

# The Constraint Satisfaction Problem: Complexity and Approximability

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## Abstract

Constraint satisfaction has always played a central role in computational complexity theory; appropriate versions of CSPs are classical complete problems for most standard complexity classes. CSPs constitute a very rich and yet sufficiently manageable class of problems to give a good perspective on general computational phenomena. For instance, they help to understand which mathematical properties make a computational problem tractable (in a wide sense, e.g., polynomial-time solvable, non-trivially approximable, fixed-parameter tractable, or definable in a weak logic). In the last 15 years, research activity in this area has significantly intensified and hugely impressive progress was made. The Dagstuhl Seminar 22201 “The Constraint Satisfaction Problem: Complexity and Approximability” was aimed at bringing together researchers using all the different techniques in the study of the CSP so that they can share their insights obtained during the past four years. This report documents the material presented during the course of the seminar.

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## 1 Executive Summary

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The *constraint satisfaction problem*, or CSP in short, provides a unifying framework in which it is possible to express, in a natural way, a wide variety of computational problems dealing with mappings and assignments, including satisfiability, graph/hypergraph colorability, and systems of equations. The CSP framework originated around 40 years ago independently in artificial intelligence, database theory, and graph theory, under three different guises, and it was realised only in the late 1990s that these are in fact different faces of the same fundamental

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\* Editor / Organizer



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problem. Nowadays, the CSP is extensively used in theoretical computer science, being a mathematical object offering a good balance between generality and structure that provides an excellent laboratory both for classification methods and for algorithmic techniques, while in AI and more applied areas of computer science this framework is widely regarded as a versatile and efficient way of modelling and solving a variety of real-world problems, such as planning and scheduling, software verification and natural language comprehension, to name just a few. An instance of CSP consists of a set of variables, a set of values for the variables, and a set of constraints that restrict the combinations of values that certain subsets of variables may take. Given such an instance, the possible questions include (a) deciding whether there is an assignment of values to the variables so that every constraint is satisfied, or optimising such assignments in various ways, or (b) finding an assignment satisfying as many constraints as possible. There are many important modifications and extensions of this basic framework, e.g., those that deal with counting assignments or involve soft or global constraints.

Constraint satisfaction has always played a central role in computational complexity theory; appropriate versions of CSPs are classical complete problems for most standard complexity classes. CSPs constitute a rich and yet sufficiently manageable class of problems to give a good perspective on general computational phenomena. For instance, they help to understand which mathematical properties make a problem tractable (in a wide sense, e.g., polynomial-time solvable or non-trivially approximable, fixed-parameter tractable or definable in a weak logic). One of the most striking features of this study is the variety of different branches of mathematics (including universal algebra and logic, combinatorics and graph theory, probability theory and mathematical programming) that are used to achieve deep insights into CSP, and this Dagstuhl Seminar aims to contribute towards further synergy in the area.

After about 15 years of intense research activity and hugely impressive progress, the culmination of the algebraic-approach to fixed-template CSPs was the resolution of the Feder-Vardi conjecture independently by Bulatov and Zhuk in 2017. While some fundamental questions (such as a fine-grained understanding of tractable CSPs) remain open, new research directions on generalizations of CSPs started emerging. The fixed-template promise CSP (PCSP) is among the most promising new directions of research motivated by better understanding computational hardness or tractability. PCSPs are a vast generalization of CSPs where each predicate has a strong and a weak form and given a CSP instance, the objective is to distinguish if the strong form can be satisfied vs. even the weak form cannot be satisfied. A prime and well-known example is the approximate graph coloring problem: distinguish  $k$ -colorable graphs from graphs that are not even  $\ell$ -colorable, for some fixed  $k < \ell$ . The main topic of this seminar is PCSPs, a highly ambitious research direction with intriguing connections to both old open problems (such as the approximate graph coloring problem) and new research directions (such as generalizations of submodularity, a key concept in optimization).

The recent flurry of activity on the topic of the seminar is witnessed by five previous Dagstuhl seminars, titled “Complexity of constraints” (06401) and “The CSP: complexity and approximability” (09441, 12541, 15301, 18231), that were held in 2006, 2009, 2012, 2015, and 2018 respectively. This seminar was a follow-up to the 2009, 2012, 2015, and 2018 seminars. Indeed, the exchange of ideas at the 2009, 2012, 2015, and 2018 seminars has led to ambitious new research projects and to establishing regular communication channels. There is clearly the potential for further systematic interaction that will keep on cross-fertilising the areas and opening new research directions. The 2022 seminar brought together 46 researchers

from different highly advanced areas of constraint satisfaction and involved many specialists who use universal-algebraic, combinatorial, geometric, and probabilistic techniques to study CSP-related algorithmic problems. 10 of the participants attended remotely owing to the global COVID pandemic. The participants presented, in 24 talks, their recent results on a number of important questions concerning the topic of the seminar. One particular feature of this seminar is a significant increase in the number of talks involving multiple subareas and approaches within its research direction – a definite sign of the growing synergy, which is one of the main goals of this series of seminars.

**Concluding remarks and future plans.** The seminar was well received as witnessed by the high rate of accepted invitations and the great degree of involvement by the participants. Because of a multitude of impressive results reported during the seminar and active discussions between researchers with different expertise areas, the organisers regard this seminar as a great success. With steadily increasing interactions between such researchers, we foresee another seminar focusing on the interplay between different approaches to studying the complexity and approximability of the CSP. Finally, the organisers wish to express their gratitude to the Scientific Directors of the Dagstuhl Centre for their support of the seminar.

## Description of the Topics of the Seminar

With the resolution of the Feder-Vardi conjecture on finite-domain CSPs by Bulatov and Zhuk in 2017, the field has moved on to more generalizations of finite-domain decision CSPs. One of the main emerging areas is that of Promise CSPs (PCSPs), which are a huge generalization of CSPs.

### Promise CSPs

The study of PCSPs is about approximability of perfectly satisfiable instances. This exciting area been the main theme of this workshop, with about half of the talks on PCSPs.

The workshop started with a 2-hour tutorial on the basics of PCSPs.

- In the first part, Opršal talked about basic tools used in the complexity analysis of PCSPs, namely minions and free structures.
- In the second part, Opršal talked about an application of topology in the study of complexity of graph colorings.

There were three topologists attending the seminar and one of them gave a talk.

- Kozlov gave a talk on uses of topology in theoretical computer science and homomorphism questions in particular.

There were 9 contributed talks on recent results on PCSPs.

- Barto talked about his work on “baby PCP”, which is a combinatorial variant of a (weaker version of) the PCP theorem.
- Brakensiek gave two talks: one on minions related to SDP relaxations for PCSPs, and one on very recent results on robust solvability of certain PCSPs.
- Butti gave a talk about Sherali-Adams relaxations for valued PCSPs.
- Ciardo gave a talk about Sherali-Adams relaxations for PCSPs, and more generally about studying hierarchies of convex relaxations via tensorisation.
- Dalmau gave a talk about his work on a condition that PCSPs solvable by local consistency algorithms have to satisfy.

- Kompatscher gave an overview of existing results on finite tractability of PCSPs.
- Mottet gave a talk about his very recent work on first-order definable PCSPs and a relationship to PCSPs solved by local consistency algorithms.
- Nagajima gave a talk on the complexity of linearly-ordered colorings.

### Approximability of CSPs

The second main theme of the seminar was recent progress in the area of approximability of CSPs. This part of the programme started with a 2-hour tutorial:

- Tulsiani gave a tutorial on how to approximate CSPs via the Sum of Squares hierarchy of SDP relaxations, including related work on high-dimensional expanders.

Three participants gave talks on approximability of CSPs.

- Bhangale gave a talk on approximability of solvable linear equations over non-Abelian groups.
- Kothari presented a refutation algorithm for unsatisfiable instances of SAT and CSP.
- Potechin talked about an algorithm for Max NAE-SAT with clauses of lengths 3 and 5.

### Infinite-domain CSPs

The area of infinite-domain CSPs has seen some exciting progress in last few years with new techniques being developed.

- Bodirsky talked about methods for transferring complexity classification result between different classes of infinite-domain CSPs.
- Nagy talked about the complexity of CSPs over random hypergraphs.
- Pinsker gave an overview of recent developments around the dichotomy conjecture for infinite-domain CSPs.

### Generalisations of CSPs

There are several interesting research directions related to CSPs, including the number of solutions, CSPs with a bound on the number of occurrences for each variable, quantified CSPs, a generalisation to the ideal membership problem, etc. The seminar featured 5 talks on these topics.

- Austrin presented an algorithmic lowerbound for perfect matching on random graphs, a canonical example of an edge CSP.
- Bulatov gave a talk the ideal membership problem.
- Chen presented his results on property testing of homomorphism inadmissibility.
- Kazemina presented a classification of counting graph homomorphism modulo 2.
- Zhuk presented a complexity classification of quantified CSPs.

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### 3 Overview of Talks

#### 3.1 Perfect Matching in Random Graphs is as Hard as Tseitin

*Per Austrin (KTH Royal Institute of Technology – Stockholm, SE)*

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**Joint work of** Per Austrin, Kilian Risse

**Main reference** Per Austrin, Kilian Risse: “Perfect Matching in Random Graphs is as Hard as Tseitin”, in Proc. of the 2022 ACM-SIAM Symposium on Discrete Algorithms, SODA 2022, Virtual Conference, pp. 979–1012, SIAM, 2022.

**URL** <https://doi.org/10.1137/1.9781611977073.43>

We study the complexity of proving that a sparse random regular graph on an odd number of vertices does not have a perfect matching, and related problems involving each vertex being matched some pre-specified number of times. We show that this requires proofs of degree  $\Omega(n/\log n)$  in the Polynomial Calculus (over fields of characteristic  $\neq 2$ ) and Sum-of-Squares proof systems, and exponential size in the bounded-depth Frege proof system. This resolves a question by Razborov asking whether the Lovász-Schrijver proof system requires  $n^\delta$  rounds to refute these formulas for some  $\delta > 0$ . The results are obtained by a worst-case to average-case reduction of these formulas relying on a topological embedding theorem which may be of independent interest.

#### 3.2 Combinatorial Value and Gap Amplification

*Libor Barto (Charles University – Prague, CZ)*

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**Joint work of** Libor Barto, Filip Bialas, Marcin Kozik

**Main reference** Libor Barto, Marcin Kozik: “Combinatorial Gap Theorem and Reductions between Promise CSPs”, in Proc. of the 2022 ACM-SIAM Symposium on Discrete Algorithms, SODA 2022, Virtual Conference, pp. 1204–1220, SIAM, 2022.

**URL** <https://doi.org/10.1137/1.9781611977073.50>

The combinatorial value of a CSP instance is a certain positive integer, which is equal to one iff the instance is satisfiable. In a joint work with Marcin Kozik we proved that a natural reduction amplifies the gap between value one and value strictly greater than one. I will talk about this result, its motivations, applications, recent developments (with Filip Bialas), and future directions.

#### 3.3 On Characterizing the Inapproximability of Satisfiable CSPs

*Amey Bhangale (University of California – Riverside, US)*

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**Joint work of** Amey Bhangale, Subhash Khot, Dor Minzer

**Main reference** Amey Bhangale, Subhash Khot: “Optimal inapproximability of satisfiable k-LIN over non-abelian groups”, in Proc. of the STOC ’21: 53rd Annual ACM SIGACT Symposium on Theory of Computing, Virtual Event, Italy, June 21–25, 2021, pp. 1615–1628, ACM, 2021.

**URL** <https://doi.org/10.1145/3406325.3451003>

In his beautiful result, Raghavendra gave a complete characterization of the inapproximability for every constraint satisfaction problem, assuming the Unique Games conjecture. This result however loses perfect completeness. In this talk, I will discuss the inapproximability of

satisfiable linear equations over non-Abelian groups. Unlike Abelian groups, it is NP-complete to decide if a given system of linear equations over non-Abelian groups is satisfiable or not. We show tight NP-hardness results for approximately solving a satisfiable system of linear equations over non-Abelian groups. I will also discuss new techniques for characterizing the inapproximability of satisfiable CSPs.

### 3.4 CSP Complexity Classification Transfer

*Manuel Bodirsky (TU Dresden, DE)*

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**Joint work of** Manuel Bodirsky, Peter Jonsson, Barnaby Martin, Antoine Mottet, Žaneta Semaniššínová

Many natural classes of computational problems can be formulated as CSPs for first-order expansions of some base structure, such as  $(\mathbb{Q}, <)$ . In this talk I will present methods how to transfer classification results for such classes from one base structure to another. This is particularly interesting if the base structure is a product structure, in which case one may hope to obtain the classification from the classifications of the factors.

Our results imply new classification results for CSPs that concern Allen's Interval Algebra, the  $n$ -dimensional Block Algebra, and the Cardinal Direction Calculus and solve some open problems about these formalisms from 1999 and 2002.

### 3.5 Robust Algorithms for Promise CSPs

*Joshua Brakensiek (Stanford University, US)*

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**Joint work of** Joshua Brakensiek, Venkatesan Guruswami, Sai Sandeep

For a constraint satisfaction problem (CSP), a robust satisfaction algorithm is one that outputs an assignment satisfying most of the constraints on instances that are near-satisfiable. It is known that the CSPs that admit efficient robust satisfaction algorithms are precisely those of bounded width.

In this talk, I will discuss our recent work on robust satisfaction algorithms for Promise CSPs, which are a vast generalization of CSPs. I will present robust SDP rounding algorithms under some general conditions, namely the existence of majority or alternating threshold polymorphisms. On the hardness front, I will prove that the lack of such polymorphisms makes the PCSP hard for a large class of symmetric Boolean predicates. Our method involves a novel method to argue SDP gaps via the absence of certain colorings of the sphere, with connections to sphere Ramsey theory.

### 3.6 Some Interesting Minions

*Joshua Brakensiek (Stanford University, US)*

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**Joint work of** Joshua Brakensiek, Venkatesan Guruswami, Sai Sandeep

In the development of the theory of Promise CSPs, it has become increasingly common to capture the structure of tractable classes of PCSPs through the existence of a minion homomorphism, where one maps a chosen minion into the polymorphisms of the particular PCSP. As the field of PCSPs is still quite young, only a handful of interesting minions are currently understood. In this talk, I will present some recently-discovered minions with interesting properties, including one that captures tractability according to the “exact” basic semi-definite program.

### 3.7 On the Complexity of CSP-Based Ideal Membership Problems

*Andrei A. Bulatov (Simon Fraser University – Burnaby, CA)*

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**Joint work of** Andrei A. Bulatov, Akbar Rafiey

**Main reference** Andrei A. Bulatov, Akbar Rafiey: “On the Complexity of CSP-based Ideal Membership Problems”, CoRR, Vol. abs/2011.03700, 2020.

**URL** <https://arxiv.org/abs/2011.03700>

In this talk we consider the Ideal Membership Problem (IMP for short), in which we are given polynomials  $f_0, f_1, \dots, f_k$  and the question is to decide whether  $f_0$  belongs to the ideal generated by  $f_1, \dots, f_k$ . In the more stringent version the task is also to find a proof of this fact. The IMP underlies many proof systems based on polynomials such as Nullstellensatz, Polynomial Calculus, and Sum-of-Squares (SOS). In such applications the IMP usually involves so called combinatorial ideals that arise from a variety of discrete combinatorial problems. This restriction makes the IMP significantly easier and in some cases allows for an efficient solution algorithm.

The first part of this paper follows the work of Mastrolilli [SODA 2019] who initiated a systematic study of IMPs arising from Constraint Satisfaction Problems (CSP) of the form  $\text{CSP}(G)$ , that is, CSPs in which the type of constraints is limited to relations from a set  $G$ .

We show that many CSP techniques can be translated to IMPs thus allowing us to significantly improve the methods of studying the complexity of the IMP. We also develop universal algebraic techniques for the IMP that have been so useful in the study of the CSP. This allows us to prove a general necessary condition for the tractability of the IMP, and three sufficient ones. The sufficient conditions include IMPs arising from systems of linear equations over  $\text{GF}(p)$ ,  $p$  prime, and also some conditions defined through special kinds of polymorphisms.

Our work has several consequences and applications. First, we introduce a variation of the IMP and based on this propose a unified framework, different from the celebrated Buchberger’s algorithm, to construct a bounded degree Groebner Basis. Our algorithm, combined with the universal algebraic techniques, leads to polynomial-time construction of Groebner Basis for many combinatorial problems.

### 3.8 Sherali-Adams Meets Weisfeiler-Leman on (Promise Valued) CSPs

*Silvia Butti (UPF – Barcelona, ES)*

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**Joint work of** Libor Barto, Silvia Butti, Víctor Dalmau

In this talk I will give an overview of recent work concerning the relationship between levels of the Sherali-Adams hierarchy for CSP and the equivalence relation induced by the 1-dimensional Weisfeiler-Leman algorithm on the set of CSP instances. In particular, I will discuss how the feasibility of a linear program obtained using the Sherali-Adams method can be decomposed into three separate components, and how this fact can be used to infer results about solvability of CSPs by distributed algorithms. I will explain how these results can be extended to the more general framework of Promise Valued CSP, and discuss how the first level of the Sherali-Adams hierarchy is no longer equivalent to the Basic Linear Programming relaxation in this broader framework.

### 3.9 Testability of Homomorphism Inadmissibility

*Hubie Chen (Birkbeck, University of London, GB)*

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**Joint work of** Hubie Chen, Yuichi Yoshida

**Main reference** Hubie Chen, Yuichi Yoshida: “Testability of Homomorphism Inadmissibility: Property Testing Meets Database Theory”, in Proc. of the 38th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems, PODS 2019, Amsterdam, The Netherlands, June 30 – July 5, 2019, pp. 365–382, ACM, 2019.

**URL** <https://doi.org/10.1145/3294052.3319679>

Traditional notions of algorithmic efficiency, such as that of polynomial time, permit each input to be read in its entirety. However, the modern necessity of performing computations on large bodies of data poses situations where inputs are so large that it is prohibitively expensive to fully read them. These situations motivate the study of algorithms that only read a part of each input. Property testing provides a framework wherein it is possible to present such algorithms.

In property testing, the input is typically read via random access queries, and an algorithm’s desideratum is to distinguish between inputs that satisfy a property and inputs that are  $\epsilon$ -far from satisfying the property. Intuitively speaking, an object is called  $\epsilon$ -far from satisfying a property if one needs to modify more than an  $\epsilon$ -fraction of the object in order for it satisfy the property; the exact definition depends on the context. The restriction on the allowed inputs leads to the possibility of algorithms that do not read the entire input.

In this work, we utilize the perspective of property testing to consider the testability of relational database queries. A primary motivation is the desire to avoid reading an entire database to decide a property hereof. We focus on conjunctive queries, which are the most basic and heavily studied database queries. Each conjunctive query can be represented as a relational structure  $\mathbf{A}$  such that deciding if the conjunctive query is satisfied by a relational structure  $\mathbf{B}$  is equivalent to deciding if there exists a homomorphism from  $\mathbf{A}$  to  $\mathbf{B}$ . We phrase our results in terms of homomorphisms. Precisely, we study, for each relational structure  $\mathbf{A}$ , the testability of homomorphism inadmissibility from  $\mathbf{A}$ . We consider algorithms that have oracle access to an input relational structure  $\mathbf{B}$  and that distinguish, with high probability, the case where there is no homomorphism from  $\mathbf{A}$  to  $\mathbf{B}$ , from the case where one needs to remove a constant fraction of tuples from  $\mathbf{B}$  in order to suppress all such homomorphisms.

We provide a complete characterization of the structures  $\mathbf{A}$  from which one can test homomorphism inadmissibility with one-sided error by making a constant number of queries to  $\mathbf{B}$ . Our characterization shows that homomorphism inadmissibility from  $\mathbf{A}$  is constant-query testable with one-sided error if and only if the core of  $\mathbf{A}$  is  $\alpha$ -acyclic. We also show that the injective version of the problem is constant-query testable with one-sided error if  $\mathbf{A}$  is  $\alpha$ -acyclic; this result generalizes existing results for testing subgraph-freeness in the general graph model.

### 3.10 Relaxing with Tensors

*Lorenzo Ciardo (University of Oxford, GB)*

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**Joint work of** Lorenzo Ciardo, Stanislav Živný

**Main reference** Albert Atserias, Víctor Dalmau: “Promise Constraint Satisfaction and Width”, in Proc. of the 2022 ACM-SIAM Symposium on Discrete Algorithms, SODA 2022, Virtual Conference, pp. 1129–1153, SIAM, 2022.

**URL** <https://doi.org/10.1137/1.9781611977073.48>

The Sherali-Adams linear programming hierarchy is a powerful algorithmic framework that is naturally applicable to the context of promise constraint satisfaction problems. This talk, based on recent joint work with Standa Živný, aims to show that the structure lying at the core of the hierarchy has a multilinear nature, as it essentially consists in a space of tensors enjoying certain increasingly tight symmetries. The geometry of this space allows then to establish non-solvability of the approximate graph coloring problem via constantly many rounds of Sherali-Adams. Besides this primary application, our tensorisation approach introduces a new tool to the study of various hierarchies of algorithmic relaxations for computational problems within (and, possibly, beyond) the context of constraint satisfaction. For example, both the algorithmic frameworks known as local consistency checking and “sum of squares” Lasserre hierarchy for the SDP relaxation can be naturally described geometrically through the tensorisation technique, by considering different spaces of tensors.

### 3.11 Promise Constraint Satisfaction and Width

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**Joint work of** Albert Atserias, Víctor Dalmau

**Main reference** Albert Atserias, Víctor Dalmau: “Promise Constraint Satisfaction and Width”, in Proc. of the 2022 ACM-SIAM Symposium on Discrete Algorithms, SODA 2022, Virtual Conference, pp. 1129–1153, SIAM, 2022.

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We study the power of the bounded-width consistency algorithm in the context of the fixed-template Promise Constraint Satisfaction Problem (PCSP). Our main technical finding is that the template of every PCSP that is solvable in bounded width satisfies a certain structural condition implying that its algebraic closure-properties include weak near unanimity polymorphisms of all large arities. While this parallels the standard (non-promise) CSP theory, the method of proof is quite different and applies even to the regime of sublinear width. We also show that, in contrast with the CSP world, the presence of weak near unanimity

polymorphisms of all large arities does not guarantee solvability in bounded width. The separating example is even solvable in the second level of the Sherali-Adams (SA) hierarchy of linear programming relaxations. This shows that, unlike for CSPs, linear programming can be stronger than bounded width. A direct application of these methods also show that the problem of  $q$ -coloring  $p$ -colorable graphs is not solvable in bounded or even sublinear width, for any two constants  $p$  and  $q$  such that  $3 \leq p \leq q$ . Turning to algorithms, we note that Wigderson’s algorithm for coloring 3-colorable graphs with  $n$  vertices is implementable in width 4. Indeed, by generalizing the method we see that, for any  $\epsilon > 0$  smaller than  $1/2$ , the optimal width for solving the problem of  $O(n^\epsilon)$ -coloring 3-colorable graphs with  $n$  vertices lies between  $n^{1-3\epsilon}$  and  $n^{1-2\epsilon}$ . The upper bound gives a simple  $\exp(\Theta(n^{1-2\epsilon} \log(n)))$ -time algorithm that, asymptotically, beats the straightforward  $\exp(\Theta(n^{1-\epsilon}))$  bound that follows from partitioning the graph into  $O(n^\epsilon)$  many independent parts each of size  $O(n^{1-\epsilon})$ .

### 3.12 Applied Topology and CSPs

*Dmitry Feichtner-Kozlov (Universität Bremen, DE)*

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In this talk we give a brief introduction to the subject of Combinatorial Algebraic Topology, which lies at the crossroads of combinatorics, topology, and computation. A special emphasis is given to the family of the so-called Hom-complexes.

Given any two graphs  $G$  and  $H$ , it is possible to construct a combinatorial cell complex (more precisely a prodsimplicial complex) whose set of vertices is the set of all graph homomorphisms from  $G$  to  $H$ , and whose cellular structure reflects the commutativity relations within various groups of homomorphisms. We illuminate various structural properties of this family of cell complexes, especially with a view towards their recent applications to the complexity theory of CSPs.

### 3.13 Modular Counting of Graph Homomorphisms

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**Joint work of** Andrei A. Bulatov, Amirhossein Kazemina

**Main reference** Andrei A. Bulatov, Amirhossein Kazemina: “Complexity classification of counting graph homomorphisms modulo a prime number”, CoRR, Vol. abs/2106.04086, 2021.

**URL** <https://arxiv.org/abs/2106.04086>

Counting graph homomorphisms and its generalizations such as the Counting Constraint Satisfaction Problem (CSP), its variations, and counting problems in general have been intensively studied since the pioneering work of Valiant. While the complexity of exact counting of graph homomorphisms (Dyer and Greenhill, 2000) and the counting CSP (Bulatov, 2013, and Dyer and Richerby, 2013) is well understood, counting modulo some natural number has attracted considerable interest as well. In their 2015 paper Faben and Jerrum suggested a conjecture stating that counting homomorphisms to a fixed graph  $H$  modulo a prime number is hard whenever it is hard to count exactly, unless  $H$  has automorphisms of certain kind. In

this talk we talk about this conjecture and how to prove it. As a part of this investigation we develop techniques that widen the spectrum of reductions available for modular counting and apply to the general CSP rather than being limited to graph homomorphisms.

### 3.14 Finitely Tractable PCSPs

*Michael Kompatscher (Charles University – Prague, CZ)*

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Besides linear programming relaxations, one of the main strategies in proving the tractability of promise constraint satisfaction problems is to reduce them to tractable CSPs. In particular,  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is called finitely tractable, if there is a finite structure  $\mathbf{C}$  such that there are homomorphisms  $\mathbf{A} \rightarrow \mathbf{C} \rightarrow \mathbf{B}$  and  $\text{CSP}(\mathbf{C})$  is tractable. In this case every algorithm for  $\text{CSP}(\mathbf{C})$  also solves  $\text{PCSP}(\mathbf{A}, \mathbf{B})$ . In this talk I will give an overview of known results on finitely tractable PCSPs and discuss some open questions.

### 3.15 Refuting Smoothed $k$ -SAT Formulas and a Proof of Feige’s Conjecture

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**Joint work of** Venkatesan Guruswami, Pravesh K. Kothari, Peter Manohar  
**Main reference** Venkatesan Guruswami, Pravesh K. Kothari, Peter Manohar: “Algorithms and Certificates for Boolean CSP Refutation: “Smoothed is no harder than Random””, CoRR, Vol. abs/2109.04415, 2021.  
**URL** <https://arxiv.org/abs/2109.04415>

I’ll present a new algorithm to refute, that is, efficiently find certificates of unsatisfiability, for smoothed instances of  $k$ -SAT and other CSPs. Smoothed instances are produced by starting with a worst-case instance and flipping each literal in every clause independently with a small constant probability. The “clause” structure in such instances is arbitrary and the only randomness is in the literal patterns. The trade-off between the density, i.e., the number of constraints in the instance, and the running time required by the algorithm matches (up to polylogarithmic factors density) the best known (and possibly sharp) trade-off for random  $k$ -SAT. As a corollary, we also obtain a simpler analysis for refuting random  $k$ -SAT formulas.

Our analysis inspires a new method that we call spectral double counting that we use to prove Feige’s 2008 conjecture on the extremal trade-off between the size of the hypergraph and its girth that generalizes the Moore bound on the girth of irregular graphs proved by Alon, Hoory, and Linial (2002). As a corollary, we obtain that smoothed instances of 3-SAT with arbitrary clause structure and  $n^{1.4} \ll n^{1.5}$ , the threshold for known efficient refutation) constraints admit polynomial-size certificates of unsatisfiability with appropriate generalization to all  $k$ -CSPs. This extends the celebrated work of Feige, Kim, and Ofek (2006) who proved a similar result for random 3-SAT. FKO’s proof uses a 2nd-moment-method based argument that strongly exploits the randomness of the clause structure. Our proof gives a simpler argument that extends to “worst-case” clause structures.

Taken together, our results show that smoothed instances with arbitrary clause structure and significantly less randomness enjoy the same thresholds for both refutation algorithms and certificates of unsatisfiability as random instances.

### 3.16 Sandwiches for PCSPs

*Antoine Mottet (TU Hamburg, DE)*

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A PCSP is said to be first-order definable if there exists a first-order sentence that holds on all Yes-instances and does not hold on No-instances. I will prove that  $\text{PCSP}(\mathbf{A}, \mathbf{B})$  is first-order definable exactly when there exists a finite sandwich  $\mathbf{A} \rightarrow \mathbf{C} \rightarrow \mathbf{B}$  where  $\mathbf{C}$  has finite duality. Using similar tools, I will show how to give a sandwich-characterisation of PCSPs solvable by local consistency.

### 3.17 Hypergraphs in the Post-Proof Era

*Tomáš Nagy (TU Wien, AT)*

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We consider CSPs over random  $k$ -uniform hypergraphs. Using the theory of smooth approximations, we can reduce the problem of finding an injective solution to such CSPs to a finite-domain CSP. On the other hand, surprisingly the problem of finding an arbitrary solution to such CSPs does not naturally reduce to a finite-domain CSP. We will present algorithmic techniques based on polymorphisms that allow us to reduce the latter problem to the aforementioned problem of finding an injective solution. This way, we confirm the Bodirsky-Pinsker conjecture for the random  $k$ -uniform hypergraphs for any  $k$ .

### 3.18 Linearly Ordered Colourings of Hypergraphs

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**Joint work of** Tamio-Vesa Nakajima, Stanislav Živný

**Main reference** Tamio-Vesa Nakajima, Stanislav Živný: “Linearly ordered colourings of hypergraphs”, CoRR, Vol. abs/2204.05628, 2022.

**URL** <https://doi.org/10.48550/arXiv.2204.05628>

A linearly ordered (LO)  $k$ -colouring of an  $r$ -uniform hyper graph assigns an integer colour from 1 to  $k$  to each vertex of the hyper graph, such that every edge contains a unique maximum colour. We will discuss two results relating to LO colouring promise problems. First, we will show the relationships between the polymorphism minions of LO 2 vs LO  $k$  colouring for all  $k \geq 3$ , and we use this to prove that LO  $k$  vs LO  $\ell$  colouring is NP-hard for uniformity  $r \geq \ell - k + 4$ . We will also discuss an algorithm that, if given an LO 2-colourable 3-uniform hypergraph, will find an LO  $O(\sqrt{(n \log \log n)/\log n})$  colouring.

### 3.19 An Introduction to Promise CSP, Free Structures, and Minions

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**Joint work of** Libor Barto, Jakub Bulín, Andrei A. Krokhin, Jakub Opršal

**Main reference** Libor Barto, Jakub Bulín, Andrei A. Krokhin, Jakub Opršal: “Algebraic Approach to Promise Constraint Satisfaction”, *J. ACM*, Vol. 68(4), pp. 28:1–28:66, 2021.

**URL** <https://doi.org/10.1145/3457606>

The promise CSP is a certain “structural approximation” variant of the CSP. The key difference from standard CSP is that the template consists of two structures  $\mathbf{A}$  and  $\mathbf{B}$ , s.t.,  $\mathbf{A}$  maps homomorphically to  $\mathbf{B}$ . The search version of promise CSP can be formulated as: given an instance  $\mathbf{X}$  of  $\text{CSP}(\mathbf{A})$  that is promised to have a solution, i.e., there is a homomorphism from  $\mathbf{X}$  to  $\mathbf{A}$ , find a homomorphism from  $\mathbf{X}$  to  $\mathbf{B}$ . For example, we could ask to find a 6 colouring of a graph that is promised to be 3 colourable. This promise CSP gained a lot of attention in recent years for a few reasons: it has substantial interactions with other variants of approximation, and it gives many interesting insights into the theory of classical CSPs.

In the tutorial I will give a brief overview of the basic abstract theory of promise CSPs with focus on several key concepts: gadget reductions, polymorphisms, minions, free structures, and their interactions with standard CSPs.

This is the first part of the tutorial on promise CSPs.

### 3.20 Topology in Approximate Graph Colouring

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**Main reference** Andrei A. Krokhin, Jakub Opršal, Marcin Wrochna, Stanislav Živný: “Topology and adjunction in promise constraint satisfaction”, *CoRR*, Vol. abs/2003.11351, 2020.

**URL** <https://arxiv.org/abs/2003.11351>

Approximate graph colouring is a prominent promise CSP, it asks to find a colouring using  $k$  colours of a graph that is 3-colourable where  $k > 3$ . It is a notorious open problem in computational complexity. We could also ask whether the problem gets easier if we strengthen the promise but insist on finding a 3-colouring. In the talk, we will discuss a method, based on topological ideas first used in graph colouring by Lovász, that shows that strengthening the promise does not make the problem easier (unless the promise implies that the graph is 2-colourable), and where these topological ideas might lead in classification of computational complexity of similar problems in the near future.

This is the second part of the tutorial on promise CSPs.

### 3.21 Surprises in Infinite-Domain Constraint Satisfaction, or: An Order Out of Nothing

*Michael Pinsker (TU Wien, AT)*

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We give an overview of recent developments around the dichotomy conjecture for CSPs of infinite-domain structures such as the order of the rationals or the random graph. It was believed until recently that in the context of CSPs, the mathematical theory for those structures with an order (such as the order of the rationals) was quite different from those without an order (such as the random graph), and that in particular the latter were always solvable by a certain natural reduction to finite-domain CSPs, whereas the former can never be solved that way. We show that this dividing line between ordered and non-ordered structures does not provide this expected dichotomy of approaches, and that orders can appear out of nothing in unordered structures. As a consequence, this forces us to give up the project of always reducing “non-ordered” CSPs to finite-domain CSPs, and to take on the challenge of directly lifting Zhuk’s finite-domain algorithmic techniques to infinite-domain CSPs.

### 3.22 On the Mysteries of MAX NAE-SAT

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**Joint work of** Joshua Brakensiek, Neng Huang, Aaron Potechin, Uri Zwick

**Main reference** Joshua Brakensiek, Neng Huang, Aaron Potechin, Uri Zwick: “On the Mysteries of MAX NAE-SAT”, in Proc. of the 2021 ACM-SIAM Symposium on Discrete Algorithms, SODA 2021, Virtual Conference, January 10 – 13, 2021, pp. 484–503, SIAM, 2021.

**URL** <https://doi.org/10.1137/1.9781611976465.30>

The NAE-SAT predicate (where NAE stands for not all equal) is a simple predicate which is 0 if all of its inputs are equal and 1 otherwise. For any fixed  $k$ , there is a  $7/8$  or better approximation algorithm for instances of MAX NAE-SAT where each clause has length exactly  $k$ . However, the approximability of MAX NAE-SAT with a mixture of different clause sizes is much less well understood. Before our work, it was an open problem whether or not there is a  $7/8$  approximation algorithm for MAX NAE-SAT where all clause lengths are allowed.

In this talk, I will prove that the approximation ratio for MAX NAE-SAT with clauses of lengths 3 and 5 is at most 0.8739 (assuming unique games is hard). I will then describe an approximation algorithm for almost satisfiable instances of MAX NAE-SAT with clauses of lengths 3 and 5 which we conjecture gives a 0.8728-approximation. After describing this algorithm, I will describe how we found this algorithm, why we conjecture that it gives a 0.8728-approximation, and why it may well be the optimal approximation algorithm for almost satisfiable instances of MAX NAE-SAT with clauses of lengths 3 and 5.

### 3.23 Approximating CSPs via the Sum of Squares Hierarchy (and the Role of Expansion)

*Madhur Tulsiani (TTIC – Chicago, US)*

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In this tutorial, we will give an overview of existing and some recent results on approximating CSPs via the Sums of Squares hierarchy. We will also examine the role played by various notions of expansion, for proving both upper and lower bounds.

### 3.24 PSpace-hard vs $\Pi_2^P$ Dichotomy for the QCSP

*Dmitriy Zhuk (Moscow State University, RU)*

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The Quantified Constraint Satisfaction Problem (QCSP) is the generalization of the Constraint Satisfaction problem (CSP) where we allow both existential and universal quantifiers. Formally, the QCSP over a constraint language  $\Gamma$  is the problem to evaluate a sentence of the form

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \dots \forall x_n \exists y_n (R_1(\dots) \wedge \dots \wedge R_s(\dots)),$$

where  $R_1, \dots, R_s$  are relations from  $\Gamma$ . While CSP remains in NP for any  $\Gamma$ , QCSP can be PSpace-hard, as witnessed by Quantified 3-Satisfiability or Quantified Graph 3-Colouring.

It turned out that there are no constraint languages whose QCSP complexity is in between PSpace and  $\Pi_2^P$ . The proof is based on a natural reduction of a QCSP instance to an exponential size CSP instance. We show that, unless the problem is PSpace-hard, for any false instance there exists a polynomial part of the CSP instance without a solution, which immediately brings us to the  $\Pi_2^P$  complexity class. In the talk we will discuss the proof as well as a concrete constraint language whose QCSP is  $\Pi_2^P$ -complete.

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