic function. f is said to be* **Scott-continuous** *if for each directed subset S we have $f(\bigvee_{S_i \in S} S_i) = \bigvee_{S_i \in S} f(S_i)$.*

▶ **Theorem 16** (Kleene Fixed Point Theorem). *Every Scott-continuous function f has the least fixpoint $\bigvee_{n\in\omega} f^n(\bot)$.*

Observe that this is a constructive formulation of a fixpoint. Cousot and Cousot proved a constructive version of Theorem 14 without using the Scott-continuity hypothesis [17]. Let $f : S \to S$ be a monotonic function. The ***lower iteration sequence*** for f starting with $x \in S$ is the sequence $\langle \mathcal{U}_\alpha \mid \alpha \in \mathsf{Ord}\rangle$ of elements of S defined by transfinite induction as follows: (i) $\mathcal{U}_0 = x$; (ii) $\mathcal{U}_{\alpha+1} = f(\mathcal{U}_\alpha)$; and, (iii) $\mathcal{U}_\lambda = \bigwedge_{\alpha<\lambda} \mathcal{U}_\alpha$ for λ a limit ordinal.

▶ **Theorem 17** ([17]). *Let $f : S \to S$ be a monotonic function. The lower iteration sequence for f starting from $\bot$ is increasing and there exists an ordinal θ (called the* **closure ordinal** *of f) such that $\mathcal{U}_\theta = \mathcal{U}_{\theta+1}$. Moreover, $\mathcal{U}_\theta$ is the least fixpoint of f.*

Note that one defines upper iteration sequence by taking supremums at limit ordinals. Dually, the greatest fixpoint is the stationary point of the decreasing upper iteration sequence starting from $\top$.

▶ Remark 18. The closure ordinal of a Scott-continuous function is at most ω.

2.3 Multiplicative additive linear logic with fixpoints

In this subsection we recall the propositional version of logic μMALL and its wellfounded proof system $\mu\mathsf{MALL}^{\mathsf{ind}}$ introduced in [7]. (See Appendix A for details)

▶ **Definition 19.** *Fix a countable set of atoms $\mathcal{A} = \{a, b, \dots\}$ and variables $\mathcal{V} = \{x, y, \dots\}$ such that $\mathcal{A} \cap \mathcal{V} = \emptyset$. μMALL* **pre-formulas** *are given by the following grammar:*

$$F, G ::= \mathbf{0} \mid \top \mid \bot \mid \mathbf{1} \mid a \mid a^\perp \mid x \mid F \parr G \mid F \otimes G \mid F \oplus G \mid F \& G \mid \mu x.F \mid \nu x.F$$

where $a \in \mathcal{A}$, $x \in \mathcal{V}$, and μ, ν bind the variable x in F. When a pre-formula is closed (i.e. no free variables), we simply call it a **formula**.

Negation, $(\bullet)^\perp$, defined as a meta-operation on pre-formulas, will be used only on formulas. As it is not part of the syntax, we do not need any positivity condition on the fixed-point expressions. As expected, least and greatest fixed points are the dual of each other.

▶ **Definition 20.** **Negation** *of a pre-formula is defined inductively as follows.*

$$\mathbf{0}^\perp = \top; \quad \top^\perp = \mathbf{0}; \quad \bot^\perp = \mathbf{1}; \quad \mathbf{1}^\perp = \bot; \quad (a)^\perp = a^\perp; \quad a^{\perp\perp} = a;$$
$$x^\perp = x; \quad (F \parr G)^\perp = F^\perp \otimes G^\perp; \quad (F \otimes G)^\perp = F^\perp \parr G^\perp; \quad (F \oplus G)^\perp = F^\perp \& G^\perp;$$
$$(F \& G)^\perp = F^\perp \oplus G^\perp; \qquad (\mu x.F)^\perp = \nu x.F^\perp; \qquad (\nu x.F)^\perp = \mu x.F^\perp.$$

Fixpoint logics have a notion of subformula that is a different from usual:

▶ **Definition 21.** *The* **Fischer-Ladner closure** $\mathbb{FL}(F)$ *of a* μMALL$_\omega$ *formula* F *is the smallest set such that:*

- $F \in \mathbb{FL}(F)$.
- $G \odot H \in \mathbb{FL}(F) \implies G, H \in \mathbb{FL}(F)$ *where* $\odot \in \{\parr, \otimes, \oplus, \&\}$.
- $\eta x.G \in \mathbb{FL}(F) \implies G(\eta x.G/x) \in \mathbb{FL}(F)$ *where* $\eta \in \{\mu, \nu\}$.

As is well-known, for any formula F, $\mathbb{FL}(F)$ *is a finite set.*

▶ **Definition 22.** *A* **proof** *of* μMALL$^{\mathsf{ind}}$ *is a finite tree generated from the inference rules of* MALL *given in Figure 1 and the following rules for fixpoint operators.*

$$\dfrac{\vdash \Gamma, F(\mu x.F/x)}{\vdash \Gamma, \mu x.F}\,(\mu) \quad ; \quad \dfrac{\vdash \Gamma, S \quad \vdash S^\perp, F(S/x)}{\vdash \Gamma, \nu x.F}\,(\nu)$$

▶ **Example 23.** Let $G = a^\perp \parr x$, $F = a \otimes (a \otimes y)$. Throughout the rest of the paper let $\Gamma_0 = \mu x.G, a \otimes \nu y.F$.

$$\dfrac{\dfrac{\dfrac{}{\vdash a^\perp, a}\,(\mathsf{id}) \qquad \dfrac{\dfrac{}{\vdash \mu x.G, (\mu x.G)^\perp}\,(\mathsf{id}) \qquad \dfrac{\dfrac{\dfrac{}{\vdash a^\perp, a}\,(\mathsf{id}) \qquad \dfrac{\dfrac{\dfrac{}{\vdash a^\perp, a}\,(\mathsf{id}) \qquad \dfrac{}{\vdash \mu x.G, (\mu x.G)^\perp}\,(\mathsf{id})}{\vdash a^\perp, \mu x.G, a \otimes (\mu x.G)^\perp}\,(\otimes)}{\vdash \mu x.G, a \otimes (\mu x.G)^\perp}\,(\mu, \parr)}{\vdash a^\perp, \mu x.G, a \otimes (a \otimes (\mu x.G)^\perp)}\,(\otimes)}{\vdash \mu x.G, a \otimes (a \otimes (\mu x.G)^\perp)}\,(\mu, \parr)}{\vdash \mu x.G, \nu y.F}\,(\nu)}{\dfrac{\vdash a^\perp, \mu x.G, a \otimes \nu y.F}{\vdash a^\perp \parr \mu x.G, a \otimes \nu y.F}\,(\parr)}\,(\otimes)$$

$$\dfrac{\vdash a^\perp \parr \mu x.G, a \otimes \nu y.F}{\vdash \Gamma_0}\,(\mu)$$

▶ **Theorem 24** ([7])**.** *The following rule is admissible in* μMALL$^{\mathsf{ind}}$.

$$\dfrac{\vdash A, B}{\vdash F^\perp(A/x), F(B/x)}\,(\mathsf{func})$$

▶ **Theorem 25** ([7, 5])**.** μMALL$^{\mathsf{ind}}$ *admits cuts.*

3 Phase semantics of μMALL$^{\mathsf{ind}}$

Fixpoints can be encoded in second order linear logic (LL^2) vis-à-vis the translation $[\mu x.F] = \forall S.?([F](S) \otimes S^\perp) \parr S$ and $[\nu x.F] = \exists S.!(S^\perp \parr [F](S)) \otimes S$. Since this translation respects provability [7] in μMALL$^{\mathsf{ind}}$, one can use LL^2 phase semantics [42] to define the phase semantics of a μMALL formula F as $[\![F]\!]$ *i.e.* the semantics of its translation into LL^2. Although the resulting semantics is sound and complete, it is barely insightful since it relies on the phase semantics of LL^2 as a black box. In this section, we essentially peek into this black box.

Observe that the set of all facts of a phase space ordered by inclusion $(\mathcal{X}, \subseteq)$ is a complete lattice. Therefore, by Theorem 14, any monotonic function $\xi : \mathcal{X} \to \mathcal{X}$ has a fixpoint. The least fixpoint $\mu\xi$ (respectively, the greatest fixpoint $\nu\xi$ by duality) is given by

$$\mu\xi = \bigcap_{X \in \mathcal{X}} \{X \mid \xi(X) \subseteq X\} \quad ; \quad \nu\xi = \left(\bigcup_{X \in \mathcal{X}} \{X \mid X \subseteq \xi(X)\}\right)^{\perp\perp}.$$

In order to extend the phase semantics of MALL to $\mu\mathsf{MALL}^{\mathsf{ind}}$ we extend valuations to variables *i.e.* for any valuation V, $\mathsf{dom}(V) = \mathcal{A} \cup \mathcal{V}$ and define $[\![F]\!]^V$ by induction on F with the usual interpretation of section 2.1 for atoms, units, and multiplicative-additive connectives and as follows for fixpoints formulas:

$$[\![\mu x.F]\!]^V = \bigcap_{X \in \mathcal{X}} \left\{ X \mid [\![F]\!]^{V[x \mapsto X]} \subseteq X \right\} \quad ; \quad [\![\nu x.F]\!]^V = \left(\bigcup_{X \in \mathcal{X}} \left\{ X \mid X \subseteq [\![F]\!]^{V[x \mapsto X]} \right\} \right)^{\perp\perp}$$

where $V[x \mapsto X](y) := \begin{cases} V(y) & \text{if } y \neq x; \\ X & \text{if } y = x. \end{cases}$.

However completeness fails for such an interpretation. Indeed, not all facts necessarily have a pre-image, therefore $[\![F]\!]^{V[x \mapsto X]}$ does not exactly correspond to syntactic substitution and *the tentative syntactic model is not a phase model.* We need to allow strict subsets of $\mathcal{X}$ for building fixpoints. Obviously, one cannot consider any subsets of $\mathcal{X}$ for this purpose and we shall require that they satisfy some closure properties. Therefore we restrict the codomain of $[\![\bullet]\!]^V$ to subspaces of $\mathcal{X}$ closed under μMALL operations. For any set of facts $\mathcal{D} \subseteq \mathcal{X}$, the set of contexts is given by the following grammar where $\odot \in \{\otimes, \parr, \&, \oplus\}$. Let $\mathbb{F}_{\mathcal{D}}$ denote the set of contexts with exactly one hole.

$$f, g ::= [\,] \mid X \in \mathcal{D} \mid f \odot g$$

For $f \in \mathbb{F}_{\mathcal{D}}$, define $\mu f = \bigcap_{X \in \mathcal{D}} \{X \mid f(X) \subseteq X\}$ and $\nu f = \left(\bigcup_{X \in \mathcal{D}} \{X \mid X \subseteq f(X)\}\right)^{\perp\perp}$.

▶ **Definition 26.** *$\mathcal{D} \subseteq \mathcal{X}$ is said to be μ-**closed** if*
- *$\{\perp, \perp^\perp, M, M^\perp\} \subseteq \mathcal{D}$;*
- *$\mathcal{D}$ is closed under the operations $\otimes, \parr, \&$, and $\oplus$; and*
- *for all $f \in \mathbb{F}_{\mathcal{D}}$, $\mu f \in \mathcal{D}$ and $\nu f \in \mathcal{D}$.*

A phase space $\mathcal{M}$ equipped with a μ-closed set of facts $\mathcal{D}$ is called a ***μ-phase space***.

A ***$\mathcal{D}$-valuation*** is a map of the form $V : \mathcal{A} \cup \mathcal{V} \to \mathcal{D}$. A μ-phase space along with a $\mathcal{D}$-valuation is called a ***μ-phase model***. The ***μ-phase semantics*** $[\![\bullet]\!]$ is a function that takes a μMALL pre-formula F and returns a fact in $\mathcal{D}$.

Note that $[\![\mu x.F]\!]^V$ and $[\![\nu x.F]\!]^V$ are defined as before except X ranges over $\mathcal{D}$. A priori, the semantics of pre-formula is only an element of $\mathcal{X}$. The closure properties of $\mathcal{D}$ ensures $[\![F]\!]^V \in \mathcal{D}$ for every formula F and $\mathcal{D}$-valuation V.

▶ **Lemma 27** (Monotonicity). *Let F be a μMALL pre-formula. If $X \subseteq Y$ then $[\![F]\!]^{V[x \mapsto X]} \subseteq [\![F]\!]^{V[x \mapsto Y]}$.*

Proof. A proof can be found in Appendix B.1.1. ◀

An application of monotonicity is showing that the interpretation of the fixpoint operators are indeed fixpoints in the mathematical sense:

▶ **Theorem 28.** *Let $\mathcal{D}$ be μ-closed and $f \in \mathbb{F}_{\mathcal{D}}$. Then μf and νf are the least and greatest fixpoints of f in $\mathcal{D}$.*

Proof. A proof can be found in Appendix B.1.2. ◀

Note that Theorem 28 cannot be proved directly by Theorem 14 since $\mathcal{D}$ is not necessarily a complete lattice. Moreover, it does not also imply that $\mathcal{D}$ is a complete lattice by the converse of Theorem 14 since we show that it has fixpoints of a particular kind of monotonic function, not any arbitrary monotonic function. Given a valuation V, define

$$V^{\perp}(p) = \begin{cases} V(p) & \text{if } p \in \mathcal{A}; \\ V(p)^{\perp} & \text{if } p \in \mathcal{V}. \end{cases}$$

We have $V^{\perp\perp} = V$ and $V[x \mapsto X]^{\perp} = V^{\perp}[x \mapsto X^{\perp}]$.

▶ **Lemma 29** (Duality preservation). *For any μMALL preformula F, $[\![F^{\perp}]\!]^{V^{\perp}} = ([\![F]\!]^{V})^{\perp}$.*

Proof. A proof can be found in Appendix B.1.3. ◀

3.1 Soundness

▶ **Theorem 30** (μMALL$^{\mathsf{ind}}$ Soundness). *If μMALL$^{\mathsf{ind}}$ $\vdash \Gamma$ then for all μ-phase models $(\mathcal{M}, \mathcal{D}, V)$, $1 \in [\![\Gamma]\!]^{V}$.*

Proof. Fix an arbitrary μ-phase model $(\mathcal{M}, \mathcal{D}, V)$. Given a proof π of $\vdash \Gamma$ we will induct on π. The proof is similar to that of Theorem 6 except for the fixpoint cases. Suppose the last rule of π is a (μ) rule. We have that $\Gamma = \Gamma', \mu x.F$. Assume that we have proved $[\![F(\mu x.F)]\!]^{V} \subseteq [\![\mu x.F]\!]^{V}$. We have the following:

$$\begin{aligned}
&\left([\![\mu x.F]\!]^{V}\right)^{\perp} \subseteq \left([\![F(\mu x.F)]\!]^{V}\right)^{\perp} && \text{[Proposition 4.3]}\\
&\Rightarrow \left([\![\Gamma']\!]^{V}\right)^{\perp}.\left([\![\mu x.F]\!]^{V}\right)^{\perp} \subseteq \left([\![\Gamma']\!]^{V}\right)^{\perp}.\left([\![F(\mu x.F)]\!]^{V}\right)^{\perp} \\
&\Rightarrow \left([\![\Gamma']\!]^{V^{\perp}}.[\![F(\mu x.F)]\!]^{V^{\perp}}\right)^{\perp} \subseteq \left([\![\Gamma']\!]^{V^{\perp}}.[\![\mu x.F]\!]^{V^{\perp}}\right)^{\perp} && \text{[Proposition 4.3]}\\
&\Leftrightarrow [\![\Gamma' \parr F(\mu x.F)]\!]^{V} \subseteq [\![\Gamma' \parr \mu x.F]\!]^{V} \\
&\Rightarrow 1 \in [\![\Gamma' \parr \mu x.F]\!]^{V} && \text{[IH]}
\end{aligned}$$

Therefore, it suffices to prove $[\![F(\mu x.F)]\!]^{V} \subseteq [\![\mu x.F]\!]^{V}$. Observe that $[\![F(\mu x.F)]\!]^{V} = [\![F]\!]^{V[x \mapsto [\![\mu x.F]\!]^{V}]}$. Let $X \in \mathcal{D}$ such that $[\![F]\!]^{V[x \mapsto X]} \subseteq X$ (we thus have $[\![\mu x.F]\!] \subseteq X$). We need to show that $[\![F]\!]^{V[x \mapsto [\![\mu x.F]\!]^{V}]} \subseteq X$. It suffices to show that $[\![F]\!]^{V[x \mapsto [\![\mu x.F]\!]^{V}]} \subseteq [\![F]\!]^{V[x \mapsto X]}$ which is true by Lemma 27.

Now suppose the last rule is a (ν) rule i.e. $\Gamma = \Gamma', \nu x.F$ such that the coinductive invariant is S. We need to show that $1 \in [\![\Gamma' \parr \nu x.F]\!]^{V}$ which by Proposition 5 is equivalent to showing $[\![\Gamma']\!]^{V^{\perp}} \subseteq [\![\nu x.F]\!]^{V}$. By hypothesis, we have that $1 \in [\![\Gamma' \parr S]\!]^{V}$ which is similarly equivalent to $[\![\Gamma']\!]^{V^{\perp}} \subseteq [\![S]\!]^{V}$. Therefore it suffices to show that $[\![S]\!]^{V} \subseteq [\![\nu x.F]\!]^{V}$:

$$\begin{aligned}
&1 \in [\![S^{\perp} \parr F(S)]\!]^{V} && \text{[IH]}\\
&\Leftrightarrow \left([\![S^{\perp}]\!]^{V}\right)^{\perp} \subseteq [\![F(S)]\!]^{V} && \text{[Proposition 5]}\\
&\Leftrightarrow [\![S]\!]^{V} \subseteq [\![F(S)]\!]^{V} = [\![F]\!]^{V[x \mapsto [\![S]\!]^{V}]} && \text{[Lemma 29]}\\
&\Rightarrow [\![S]\!]^{V} \subseteq [\![\nu x.F]\!]^{V}
\end{aligned}$$
◀

3.2 Completeness

Completeness for fixpoint logics are generally quite difficult since analyticity does not guarantee a subformula property. One is faced with a similar *cul de sac* in proving the cut-elimination of μMALL$^{\mathsf{ind}}$ since it is not straightforward to define the notion of the complexity of the cut formula which reduces with each step of cut-elimination. This problem is solved in [5] by invoking a technique similar to the Tait-Girard reducibility candidates (originally formulated to establish certain properties of various typed lambda calculi [48, 29]). Recall that the completeness of phase semantics gives cut admissibility for free. Therefore, it is not surprising that in order to prove completeness one needs to invoke reducibility candidates.

▶ **Definition 31.** *Let $(\mathcal{M}, \mathcal{D}, V)$ be a μ-phase model. Given a μMALL formula F, the* **reducibility candidates** *of F, denoted $\langle F\rangle$, is given by $\{X \in \mathcal{X} \mid F^\perp \in X \subseteq \mathsf{Pr}_{cf}(F)\}$.*

▶ **Proposition 32.** $X \in \langle F\rangle \iff X^\perp \in \langle F^\perp\rangle$

Proof. We first note that $\mathsf{Pr}_{cf}(\bullet)$ can be straightforwardly generalised to sets of formulas as follows: $\mathsf{Pr}_{cf}(\{F_1, \ldots, F_n\}) = \bigcup_{i\in[n]} \mathsf{Pr}_{cf}(F_i)$. Let $X \in \langle F\rangle$. Then $\{F^\perp\} \subseteq X \implies X^\perp \subseteq \{F^\perp\}^\perp = \mathsf{Pr}_{cf}(F^\perp)$. Also, $X \subseteq \mathsf{Pr}_{cf}(F) \implies \mathsf{Pr}_{cf}(F)^\perp = \mathsf{Pr}_{cf}(\mathsf{Pr}_{cf}(F)) \subseteq X$. But $F \in \mathsf{Pr}_{cf}(\mathsf{Pr}_{cf}(F))$. Hence done. ◀

We are now ready to define the μ-syntactic model. Recall that $\mathsf{Pr}_{cf}(F)$ is the set of all sequents Γ such that $\vdash \Gamma, F$ is *cut-free provable.*

▶ **Definition 33.** *The* **μ-syntactic model***, denoted $(\mu\mathsf{MALL}^\bullet, \emptyset, \cdot, \perp\!\!\!\perp, V)$, is defined as:*

- *$(\mu\mathsf{MALL}^\bullet, \emptyset, \cdot)$ is the free commutative monoid generated by all formulas.*
- *$\perp\!\!\!\perp = \mathsf{Pr}_{cf}(\perp)$.*
- *$V(p) = \mathsf{Pr}_{cf}(p)$ for all $p \in \mathcal{A} \cup \mathcal{V}$.*
- *$\mathcal{D} = \bigcup_{F\in Form}\langle F\rangle$ where $Form$ is the set of all μMALL formulas.*

Observe that $\perp\!\!\!\perp = \mathsf{Pr}_{cf}(\perp) \in \mathcal{D}$ and that $\mathcal{D}$ indeed contains $\perp\!\!\!\perp^\perp$, $\mu\mathsf{MALL}^\bullet$ and $\mu\mathsf{MALL}^{\bullet\perp}$.

▶ **Lemma 34** (Adequation Lemma for $\mu\mathsf{MALL}^{\mathsf{ind}}$)**.** *Let $F(\overline{x})$ be a pre-formula, $\overline{G} \equiv G_1, \ldots, G_m$ with $\overline{x} \equiv x_1, \ldots, x_m$ be an m-tuple of pre-formulas and let $\overline{X} \equiv X_1 \ldots X_m$ be an m-tuple of facts such that $X_i \in \langle G_i\rangle$. We have $[\![F]\!]^{V[\overline{x}\mapsto\overline{X}]} \subseteq \mathsf{Pr}_{cf}(F(\overline{G}/\overline{x}))$.*

Proof. By induction on F. Multiplicative additive cases are treated as in the proof of Lemma 10; we detail fixed-point cases only.

Case 1. Suppose $F = \mu y.F'$. Let $\xi = \mu y.F'(\overline{G}/\overline{x})$. In the proof of Theorem 30, we showed that for any formula F_0, $\mathsf{Pr}_{cf}(F_0(\mu x.F_0/x)) \subseteq \mathsf{Pr}_{cf}(\mu x.F_0)$. Therefore, $\mathsf{Pr}_{cf}(F'(\overline{G}/\overline{x}, \xi/y)) \subseteq \mathsf{Pr}_{cf}(\xi)$. Let $Y^* = [\![F']\!]^{V[\overline{x}\mapsto\overline{X}, y\mapsto\mathsf{Pr}_{cf}(\xi)]}$. By induction hypothesis and that fact that $\mathsf{Pr}_{cf}(F_0) \in \langle F_0\rangle$ for all formulas F_0, we have the following.

$$Y^* \subseteq \mathsf{Pr}_{cf}(F'(\overline{G}/\overline{x}, \xi/y)) \tag{1}$$

Therefore, it is enough to show that $[\![\mu y.F']\!]^{V[\overline{x}\mapsto\overline{X}]} \subseteq Y^*$. Take $\Gamma \in [\![\mu y.F']\!]^{V[\overline{x}\mapsto\overline{X}]}$. For any fact $Y \in \mathcal{D}$, to show $\Gamma \in Y$ it is enough to show $[\![F']\!]^{V[\overline{x}\mapsto\overline{X}, y\mapsto Y]} \subseteq Y$. Therefore we need to check that $[\![F']\!]^{V[\overline{x}\mapsto\overline{X}, y\mapsto Y^*]} \subseteq Y^* = [\![F']\!]^{V[\overline{x}\mapsto\overline{X}, y\mapsto\mathsf{Pr}_{cf}(\xi)]}$. This follows by Lemma 27 from 1.

Case 2. Suppose $F = \nu y.F'$. For any fact Y, define $Z_Y = [\![F']\!]^{V[\overline{x}\mapsto\overline{X}, y\mapsto Y]}$. Let $\Gamma \in \bigcup\{Y \in \mathcal{D} \mid Y \subseteq Z_Y\}$. Therefore, there exists, $Y^* \in \mathcal{D}$ such that $\Gamma \in Y^* \subseteq Z_{Y^*}$. Since $Y^* \in \mathcal{D}$, $Y^* \in \langle\xi\rangle$ for some formula ξ. By induction hypothesis, $[\![F']\!]^{V[\overline{x}\mapsto\overline{X}, y\mapsto Y^*]} \subseteq \mathsf{Pr}_{cf}(F'(\overline{G}/\overline{x}, \xi/y))$.

We will now show that $\mathsf{Pr}_{cf}(F'(\overline{G}/\overline{x}, \xi/y)) \subseteq \mathsf{Pr}_{cf}(F(\overline{G}/\overline{x}))$. Let $\Delta \in \mathsf{Pr}_{cf}(F'(\overline{G}/\overline{x}, \xi/y))$. If we show that $\xi^\perp \in \mathsf{Pr}_{cf}(F'(\overline{G}/\overline{x}, \xi/y))$, we have the following.

$$\dfrac{\vdash \Delta, F'(\overline{G}/\overline{x},\xi/y) \quad \dfrac{\vdash \xi^\perp, F'(\overline{G}/\overline{x},\xi/y)}{\vdash (F'(\overline{G}/\overline{x},\xi/y))^\perp, F'(\overline{G}/\overline{x}, F'(\overline{G}/\overline{x},\xi/y)/y)}\,(\mathsf{func})}{\vdash \Delta, \nu x.F'(\overline{G}/\overline{x})}\,(\nu)$$

In order to show $\xi^\perp \in \mathsf{Pr}_{cf}(F'(\overline{G}/\overline{x},\xi/y))$, we use the induction hypothesis to reduce the problem to showing $\xi^\perp \in [\![F']\!]^{V[\overline{x}\mapsto\overline{X},y\mapsto Y^*]}$ which is true since $Y^* \in \langle\xi\rangle$. Therefore we have,

$$\begin{aligned}\Gamma \in \mathsf{Pr}_{cf}(\nu y.F'(\overline{G}/\overline{x})) &\Rightarrow \bigcup\{Y \in \mathcal{D} \mid Y \subseteq Z_Y\} \subseteq \mathsf{Pr}_{cf}(\nu y.F'(\overline{G}/\overline{x}))\\ &\Rightarrow \left(\bigcup\{Y \in \mathcal{D} \mid Y \subseteq Z_Y\}\right)^{\perp\perp} \subseteq \mathsf{Pr}_{cf}(\nu y.F'(\overline{G}/\overline{x}))\end{aligned}$$

This concludes our proof. ◀

▶ **Lemma 35.** *Using notations of the previous lemma, we have that* $F^\perp(\overline{G}/\overline{x}) \in [\![F]\!]^{V[\overline{x}\mapsto\overline{X}]}$.

Proof. Observe that Lemma 34 and Proposition 32 imply $[\![F^\perp]\!]^{V[\overline{x}\mapsto\overline{X^\perp}]} \subseteq \mathsf{Pr}_{cf}(F^\perp(\overline{G^\perp}/\overline{x}))$. Therefore, $\{F^\perp(\overline{G^\perp}/\overline{x})\}[\![F^\perp]\!]^{V[\overline{x}\mapsto\overline{X^\perp}]} \subseteq \{F^\perp(\overline{G^\perp}/\overline{x})\}.\mathsf{Pr}_{cf}(F^\perp(\overline{G^\perp}/\overline{x})) \subseteq \mathsf{Pr}_{cf}(\perp) = \perp\!\!\!\perp$. By Proposition 4.1, $F^\perp(\overline{G^\perp}/\overline{x}) \subseteq \left([\![F^\perp]\!]^{V[\overline{x}\mapsto\overline{X^\perp}]}\right)^\perp$ which is $[\![F]\!]^{V[\overline{x}\mapsto\overline{X}]}$ by Lemma 29. ◀

Observe that by Lemma 34 and Lemma 35 we have $[\![F]\!]^{V[x\mapsto X]} \in \langle F(\overline{G}/\overline{x})\rangle$. This ensures that $\mathcal{D}$ is μ-closed. Consequently, $(\mu\mathsf{MALL}^\bullet, \emptyset, \cdot, \perp\!\!\!\perp, V)$ is a μ-phase space model.

▶ **Theorem 36** ($\mu\mathsf{MALL}^{\mathsf{ind}}$ cut-free completeness)**.** *If for any μ-phase model $(\mathcal{M},\mathcal{D},V)$, $1 \in [\![\Gamma]\!]^V$ then $\mu\mathsf{MALL}^{\mathsf{ind}} \vdash_{cf} \Gamma$.*

▶ **Corollary 37.** $\mu\mathsf{MALL}^{\mathsf{ind}}$ *admits cuts.*

▶ Remark 38. Note that this is an alternate proof of Theorem 25.

3.3 Closure ordinals

Closure ordinals are a standard measure of the complexity of any (class of) monotone functions. The closure ordinal of a fixed-point formula is essentially the closure ordinal of the corresponding monotone function in the truth semantics. In [21], the closure ordinal is construed as a function of the size of the finite model. The study of closure ordinals of modal logic formulas is a young and exciting area of research [18, 2, 31]. It departs from the previous notion of closure ordinals in its model-independence. In this case, closure ordinal really serves as a measure of complexity of a formula.

▶ **Definition 39.** *Let F be a pre-formula such that $x \in \mathsf{fv}(F)$ and let $\mathbb{M} = (\mathcal{M},\mathcal{D},V)$ be a μ-phase model. We define the* **closure ordinal** *of F with respect to x and $\mathbb{M}$, denoted $\mathcal{O}_{\mathbb{M}}(F)$, as the closure ordinal of $\lambda X.[\![F]\!]^{V[x\mapsto X]}$. The closure ordinal of F with respect to x (across all models) is defined as $\mathcal{O}(F) := \sup_{\mathbb{M}}\{\mathcal{O}_{\mathbb{M}}(F)\}$. Finally, F is said to be* **constructive** *if its closure ordinal is at most ω.*

For any pre-formula F and phase model $\mathbb{M}$, $\mathcal{O}_{\mathbb{M}}(F)$ exists since $\lambda X.[\![F]\!]^{V[x\mapsto X]}$ is monotonic. Consequently, the supremum $\mathcal{O}(F)$ exists since the class of all μ-phase models is indeed a set.

▶ **Example 40.** Consider the pre-formula $a \& x$. Let $\langle \mathcal{U}^{a\&x}_\alpha \mid \alpha \in \mathsf{Ord}\rangle$ be the iteration sequence of its approximations. $\mathcal{U}^{a\&x}_0 = \emptyset^{\perp\perp}$ and $\mathcal{U}^{a\&x}_1 = [\![a\&x]\!]^{V[x\mapsto \mathcal{U}^{a\&x}_0]} = V(a) \cap \emptyset^{\perp\perp}$. Therefore, $\mathcal{O}(a\&x) = 0$.

In the tradition of μ-calculus, the name "constructive" is used loosely here, motivated by the observation that if $\mathcal{O}(F)$ is a finite ordinal strictly below ω for any pre-formula F, then $\mu x.F$ is provably equivalent to $F^{\mathcal{O}(F)}(0)$. Therefore, the class of μMALL formulas with closure ordinal strictly less than ω can be embedded in MALL and enjoys several good properties like finite model property and decidability. Observe that if the interpretation of a formula in any phase model is Scott-continuous, then it is constructive by Theorem 16. The converse does not hold in general. In the following section we consider a proof system of μMALL where fixed points are approximated by their ωth approximation.

4 μMALL$_\omega$: an infinitary proof system

For a constructive formula F,

$$[\![\mu x.F]\!]^V = \left(\bigcup_{n\geq 0} [\![F^n(\mathbf{0})]\!]^V\right)^{\perp\perp} \quad ; \quad [\![\nu x.F]\!]^V = \bigcap_{n\geq 0} [\![F^n(\top)]\!]^V \tag{2}$$

Therefore, syntactically, $\mu x.F$ (respectively $\nu x.F$) is equivalent to an infinitary $\oplus$-formula $\bigoplus_{n\in\omega} F^n(\mathbf{0})$ (respectively, an infinitary $\&$-formula $\&_{n\in\omega} F^n(\top)$). We enrich the language of μMALL with the operators μ^n and ν^n for all $n \in \omega$. We call this language as well as the corresponding proof system, μMALL$_\omega$. (See Appendix A for details.)

▶ **Definition 41.** *A μMALL$_\omega$ proof is a wellfounded (possibly infinitely branching) tree generated from the inference rules of* MALL *given in Figure 1 and the following rules for fixpoint operators where $\eta \in \{\mu, \nu\}$.*

$$\frac{\vdash \Gamma, \mathbf{0}}{\vdash \Gamma, \mu^0 x.F}\,(\mu^0) \quad ; \quad \frac{\vdash \Gamma, F(\eta^n x.F/x)}{\vdash \Gamma, \eta^{n+1} x.F}\,(\eta^{n+1}) \quad ; \quad \frac{\vdash \Gamma, \mu^n x.F}{\vdash \Gamma, \mu x.F}\,(\mu^\omega_n)$$

$$\frac{\vdash \Gamma, \top}{\vdash \Gamma, \nu^0 x.F}\,(\nu^0) \quad ; \quad \frac{\vdash \Gamma, \nu^0 x.F \quad \vdash \Gamma, \nu^1 x.F \quad \vdash \Gamma, \nu^2 x.F \quad \ldots}{\vdash \Gamma, \nu x.F}\,(\nu^\omega)$$

▶ **Example 42.** Let $H = (\mu x.a \parr x)^\perp \parr (a^{\perp p} \parr \mathbf{0})$ for some $p \in \omega$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash a^\perp, a}\,(\mathsf{id}) \quad \dfrac{\vdash \mu^p x.a^\perp \parr x, (a^{\perp p} \parr \mathbf{0})^\perp \quad \dfrac{\dfrac{\{\vdash \nu^n y.F, \mu^{2n} x.a \parr x\}_{n\in\omega}}{\{\vdash \nu^n y.F, \mu x.a \parr x\}_{n\in\omega}}\,(\mu^\omega_{2n})}{\vdash \nu y.F, \mu x.a \parr x}\,(\nu^\omega)}{\vdash \mu^p x.a^\perp \parr x, \nu y.F, H^\perp}\,(\otimes)}{\vdash a^\perp, \mu^p x.a^\perp \parr x, a \otimes (\nu y.F), H^\perp}\,(\otimes)}{\vdash a^\perp \parr (\mu^p x.a^\perp \parr x), a \otimes (\nu y.F), H^\perp}\,(\parr)}{\vdash \mu^{p+1} x.a^\perp \parr x, a \otimes (\nu y.F), H^\perp}\,(\mu^{p+1})}{\vdash \Gamma_0, H^\perp}\,(\mu^\omega_{p+1})$$

It is easy to show that for all $p, n \in \omega$, $\vdash \mu^p x.a^\perp \parr x, (a^{\perp p} \parr \mathbf{0})^\perp$ and $\vdash \nu^n y.F, \mu^{2n} x.a \parr x$ are provable by induction on p and n respectively.

Note that the set of inference rules (as schema) is infinite since $\{\mu^\omega_n\}_{n\in\omega}$ is a collection of ω-many rules. Such infinitary systems (called Tait-style systems) where proof trees are wellfounded with possible infinite branching are studied in various areas of logic *viz.* arithmetic [15, 41] and fixpoint logics [35, 32]. It is quite tricky to define a proper notion of the complexity of a fixpoint formula in such settings. Closely based on [32], our notion of the rank of a $\mu\mathsf{MALL}_\omega$ formula is a finite sequences of ordinals.

First we will set up some notion. If $\alpha_1,\dots,\alpha_n$ are ordinals, we write $\langle\alpha_1,\dots,\alpha_n\rangle$ for the sequence σ whose length $|\sigma|$ is n and whose ith component σ_i is the ordinal α_i. Let $<_{lex}$ be the strict lexicographical ordering of finite sequences of ordinals and $\leq_{lex}$ its reflexive closure. Note that $<_{lex}$ is a well-ordering on any set of sequences of bounded lengths but not a well-ordering in general. In particular, $\langle 1\rangle,\langle 0,1\rangle,\langle 0,0,1\rangle,\dots$ is an infinite descending chain in $<_{lex}$. Given two finite sequences of ordinals σ,τ, we define the component-wise ordering $\trianglelefteq$ as σ,τ iff $|\sigma|\leq|\tau|$ and $(\sigma)_i\leq(\tau)_i$ for all $1\leq i\leq|\sigma|$. Clearly, the relation $\trianglelefteq$ is transitive. We denote the standard concatenation of sequences by $*$. Finally we define a component-wise maximum operation $\sqcup$ by setting: (i) $\sigma\sqcup\langle\rangle:=\langle\rangle$; (ii) if $\sigma=\langle b_1,\dots,b_m\rangle$ and $\tau=\langle b'_1,\dots,b'_n\rangle$, then

$$\sigma\sqcup\tau=\begin{cases}\langle\max(b_1,b'_1),\dots,\max(b_m,b'_m),b'_{m+1},\dots,b'_n\rangle & \text{if } m\leq n;\\ \langle\max(b_1,b'_1),\dots,\max(b_n,b'_n),b_{n+1},\dots,b_m\rangle & \text{otherwise.}\end{cases}$$

Now we are ready to define the rank of a $\mu\mathsf{MALL}_\omega$ formula. The rank of every $\mu\mathsf{MALL}_\omega$ formula will be a finite sequence of ordinals less than or equal to ω.

▶ **Definition 43.** *The* **rank** *of a $\mu\mathsf{MALL}_\omega$ formula F, denoted $\mathsf{rk}(F)$, is defined by induction on F as follows.*

- *if F is an atom, a variable, or a unit, then $\mathsf{rk}(F)=\langle 0\rangle$;*
- *if $F=G\odot G'$, then $\mathsf{rk}(F)=(\mathsf{rk}(G)\sqcup\mathsf{rk}(G'))*\langle 0\rangle$ where $\odot\in\{\parr,\otimes,\oplus,\&\}$;*
- *if $F=\eta^n x.G$, then $\mathsf{rk}(F)=\mathsf{rk}(G)*\langle n\rangle$ where $\eta\in\{\mu,\nu\}$;*
- *if $F=\eta x.G$, then $\mathsf{rk}(F)=\mathsf{rk}(G)*\langle\omega\rangle$ where $\eta\in\{\mu,\nu\}$.*

▶ **Example 44.** Consider $H=(\nu x.a^\perp\otimes x)\parr(a^p\parr\mathbf{0})$ from Example 42.

$$\begin{aligned}\mathsf{rk}(H)\;&=\mathsf{rk}(\nu x.a^\perp\otimes x)\sqcup\mathsf{rk}(a^p\parr\mathbf{0})*\langle 0\rangle\;=\left(\left(\mathsf{rk}(a^\perp\otimes x)*\langle\omega\rangle\right)\sqcup\langle\overbrace{0,\dots,0}^{p+1}\rangle\right)*\langle 0\rangle\\&=\left(\langle 0,0,\omega\rangle\sqcup\langle\overbrace{0,\dots,0}^{p+1}\rangle\right)*\langle 0\rangle\quad=\langle 0,0,\omega,\overbrace{0,\dots,0}^{\max(1,p-1)}\rangle\end{aligned}$$

▶ **Lemma 45.** *Let F be a $\mu\mathsf{MALL}_\omega$ pre-formula such that $x\in\mathsf{fv}(F)$. Let ξ be a preformula such that $\mathsf{rk}(F)\trianglelefteq\mathsf{rk}(\xi)$. Then, there exists a finite (possibly empty) sequence of ordinals σ such that $\mathsf{rk}(F(\xi/x))=\mathsf{rk}(\xi)*\sigma$.*

Proof. A proof can be found in Appendix B.2.1. ◀

▶ **Theorem 46.** *The following hold for any $\mu\mathsf{MALL}_\omega$ formula F:*

1. $\mathsf{rk}(F)<_{lex}\mathsf{rk}(F\odot G)$ *and* $\mathsf{rk}(G)<_{lex}\mathsf{rk}(F\odot G)$*;*
2. $\mathsf{rk}(\mathbf{0})<_{lex}\mathsf{rk}(\mu^0x.F)$, $\mathsf{rk}(\top)<_{lex}\mathsf{rk}(\nu^0x.F)$*;*
3. $\mathsf{rk}(F(\eta^n x.F/x))<_{lex}\mathsf{rk}(\eta^{n+1}x.F)$ *for all* $n<\omega$*;*
4. $\mathsf{rk}(\eta^n x.F)<_{lex}\mathsf{rk}(\eta x.F)$ *for all* $n<\omega$.

Proof. The first, second, and fourth assertions are immediate from Definition 43. For the third one, we have two cases:

- Suppose $x \notin \mathsf{fv}(F)$. Then, $F(\eta^n x.F/x) = F$ and the result follows from Definition 43.
- Otherwise, note that we have $\mathsf{rk}(F) \trianglelefteq \mathsf{rk}(\eta^n x.F)$. By Lemma 45, $\mathsf{rk}(F(\eta^n x.F/x)) = \mathsf{rk}(\eta^n x.F) * \sigma = \mathsf{rk}(F) * \langle n\rangle * \sigma$ for some σ. But $\mathsf{rk}(\eta^{n+1}x.F) = \mathsf{rk}(F) * \langle n+1\rangle$.

Hence we are done. ◀

▶ **Definition 47.** *The* **strong closure** $\mathbb{SC}(F)$ *of a* $\mu\mathsf{MALL}_\omega$ *formula* F *is the least set s.t.:*

- $F \in \mathbb{SC}(F)$*;*
- $G \odot H \in \mathbb{SC}(F) \implies \{G, H\} \subset \mathbb{SC}(F)$ *where* $\odot \in \{⅋, \otimes, \oplus, \&\}$*;*
- $\mu^0 x.G \in \mathbb{SC}(F) \implies \mathbf{0} \in \mathbb{SC}(F)$*;*
- $\nu^0 x.G \in \mathbb{SC}(F) \implies \top \in \mathbb{SC}(F)$*;*
- $\eta^{n+1}x.G \in \mathbb{SC}(F) \implies G(\eta^n x.G/x) \in \mathbb{SC}(F)$ *for all* $n \in \omega$ *and* $\eta \in \{\mu, \nu\}$*;*
- $\eta x.G \in \mathbb{SC}(F) \implies \eta^n x.G \in \mathbb{SC}(F)$ *for all* $n \in \omega$ *and* $\eta \in \{\mu, \nu\}$*.*

Define F^- to be image of F under the forgetful functor that erases the explicit approximations occurring in F. For example, $(a \otimes \mu^6 x.\nu y.x \oplus y)^- = a \otimes \mu x.\nu y.x \oplus y$.

▶ **Theorem 48.** *For any formula* F*, the set* $\{\mathsf{rk}(G) \mid G \in \mathbb{SC}(F)\}$ *is a well-order with respect to the* $<_{lex}$ *ordering.*

Proof. By contradiction. Note that $|\mathsf{rk}(G)| = |\mathsf{rk}(G^-)|$. Furthermore, if $G \in \mathbb{SC}(F)$ then $G^- \in \mathbb{FL}(F)$. But $\mathbb{FL}(F)$ is a finite set, therefore the set $\{|\mathsf{rk}(G)| \mid G \in \mathbb{SC}(F)\}$ is finite.

Assume there exists $\{\sigma_i\}_{i\in I}$ an infinite descending chain in $\{\mathsf{rk}(G) \mid G \in \mathbb{SC}(F)\}$. By the Infinite Ramsey Theorem, there is an infinite subsequence $\{\sigma_i\}_{i\in\omega}$ such that for all j, $i_j \in I$ and for all j, j', $|\sigma_{i_j}| = |\sigma_{i_{j'}}| = n$. Then, there exist $k \leq n$ and $N \in \mathbb{N}$ such that $\{(\sigma_{i_j})_k\}_{j>N}$ is a descending chain. This contradicts the wellfoundedness of natural numbers. ◀

4.1 Soundness and completeness of $\mu\mathsf{MALL}_\omega$

The phase semantics for $\mu\mathsf{MALL}_\omega$ is much simpler to define. Like MALL the semantics can be defined given a phase space and a valuation (without the extra structure over the set of facts and extension of valuations to variables as in $\mu\mathsf{MALL}^{\mathsf{ind}}$). Given a phase space $\mathcal{M}$ and a valuation function $V : \mathcal{A} \to \mathcal{X}$, we define the interpretation of multiplicative-additive connectives and units as usual, of $\mu x.F$ and $\nu x.F$ as in Equation (2), and of $\eta^n x.F$ (for $\eta \in \{\mu, \nu\}$) as follows: $[\![\mu^0 x.F]\!] = [\![\mathbf{0}]\!]$, $[\![\nu^0 x.F]\!] = [\![\top]\!]$, and $[\![\eta^{n+1}x.F]\!] = [\![F(\eta^n x.F/x)]\!]$.

▶ **Theorem 49** ($\mu\mathsf{MALL}_\omega$ soundness)**.** *If* $\vdash \Gamma$ *then for all phase models* $(\mathcal{M}, V)$*,* $1 \in [\![\Gamma]\!]^V$*.*

The proof of Theorem 49 is by a straightforward induction on the structure of the proof.

▶ **Lemma 50** (Adequation Lemma for $\mu\mathsf{MALL}_\omega$)**.** *For all formula* F*,* $[\![F]\!]^V \subseteq \mathsf{Pr}_{cf}(F)$*.*

Proof. By induction on $\mathsf{rk}(F)$. The base case is when F is an atom or a unit in which case by definition $[\![F]\!] = \mathsf{Pr}_{cf}(F)$. Suppose $F = G \odot H$ for $\odot \in \{\otimes, ⅋, \&, \oplus\}$. By Theorem 46.1, $\mathsf{rk}(G) <_{lex} \mathsf{rk}(F)$ and $\mathsf{rk}(H) <_{lex} \mathsf{rk}(F)$. By IH, $[\![G]\!]^V \subseteq \mathsf{Pr}_{cf}(G)$ and $[\![H]\!]^V \subseteq \mathsf{Pr}_{cf}(H)$. Using standard techniques in the proof of Lemma 10 (cf. [30]), we can conclude that $[\![F]\!] = \mathsf{Pr}_{cf}(F)$.

The case $F = \eta^0 x.G$ is trivial. Suppose $F = \eta^{n+1}x.G$ where $\eta \in \{\mu, \nu\}$. By Theorem 46.3, $\mathsf{rk}(G(\eta^n x.G)) <_{lex} \mathsf{rk}(\eta^{n+1}x.G)$. By IH, $[\![F]\!] = [\![G(\eta^n x.G)]\!]^V \subseteq \mathsf{Pr}_{cf}(G(\eta^n x.G)) \subseteq \mathsf{Pr}_{cf}(F)$.

Suppose $F = \mu x.G$. Noting that $\mathsf{Pr}_{cf}(F)$ is a fact for all F, it is enough to show that $\bigcup_{n\geq 0}[\![\mu^n x.G]\!] \subseteq \mathsf{Pr}_{cf}(\mu x.G)$. By Theorem 46.4, $\mathsf{rk}(\mu^n x.G) <_{lex} \mathsf{rk}(F)$. By IH, for all n, $[\![\mu^n x.G]\!] \subseteq \mathsf{Pr}_{cf}(\mu^n x.G)$. Observe that $\mathsf{Pr}_{cf}(\mu^n x.G) \subseteq \mathsf{Pr}_{cf}(\mu x.G)$ for all n. Therefore, $\bigcup_{n\geq 0} \mathsf{Pr}_{cf}(\mu^n x.G) \subseteq \mathsf{Pr}_{cf}(\mu x.G)$. The case for $F = \nu x.G$ works similarly. ◀

▶ **Theorem 51** (μMALL$_\omega$ cut-free completeness). *If for any phase model $(\mathcal{M}, V)$, $1 \in [\![\Gamma]\!]^V$ then $\mu\mathsf{MALL}_\omega \vdash_{cf} \Gamma$.*

▶ **Corollary 52.** *$\mu\mathsf{MALL}_\omega$ admits cuts.*

Standard cut-elimination techniques for Tait-like systems employ techniques from Schütte [45] where proofs are assigned a cut rank. One shows that if there is a proof π of a sequent $\vdash \Gamma$ with cut-rank $\mathsf{rk}(\pi) > 0$ then there is a proof π' of $\vdash \Gamma$ such that $\mathsf{rk}(\pi') = 0$ (possibly incurring a blowup in the size of the proof). Cut-admissibility has been proved previously for Tait-style systems of fixpoints logics in [43, 14] using this technique. Semantic proofs of cut-admissibility have been explored in various logics [46, 39, 4] but, to our knowledge, this is the first semantic proof of cut-admissibility in a Tait-style system.

4.2 Finite model property

In this subsection, we show that $\mu\mathsf{MALL}_\omega$ does not have the finite model property. Since MALL has finite model property, this implies that $\mu\mathsf{MALL}_\omega$ cannot be embedded in MALL.

▶ **Lemma 53.** *$\vdash \Gamma_0$ is not provable in $\mu\mathsf{MALL}_\omega$.*

Proof. A proof can be found in Appendix B.2.2. ◀

▶ **Corollary 54.** *$\mu\mathsf{MALL}_\omega$ does not prove the same theorems as $\mu\mathsf{MALL}^{\mathsf{ind}}$.*

Proof. From Example 23 obtain $\mu\mathsf{MALL}^{\mathsf{ind}} \vdash \Gamma_0$. The result now follows from Lemma 53. ◀

▶ **Theorem 55.** *$\mu\mathsf{MALL}_\omega$ does not have finite model property.*

Proof. The proof goes by contradiction. A finite phase model $(\mathcal{M}, V)$ has finitely many facts. Therefore, by pigeonhole principle, there exists p, q such that $p < q$ and $[\![a^p \parr \mathbf{0}]\!]^V = [\![a^q \parr \mathbf{0}]\!]^V$. Therefore, $[\![\mu x.a \parr x]\!]^V = [\![a^p \parr \mathbf{0}]\!]^V$. Consequently, $1 \in [\![H]\!]^V$ where $H = (\mu x.a \parr x)^\perp \parr (a^{\perp p} \parr \mathbf{0})$. By Theorem 51, $\mu\mathsf{MALL}_\omega \vdash H$. In Example 42, we show that $\mu\mathsf{MALL}_\omega \vdash \Gamma_0, H^\perp$. By an application of the cut-rule, we have $\mu\mathsf{MALL}_\omega \vdash \Gamma_0$ is provable. This is a contradiction by Lemma 53. ◀

5 Conclusion

In this work, we provided a sound and complete provability semantics for linear logic with fixpoints (μMALL) for the proof system with explicit induction due to Baelde and Miller [5, 7]. The completeness proof goes via an adaptation of Tait-Girard reducibility candidates. We then introduced a Tait-style proof system where fixpoints are approximated by their ω-th approximants ($\mu\mathsf{MALL}_\omega$) and we get a direct proof of completeness. Finally, we show that $\mu\mathsf{MALL}_\omega$ does not have finite model property and hence is a logic with non-trivial expressivity.

We conclude by mentioning pertinent directions for future work.

- Exploring the phase semantics of $\mu\mathsf{MALL}^{\circlearrowleft}$ and $\mu\mathsf{MALL}^{\infty}$, circular and non-wellfounded systems for μMALL respectively [6, 23], are relevant questions especially because that would help settle the Brotherston-Simpson conjecture by semantic techniques. If the conjecture is true, then it suffices to show that $\mu\mathsf{MALL}^{\circlearrowleft}$ is sound with respect to μ-phase models. If the conjecture is false, there are two possibilities for using phase semantics. Firstly, if we have a $\mu\mathsf{MALL}^{\circlearrowleft}$ theorem F which we wish to prove is not a $\mu\mathsf{MALL}^{\mathsf{ind}}$ theorem, then we can use the μ-phase model. Secondly, if one can devise the phase semantics of $\mu\mathsf{MALL}^{\circlearrowleft}$, then the proof is reduced to checking that the space of μ-phase models and its counterpart for $\mu\mathsf{MALL}^{\circlearrowleft}$ are the same.

- The computation of closure ordinals in phase semantics seems quite difficult and deserves a closer study. An important problem is to compute an upper bound possibly by embedding $\mu\mathsf{MALL}^{\mathsf{ind}}$ in some arithmetic theory.
- Techniques involving cut ranks used to obtain cut-admissibility also provide upper bounds on the size of the cut-free proof. It would be interesting to see if Theorem 51 can be refined to obtain such bounds.
- $\mu\mathsf{MALL}^{\mathsf{ind}}$ has the focusing property [7] but assigning polarities to fixed point operators is not a priori clear. In fact, it holds for both possible assignments (the proofs being quite different). In the implicit case, one can syntactically argue that μ has to be positive (consequently ν should be negative). Categorical semantics of polarised $\mu\mathsf{MALL}^{\mathsf{ind}}$ informs us that μ should indeed be positive [25]. Can phase semantics also shed light on the polarisation of fixed points?
- We showed that $\mu\mathsf{MALL}^{\mathsf{ind}} \not\subseteq \mu\mathsf{MALL}_\omega$. Noting that the Park's rule can be simulated with the (ν_ω) rule, we conjecture that $\mu\mathsf{MALL}_\omega \not\subseteq \mu\mathsf{MALL}^{\mathsf{ind}}$. Moreover, $\mu\mathsf{MALL}^{\mathsf{ind}}$ can encode the provability of linear logic exponentials but this is not clear for $\mu\mathsf{MALL}_\omega$. We conjecture that $\mu\mathsf{MALL}_\omega$ cannot simulate *digging*, but it can, indeed, encode *soft promotion* and *multiplexing*, thereby being at least as powerful as soft linear logic [37].

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A Summary of μMALL systems used in the paper

μMALL preformulas:

$$F, G \quad ::= \quad \mathbf{0} \mid \top \mid \bot \mid \mathbf{1} \mid a \mid a^{\perp} \mid F \parr G \mid F \otimes G \mid F \oplus G \mid F \& G \\ \mid x \mid \mu x.F \mid \nu x.F$$

μMALL$^{\mathsf{approx}}$, preformulas with fixed-point approximants:

$$F, G \quad ::= \quad \mathbf{0} \mid \top \mid \bot \mid \mathbf{1} \mid a \mid a^{\perp} \mid F \parr G \mid F \otimes G \mid F \oplus G \mid F \& G \\ \mid x \mid \mu x.F \mid \nu x.F \mid \mu^n x.F \mid \nu^n x.F \qquad n \in \mathbb{N}$$

	Formula language	Inference rules	Proof trees
μMALL$^{\mathsf{ind}}$	μMALL	Definition 22	finite
μMALL$_{\omega}$	μMALL$^{\mathsf{approx}}$	Definition 41	wellfounded infinitely branching
μMALL$^{\infty}$	μMALL	Figure 1 of [6]	non-wellfounded finitely branching
μMALL$^{\circlearrowleft}$	μMALL	Figure 1 of [6]	non-wellfounded regular

▶ Remark. Note that, contrary to the convention used in the present paper, in [6] notation μMALL$_{\omega}$ is used to refer to the circular proof system (that is regular non-wellfounded trees).

B Detailed proofs and clarifications

B.1 Proofs of Section 3

B.1.1 Proof of Lemma 27

Before we prove the lemma, we will prove the following claim.

▷ Claim 56. If S, T, U, V are subsets of M such that $S \subseteq T$ and $U \subseteq V$ then $SU \subseteq TV$.

Proof. Suppose $x = su \in SU$ such that $s \in S$ and $u \in U$. Then, $s \in T$ and $u \in V$. Hence $x = st \in TV$. ◁

Proof. By induction on F. The base case is when F is an atom, a variable, or a unit which are trivial.

- Suppose $F = p \in \mathcal{A} \cup \{\bot, \mathbf{1}, \mathbf{0}, \top\}$. Then, $[\![F]\!]^{V[x \mapsto X]} = [\![F]\!]^{V[x \mapsto Y]} = V(p)$.
- Suppose $F = y \in \mathcal{V}$. There are two cases. If $y \neq x$, then $[\![F]\!]^{V[x \mapsto X]} = [\![F]\!]^{V[x \mapsto Y]} = V(y)$. Otherwise, we have $[\![F]\!]^{V[x \mapsto X]} = X \subseteq Y = [\![F]\!]^{V[x \mapsto Y]}$.

There are several subcases for the induction case.

- Suppose $F = G \otimes G'$.

$$\begin{aligned}
&[\![G]\!]^{V[x\mapsto X]} \subseteq [\![G]\!]^{V[x\mapsto Y]}; [\![G']\!]^{V[x\mapsto X]} \subseteq [\![G']\!]^{V[x\mapsto Y]} && \text{[By IH]}\\
&\Rightarrow [\![G]\!]^{V[x\mapsto X]}[\![G']\!]^{V[x\mapsto X]} \subseteq [\![G]\!]^{V[x\mapsto Y]}[\![G']\!]^{V[x\mapsto Y]} && \text{[By Claim 56]}\\
&\Rightarrow ([\![G]\!]^{V[x\mapsto X]}[\![G']\!]^{V[x\mapsto X]})^{\perp\perp} \subseteq ([\![G]\!]^{V[x\mapsto Y]}[\![G']\!]^{V[x\mapsto Y]})^{\perp\perp} && \text{[By Proposition 4]}\\
&\Rightarrow [\![F]\!]^{V[x\mapsto X]} \subseteq [\![F]\!]^{V[x\mapsto Y]}
\end{aligned}$$

- Suppose $F = G⅋G'$.

$$\begin{aligned}
&[\![G]\!]^{V[x\mapsto X]} \subseteq [\![G]\!]^{V[x\mapsto Y]}; [\![G']\!]^{V[x\mapsto X]} \subseteq [\![G']\!]^{V[x\mapsto Y]} && \text{[By IH]}\\
&\Rightarrow ([\![G]\!]^{V[x\mapsto Y]})^{\perp} \subseteq ([\![G]\!]^{V[x\mapsto X]})^{\perp}; ([\![G']\!]^{V[x\mapsto Y]})^{\perp} \subseteq ([\![G']\!]^{V[x\mapsto X]})^{\perp} && \text{[By Proposition 4]}\\
&\Rightarrow ([\![G]\!]^{V[x\mapsto Y]})^{\perp}([\![G']\!]^{V[x\mapsto Y]})^{\perp} \subseteq ([\![G]\!]^{V[x\mapsto X]})^{\perp}([\![G']\!]^{V[x\mapsto X]})^{\perp} && \text{[By Claim 56]}\\
&\Rightarrow \left(([\![G]\!]^{V[x\mapsto X]})^{\perp}([\![G']\!]^{V[x\mapsto X]})^{\perp}\right)^{\perp} \subseteq \left(([\![G]\!]^{V[x\mapsto Y]})^{\perp}([\![G']\!]^{V[x\mapsto Y]})^{\perp}\right)^{\perp} && \text{[By Proposition 4]}\\
&\Rightarrow [\![F]\!]^{V[x\mapsto X]} \subseteq [\![F]\!]^{V[x\mapsto Y]}
\end{aligned}$$

- Suppose $F = G\&G'$.

$$\begin{aligned}
&[\![G]\!]^{V[x\mapsto X]} \subseteq [\![G]\!]^{V[x\mapsto Y]}; [\![G']\!]^{V[x\mapsto X]} \subseteq [\![G']\!]^{V[x\mapsto Y]} && \text{[By IH]}\\
&\Rightarrow [\![G]\!]^{V[x\mapsto X]} \cap [\![G']\!]^{V[x\mapsto X]} \subseteq [\![G]\!]^{V[x\mapsto Y]} \cap [\![G']\!]^{V[x\mapsto Y]}\\
&\Rightarrow [\![F]\!]^{V[x\mapsto X]} \subseteq [\![F]\!]^{V[x\mapsto Y]}
\end{aligned}$$

- Suppose $F = G \oplus G'$.

$$\begin{aligned}
&[\![G]\!]^{V[x\mapsto X]} \subseteq [\![G]\!]^{V[x\mapsto Y]}; [\![G']\!]^{V[x\mapsto X]} \subseteq [\![G']\!]^{V[x\mapsto Y]} && \text{[By IH]}\\
&\Rightarrow [\![G]\!]^{V[x\mapsto X]} \cup [\![G']\!]^{V[x\mapsto X]} \subseteq [\![G]\!]^{V[x\mapsto Y]} \cup [\![G']\!]^{V[x\mapsto Y]}\\
&\Rightarrow ([\![G]\!]^{V[x\mapsto X]} \cup [\![G']\!]^{V[x\mapsto X]})^{\perp\perp} \subseteq ([\![G]\!]^{V[x\mapsto Y]} \cup [\![G']\!]^{V[x\mapsto Y]})^{\perp\perp} && \text{[By Proposition 4]}\\
&\Rightarrow [\![F]\!]^{V[x\mapsto X]} \subseteq [\![F]\!]^{V[x\mapsto Y]}
\end{aligned}$$

- Suppose $F = \mu y.G$. Observe that $y \neq x$ since we assumed that x is not a bound variable in F. Now by hypothesis, for any fact Z, $[\![G]\!]^{V[x\mapsto X, y\mapsto Z]} \subseteq [\![G]\!]^{V[x\mapsto Y, y\mapsto Z]}$. Therefore for every Z such that $[\![G]\!]^{V[x\mapsto Y, y\mapsto Z]} \subseteq Z$ we have $[\![G]\!]^{V[x\mapsto X, y\mapsto Z]} \subseteq Z$. Therefore, $\{Z \mid [\![G]\!]^{V[x\mapsto Y, y\mapsto Z]} \subseteq Z\} \subseteq \{Z \mid [\![G]\!]^{V[x\mapsto X, y\mapsto Z]} \subseteq Z\}$. Hence, $\bigcap_{[\![G]\!]^{V[x\mapsto X, y\mapsto Z]} \subseteq Z}\{Z\} \subseteq \bigcap_{[\![G]\!]^{V[x\mapsto Y, y\mapsto Z]} \subseteq Z}\{Z\}$. We conclude $[\![F]\!]^{V[x\mapsto X]} \subseteq [\![F]\!]^{V[x\mapsto Y]}$.
- Suppose $F = \nu y.G$. As before we comment that $y \neq x$ and therefore by hypothesis, for any fact Z, $[\![G]\!]^{V[x\mapsto X, y\mapsto Z]} \subseteq [\![G]\!]^{V[x\mapsto Y, y\mapsto Z]}$. Therefore for every Z such that $Z \subseteq [\![G]\!]^{V[x\mapsto X, y\mapsto Z]}$ we have $Z \subseteq [\![G]\!]^{V[x\mapsto Y, y\mapsto Z]}$. Therefore, $\{Z \mid Z \subseteq [\![G]\!]^{V[x\mapsto X, y\mapsto Z]}\} \subseteq \{Z \mid Z \subseteq [\![G]\!]^{V[x\mapsto Y, y\mapsto Z]}\}$. Hence, $\bigcup_{Z \subseteq [\![G]\!]^{V[x\mapsto X, y\mapsto Z]}}\{Z\} \subseteq \bigcup_{Z \subseteq [\![G]\!]^{V[x\mapsto Y, y\mapsto Z]}}\{Z\}$. Applying Proposition 4 twice, we conclude $[\![F]\!]^{V[x\mapsto X]} \subseteq [\![F]\!]^{V[x\mapsto Y]}$. ◀

B.1.2 Proof of Theorem 28

Proof. We show it for μf. First of all, $\{X \mid f(X) \subseteq X\}$ is non-empty since $M^{\perp} \in \mathcal{D}$. First, we show that it is indeed a fixpoint. Observe that $\mu_f \subseteq X$ for any $X \in \mathcal{D}$ which is a pre-fixpoint of f. By Lemma 27, one can apply f on both sides. So $f(\mu f) \subseteq f(X)$ for all $X \in \mathcal{D}$ satisfying $f(X) \subseteq X$ and therefore $f(\mu f) \subseteq \cap_{X\in\mathcal{D}}\{X \mid f(X) \subseteq X\} = \mu f$. So μf is a prefixpoint.

But then, since $\mu_f \in \mathcal{D}$, and thanks to the closure properties of D, so is $f(\mu_f)$. By monotonicity of f, one gets that $f(f(\mu_f)) \subseteq f(\mu_f)$, ensuring that $f(\mu_f)$ is a prefixpoint of f. But μf is the least prefixpoint; so, we conclude that $\mu_f \subseteq f(\mu_f)$. Therefore, $\mu_f = f(\mu_f)$. Finally recall $\mu f \subseteq X$ for any pre-fixpoint X in $\mathcal{D}$, so it is the least fixpoint in $\mathcal{D}$. ◀

B.1.3 Proof of Lemma 29

Proof. By induction on F. The base case is when F is an atom, a variable or a unit.
- Suppose $F = a \in \mathcal{A}$. Then, $[\![F^{\perp}]\!]^{V^{\perp}} = V^{\perp}(a^{\perp}) = V(a^{\perp}) = V(a)^{\perp} = ([\![F]\!]^{V})^{\perp}$.
- Suppose $F = x \in \mathcal{V}$. Then, $[\![F^{\perp}]\!]^{V^{\perp}} = V^{\perp}(x^{\perp}) = V^{\perp}(x) = V(x)^{\perp} = ([\![F]\!]^{V})^{\perp}$.
- The case for the units is easy.

There are several subcases for the induction case.

- Suppose $F = G \otimes G'$.

$$\begin{aligned}
[\![(G \otimes G')^\perp]\!]^{V^\perp} &= [\![G^\perp \parr G'^\perp]\!]^{V^\perp} \\
&= (([\![G^\perp]\!]^{V^\perp})^\perp ([\![G'^\perp]\!]^{V^\perp})^\perp)^\perp \\
&= ([\![G^{\perp\perp}]\!]^V [\![G'^{\perp\perp}]\!]^V)^\perp && \text{[By IH]} \\
&= ([\![G]\!]^V [\![G']\!]^V)^\perp \\
&= ([\![G]\!]^V [\![G']\!]^V)^{\perp\perp\perp} && \text{[By Proposition 4]} \\
&= ([\![(G \otimes G')]\!]^V)^\perp
\end{aligned}$$

Negating both sides we have, $([\![(G \otimes G')^\perp]\!]^{V^\perp})^\perp = ([\![(G \otimes G')]\!]^V)^{\perp\perp}$. Hence, $([\![G^\perp \parr G'^\perp]\!]^{V^\perp})^\perp = [\![(G^\perp \parr G'^\perp)^\perp]\!]^V$. This takes care of the case when the outermost connective of F is a $\parr$.

- Suppose $F = G' \oplus G'$.

$$\begin{aligned}
[\![(G' \oplus G')^\perp]\!]^{V^\perp} &= [\![G^\perp \& G'^\perp]\!]^{V^\perp} \\
&= [\![G^\perp]\!]^{V^\perp} \cap [\![G'^\perp]\!]^{V^\perp} \\
&= ([\![G]\!]^V)^\perp \cap ([\![G']\!]^V)^\perp && \text{[By IH]} \\
&= ([\![G]\!]^V \cup [\![G']\!]^V)^\perp \\
&= ([\![G]\!]^V \cup [\![G']\!]^V)^{\perp\perp\perp} && \text{[By Proposition 4]} \\
&= ([\![G \oplus G']\!]^V)^\perp
\end{aligned}$$

As in the previous case, negating both sides, we derive the case when the outermost connective of F is a $\&$.

- Suppose $F = \mu x.G$.

$$\begin{aligned}
[\![(\mu x.G)^\perp]\!]^{V^\perp} &= [\![\nu x.G^\perp]\!]^{V^\perp} \\
&= \left(\bigcup_{X \in \mathcal{D}} \left\{X \mid X \subseteq [\![G^\perp]\!]^{V^\perp[x \mapsto X]}\right\}\right)^{\perp\perp} \\
&= \left(\bigcup_{X \in \mathcal{D}} \left\{X \mid X \subseteq [\![G^\perp]\!]^{(V[x \mapsto X^\perp])^\perp}\right\}\right)^{\perp\perp} \\
&= \left(\bigcup_{X \in \mathcal{D}} \left\{X \mid X \subseteq ([\![G]\!]^{V[x \mapsto X^\perp]})^\perp\right\}\right)^{\perp\perp} && \text{[By IH]} \\
&= \left(\bigcap_{X \in \mathcal{D}} \left\{X^\perp \mid X \subseteq ([\![G]\!]^{V[x \mapsto X^\perp]})^\perp\right\}\right)^{\perp} && \text{[By Proposition 4]} \\
&= \left(\bigcap_{X \in \mathcal{D}} \left\{X^\perp \mid [\![G]\!]^{V[x \mapsto X^\perp]} \subseteq X^\perp\right\}\right)^{\perp} && \text{[By Proposition 4]} \\
&= \left(\bigcap_{X \in \mathcal{D}} \left\{X \mid [\![G]\!]^{V[x \mapsto X]} \subseteq X\right\}\right)^{\perp} && \text{[Closure property of } \mathcal{D}\text{]} \\
&= ([\![\mu x.G]\!]^V)^\perp
\end{aligned}$$

As in the previous case, negating both sides, we derive the case when the outermost operator of F is a ν. ◀

B.2 Proofs of Section 4

B.2.1 Proof of Lemma 45

Proof. We induct on $|\mathsf{rk}(F)|$. The base case is when $|\mathsf{rk}(F)| = 1$. Since $x \in \mathsf{fv}(F)$, F cannot be an atom or a unit. Therefore, $F = x$. Plugging $\sigma = \langle\ \rangle$, we are done. The induction case has several subcases.

- Suppose $F = G \odot G'$ where $\odot \in \{\otimes, ⅋, \&, \oplus\}$. We have two cases. Either $x \in \mathsf{fv}(G) \cap \mathsf{fv}(G')$ or x is free in only one of them. Note that $\mathsf{rk}(G), \mathsf{rk}(G') \trianglelefteq \mathsf{rk}(F)$. Therefore if x is free in them, the induction hypothesis can be fired. In the first case, we have $\mathsf{rk}(G(\xi/x)) = \mathsf{rk}(\xi) * \sigma_1$ and $\mathsf{rk}(G'(\xi/x)) = \mathsf{rk}(\xi) * \sigma_2$. Therefore,

$$\begin{aligned}\mathsf{rk}(F(\xi/x)) &= \mathsf{rk}(G(\xi/x) \odot \mathsf{rk}(G'(\xi/x)))\\ &= (\mathsf{rk}(G(\xi/x)) \sqcup \mathsf{rk}(G'(\xi/x))) * \langle 0\rangle\\ &= ((\mathsf{rk}(\xi) * \sigma_1) \sqcup (\mathsf{rk}(\xi) * \sigma_2)) * \langle 0\rangle\\ &= \mathsf{rk}(\xi) * (\sigma_1 \sqcup \sigma_2) * \langle 0\rangle\end{aligned}$$

  Therefore by plugging $\sigma = (\sigma_1 \sqcup \sigma_2) * \langle 0\rangle$, we are done. In the other case, wlog assume $x \notin \mathsf{fv}(G')$. Therefore, we have $G'(\xi/x) = G'$. Firing the induction hypothesis for G, we have $\mathsf{rk}(G(\xi/x)) = \mathsf{rk}(\xi) * \sigma_1$ as before. Therefore,

$$\begin{aligned}\mathsf{rk}(F(\xi/x)) &= \mathsf{rk}(G(\xi/x) \odot \mathsf{rk}(G'(\xi/x)))\\ &= (\mathsf{rk}(G(\xi/x)) \sqcup \mathsf{rk}(G')) * \langle 0\rangle\\ &= ((\mathsf{rk}(\xi) * \sigma_1) \sqcup \mathsf{rk}(G')) * \langle 0\rangle\\ &= \mathsf{rk}(\xi) * \sigma_1 * \langle 0\rangle && [\text{Since } \mathsf{rk}(G') \trianglelefteq \mathsf{rk}(\xi)]\end{aligned}$$

  Therefore by plugging $\sigma = \sigma_1 * \langle 0\rangle$, we are done.
- Suppose $F = \eta^\beta y.G$ where $\eta \in \{\mu, \nu\}$, $\beta < \omega$, and $y \neq x$. Clearly, $x \in \mathsf{fv}(G)$ and $\mathsf{rk}(G) \trianglelefteq \mathsf{rk}(F)$. Therefore, by hypothesis, $\mathsf{rk}(G(\xi/x)) = \mathsf{rk}(\xi) * \sigma'$.

$$\begin{aligned}\mathsf{rk}(F(\xi/x)) &= \mathsf{rk}(\eta^\beta y.G(\xi/x))\\ &= \mathsf{rk}(G(\xi/x)) * \langle\beta\rangle\\ &= (\mathsf{rk}(\xi) * \sigma' * \langle\beta\rangle\end{aligned}$$

By plugging $\sigma = \sigma' * \langle\beta\rangle$, we are done. The case $F = \eta y.G$ goes exactly similarly. ◀

B.2.2 Proof of Lemma 53

Proof. We recall that $\Gamma_0 = \mu x.G, a \otimes \nu y.F$ and we will show that this sequent is not provable in $\mu\mathsf{MALL}_\omega$. Suppose there is a proof. By Corollary 52, we can assume that this proof is cut-free. Therefore, the only possibilities for the first rule are $(\otimes)$ or one of $\{(\mu^\omega_n)\}_{n\in\omega}$. If it is the former, then the left premisse has to contain $\mu x.G$ since $\vdash a$ cannot be proved. Consequently, the right premisse is $\vdash \nu y.F$. If it were provable, so would be $\vdash \nu^n x.F$ for all $n \in \omega$. It is easy to observe that $\vdash \nu^n x.F$ is not provable for all $n > 0$.

Now suppose the first rule is a (μ^{ω}_{n}) for some $n \in \omega$. We note that the (μ^{n}) rule is invertible. Morever, the $(\parr)$ rule is also invertible. Therefore, it suffices to show that $\vdash (a^{\perp})^{n}, \mathbf{0}, a \otimes \nu y.F$ is not provable. The only possible rule here is the $(\otimes)$. The only splitting of the context which renders the left premisse provable is one where the right premisse is $\vdash \overbrace{a^{\perp}, \dots, a^{\perp}}^{n-1}, \mathbf{0}, \nu y.F$. The only possible rule that can be applied here is the (ν^{ω}) rule. Consider any premisse other than the $\lfloor \frac{n-1}{2} \rfloor$th premisse. Since the number of a and $a^{\perp}$ are different in it, it is not provable by Example 7. ◀