List Agreement Expansion from Coboundary Expansion

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⎯ Abstract ⎯⎯⎯⎯⎯⎯⎯⎯

One of the key components in PCP constructions are agreement tests. In agreement test the tester is given access to subsets of fixed size of some set, each equipped with an assignment. The tester is then tasked with testing whether these local assignments agree with some global assignment over the entire set. One natural generalization of this concept is the case where, instead of a single assignment to each local view, the tester is given access to l different assignments for every subset. The tester is then tasked with testing whether there exist l global functions that agree with all of the assignments of all of the local views. In this work we present sufficient condition for a set system to exhibit this generalized definition of list agreement expansion. This is, to our knowledge, the first work to consider this natural generalization of agreement testing.

Despite initially appearing very similar to agreement expansion in definition, proving that a set system exhibits list agreement expansion seem to require a different set of techniques. This is due to the fact that the natural extension of agreement testing (i.e. that there exists a pairing of the lists such that each pair agrees with each other) does not suffice when testing for list agreement as list agreement crucially relies on a global structure. It follows that if a local assignments satisfy list agreement they must not only agree locally but also exhibit some additional structure. In order to test for the existence of this additional structure we use the connection between covering spaces of a high dimensional complex and its coboundaries. Specifically, we use this connection as a form of "decoupling".

Moreover, we show that any set system that exhibits list agreement expansion also supports direct sum testing. This is the first scheme for direct sum testing that works regardless of the parity of the sizes of the local sets. Prior to our work the schemes for direct sum testing were based on the parity of the sizes of the local tests.

1 Introduction

Agreement testing

Agreement testing is an important tool that is central to many PCP constructions. In agreement testing one is given access to a set of functions $\{f_s\}_{s \in S}$ that are thought of as "local views" of some global function $F : \bigcup_{s \in S} s \to \{0, 1\}$. These are local views in the sense that every f_s is a function $f_s : s \to \{0, 1\}$ and for every s: $F|_s = f_s$. If a set of local functions $\{f_s\}_{s \in S}$ meets the above criterion we call that set an agreeing set of functions. An agreement test is a probabilistic algorithm that picks two sets s_1, s_2 of size k that intersect each other on

ξ elements and checks whether $f_{s_1}|_{s_1 \cap s_2} = f_{s_2}|_{s_1 \cap s_2}$. A structure is said to support agreement testing (alternatively, a structure is an agreement expander) if the following two properties hold:

1. The agreement test always accepts agreeing sets of functions.
2. If the agreement test rejects a set with probability ε then there is a global function that disagrees with at most $O\left(\frac{1}{1-\frac{\xi}{k}}\varepsilon\right)$ of the local functions.

Most works that pertain to agreement testing are interested in the case where the intersection between sets is small, specifically $\xi = \frac{k}{2}$. This is because in those cases the proportion between k and ξ is constant and any local assignment that is rejected with probability ϵ has a global assignment which disagrees with $O(\epsilon)$ of the local assignments. In this work, however, we are interested in a slightly different definition of agreement expansion called 1-*up agreement expansion*. In 1-up agreement the intersection between pairs of sets contain all but one element from each set (i.e. $\xi = k - 1$). Also note that many complexes (and even some sparse complexes [9]) exhibit 1-up agreement expansion (as well complexes such as the complete complex).

In [9] Dinur and Kaufman proved that some high dimensional expanders support agreement testing and term these *agreement expanders*. Since then agreement expanders have proven to be extremely useful in various contexts: From derandomization for direct product testing [9] to conversion of local tests to robust tests [8] and others (for more examples see [5, 4]).

List agreement expansion

In this paper we present a new, natural generalization of agreement testing in which, instead of being given a single function for each set s, we are given a list of l functions and we want to test whether these functions are local views of l global functions (where l is some constant). By that we mean that there exist l global functions $F_1, \cdots, F_l : \bigcup_{s \in S} s \to \{0, 1\}$ and a permutation π_s for every s such that every local view $f_s^i : s \to \{0, 1\}$ agrees with the global function $F_{\pi_s(i)}$ (i.e. $F_{\pi_s(i)}|_s = f_s^i$). In this paper we ask whether there are structures that support list agreement tests. We term such structures *list agreement expanders*.

▶ **Definition 1** (List Agreement expansion, informal. For formal see 26). *We say that a set system exhibits list agreement expansion if there exists a tester that, given access to a set of assignments $\mathscr{F} = \{\mathscr{F}_i^s\}_{s \in S, i \in [l]}$ such that $\mathscr{F}_i^s : s \to \{0, 1\}$, queries Q of the local assignments[1] $\mathscr{F}_{i_1}^{s_1}, \cdots, \mathscr{F}_{i_Q}^{s_Q}$ and accepts or rejects such that the following holds:*
1. *Always accepts if there exists l functions $F_1, \cdots, F_l : \bigcup_{s \in S} s \to \{0, 1\}$ and a permutation π_s for every s such that every local view $f_s^i : s \to \{0, 1\}$ agrees with the global function $F_{\pi_s(i)}$ (i.e. $F_{\pi_s(i)}|_s = f_s^i$).*
2. *If the tester rejects with probability ϵ then $O(\epsilon)$ of the assignment in $\mathscr{F}$ can be changed such that the property stated above will hold.*

List agreement testing compared to agreement testing

At first list agreement testing might seem fairly reminiscent of agreement testing and, while the definitions are similar, list agreement testing seem to require different tools altogether. In the list agreement testing paradigm one is not only concerned with having agreement between local assignments, but also that these agreements are structured in the right way. One key example of agreements that are not well structured is the following: Consider a cycle of odd length

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[1] The algorithm queries one of the local assignments in one of the lists.

with the following 2-assignments on every edge: $\mathscr{F}^{\{u,v\}} = \{[u = 1, v = 0], [u = 0, v = 1]\}$. Note that any two edges that share a vertex agree on the intersection[2]. That being said, $\mathscr{F}$ is not agreeing (since if it was an agreeing assignment the graph would be bi-partite[3]).

Conditions under which list agreement testing is possible

As we discussed, list agreement not only requires agreement but also some additional structure. We will show that this additional structure comes in the form of *coboundary expansion* - A topological notion of expansion. Our construction of a list agreement tester will rely on coboundary expansion as a way to decouple the l instances of agreement testing and then use agreement expansion in order to achieve local agreement. We found that the global structure required in order to have l global cochains is, in a sense, equivalent to a coboundary of a different complex. That complex is induced by the original complex and the agreement test. We show that that the coboundaries of the induced complex are testable using the fact that the original complex is *both* a coboundary expander and a 1-up agreement expander. Effectively, we use the coboundary expansion in order to derive the *global* structure (i.e. that the local assignments can indeed be "glued together" into l global functions) while using the agreement expansion in order to derive the *local* structure (i.e. that the local agreements agree with each other).

As we previously hinted at, list agreement offers a very descriptive language which can, at times, be considerably richer than regular agreement. This richer structure allows us to, for example, describe the question of whether a subgraph[4] of the complex's underlying graph is two sided or not. One such subgraph of particular interest is a cycle as list agreement allows us to describe the question of whether a cycle is of odd length merely by knowing which vertices are a part of the cycle. This, in effect, yields a non-constant lower bound on the number of queries required in order to test list agreement in the general case as testing whether a cycle is odd cannot be done locally (More details on this can be found in the full version of this paper).

In order to overcome this limitation we introduce a restriction on list agreement. Namely, we require that the local assignments given to each set have some small distance separating them. Using this small distance we show that testing list agreement is possible with a constant number of queries.

Now that we have an understanding of list agreement expansion we can present our main theorem:

▶ **Theorem 2** (Main Theorem, informal. For formal see 51). *Any simplicial complex that has sufficient expansion properties (namely coboundary expansion and a 1-up agreement expansion) supports list agreement testing using 3l queries[5] (where l is the length of the list) under small distance assumptions on the local assignments[6].*

It is important to note that there are simplicial complexes that meet the Theorem's criteria. For example, the spherical buildings and the complete complex have sufficient expanding conditions for Theorem 2 (See [11] and [17]).

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[2] In the sense that there is a permutation π such that the i-th assignment of one of the faces agrees with the $\pi(i)$-th assignment of the other.

[3] One can interpret the local lists as "u and v are on different sides of the graph".

[4] Here a subgraph is determined by picking a subset of the vertices and all the edges that connect them.

[5] The test queries all the local assignments of three faces.

[6] More specifically we assume that the local assignments differ on at least two vertices. This assumption cannot be removed in the domain of complexes that we examine. For further details, see the full version.

Direct sum testing

Another natural (and extremely useful) construction in hardness amplification is the direct sum.

▶ **Definition 3** (Direct sum). *Given a function $f : S \to \{0, 1\}$ (where S is an arbitrary set) its k-fold direct sum is a function $F : \binom{S}{k} \to \{0, 1\}$ such that: $F(A) = \sum_{a \in A} f(a)$.*

Direct sums are useful in a variety of contexts, from Yao's XOR Lemma [22] which states that if a function is hard to approximate then its direct sum is exponentially harder, to the hardness of approximation of problems in $P^{NP} \parallel$[7] [15].

A unified framework for direct sum testing

It is natural to ask how, given a function, can one make sure that it is indeed a direct sum in a derandomized fashion. There have been several works on derandomizing direct sum testing [2, 12] but the tests presented in them for constant values of k were heavily dependent on the parity of k. In the full version of this work, we show how to use list agreement expanders in order to provide a new natural test for whether a function F is a direct-sum (while having stronger assumptions on the expansion of the complex). Our testing framework is the first that can be applied to any value of k regardless of its parity. In addition, our framework shaves off an $O(k)$ factor in the query complexity for the case when k is odd (compared to [12]). This is discussed further in the full version of this paper.

1.1 High Dimensional Expansion Toolset

We will now move on to presenting the main toolset we use in the proof:

Simplicial complexes

Simplicial complexes are generalizations of graphs to higher dimensions. A simplicial complex is a hyper-graph with closure property, i.e. if σ is a hyper-edge then any subset of σ is also an hyper-edge in the hyper-graph. We term the hyper-edges of a simplicial complex as its faces and define the dimension of a face σ to be $|\sigma| - 1$. We denote the set of i dimensional faces by $X(i)$. We also define the dimension of a simplicial complex as the dimension of its maximal face. For example, any non-empty graph is a 1-dimensional simplicial complex[8], its vertices are the 0-dimensional faces and edges are its 1-dimensional faces. Note that in connected graphs all of the maximal faces are of the same dimension as every vertex is part of an edge (otherwise there exists an isolated edge and the graph is not connected), when this holds we say that complex is *pure*. We limit our discussion to pure simplicial complexes. It is often convenient to think of high dimensional faces as geometrical shapes, for example we think of a 2-dimensional face as a triangle, a 3-dimensional face as a pyramid etc. In many cases we will be interested in weighted simplicial complexes. In weighted simplicial complexes a weight function is given to the top dimensioanl faces. This weight function is positive and sums up to 1. The weight of lower dimensional faces is determined by the weight of the top dimensional faces in which they are contained. It is important to note that even if the top dimensioanl faces all have the same weight, the same does not necessarily hold for faces of a lower dimension.

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[7] $P^{NP} \parallel$ is the set of problems that can be solved in polynomial time with oracle access to a problem in NP such that all the queries to the oracle are performed in parallel.

[8] We add the empty face to the graph.

High dimensional expanders

High dimensional expanders are generalizations of one-dimensional expanders (i.e. graph expanders) to higher dimensions. Unlike the one dimensional case, there are multiple definitions for the notion of expansion in higher dimensions. Moreover, the connections between the various generalizations of expansion is unknown. In this work we will use two of these definitions: *coboundary expansion* and *agreement expansion*. We are going to think of both definitions of expansion as measures of how well certain properties can be tested on the complex.

Coboundary expansion

Coboundary expansion is a natural generalization of the combinatorial expansion to higher dimensions. Specifically, one dimensional expansion can be thought of how well the following test tests whether a cut in the graph is trivial: Pick a random edge and accept iff it does not cross the cut. Note that this corresponds directly with the graph's cheeger constant as the cut can be thought of as a set and the number of edges that cross the cut are exactly the outgoing edges of the set. Therefore the cheeger constant of the graph determines *exactly* how well the test performs. Coboundary expansion is a generalization of the cheeger constant that includes the second dimension as well. In the second dimension the property being tested is whether a given subset of edges represents a cut (i.e. whether there is an underlying cut such that an edge is in the set iff it crosses the cut). The test being measured is the test that picks a triangle and accepts if the number of edges that are both in the set and the triangle is even. In a coboundary expander both tests (i.e. the test in the first and second dimensions) have a large soundness parameter (larger than some constant). This view of coboundary expansion was first introduced in [16].

Agreement expanders

Agreement expanders are (possibly sparse) simplicial complexes and a distribution $\mathcal{D}$ on the k-dimensional faces such that, for any set of assignments to the k-dimensional faces of the simplicial complex, if most pairs $\sigma_1, \sigma_2 \sim \mathcal{D}$ agree on their intersection then the local assignments are close to agreeing with some global assignment (for any k). In this work we are going to use a strengthening of agreement expansion, namely 1-*up agreement expanders*, 1-up agreement expanders are agreement expanders where the distribution $\mathcal{D}$ matches the procedure of picking a $(k+1)$-dimensional face and then randomly selecting two k-faces that are contained in it. In [9] Dinur and Kaufman show that there exists a family of bounded degree 1-up agreement expanders.

Covers

Covers of a simplicial complex X are simplicial complexes Y that, for every vertex v of X contain l vertices $[v, 1], \ldots, [v, l]$, In addition, $\sigma = (v_1, \ldots, v_i)$ is a face in X iff Y has l disjoint faces of the form $([v_1, j_1], \ldots, [v_i, j_i])$. For example: a 2-cover of a 1-dimensional complex (i.e. a graph) has, for every vertex v, two vertices $[v, 0], [v, 1]$. In addition, for every edge (v, u) in the graph the cover has two edges, either $([v, 0], [u, 0]), ([v, 1], [u, 1])$ or $([v, 0], [u, 1]), ([v, 1], [u, 0])$. In this work we are going to discuss a special subset of the covering spaces of a simplicial complex. Specifically we are going to be interested in the trivial covering spaces, i.e. covering spaces that are comprised of l disjoint copies of the original complex. We end our discussion of covering spaces by noting that the definition of a cover applies to general topological spaces and not necessarily to simplicial complexes.

Near Covers

Near Covers are spaces that are close to being covering spaces of a given simplicial complex. Near covers are a relaxation of covers where only the first two dimensions are required to be covered properly. More specifically, every vertex v is covered by l vertices and every edge $\{v, u\}$ is covered by l edges. The higher dimensional faces are only covered if the lower dimensioanl faces are structured in a way that allows them to be covered. For example consider a simplicial complex comprised of a single two dimensional face $\{v_1, v_2, v_3\}$. Consider the near cover of said complex whose 1-dimensioanl faces are the following cycle of length 6:

$$\left\{ \begin{matrix} \{[v_1,0],[v_2,0]\},\{[v_2,0],[v_3,0]\},\{[v_3,0],[v_1,1]\}, \\ \{[v_1,1],[v_2,1]\},\{[v_2,1],[v_3,1]\},\{[v_3,1],[v_1,0]\} \end{matrix} \right\}$$

Note that every vertex and every edge are covered and yet the single 2-dimensional face is not and thus this is a near cover that is not a cover. It is important to note that every cover is a near cover and that sometimes *genuine cover* is used instead of cover to emphasize that a near cover is indeed a cover.

1.2 Proof Strategy

In this work we are interested in testing whether a set system S exhibits list agreement expansion. Agreement expansion stems directly from the rapid convergence of random walks that local spectral expanders exhibit. While we still test for agreement (and therefore require the same spectral expansion assumptions on our set system) we also recall that list agreement requires some additional structure over agreement. Specifically, not only does list agreement require local agreement it also requires the local agreements to exist in such a way that l different global functions are formed.

Let us now characterize this additional structure further. Consider a set of assignments $\mathscr{F}$ that gives every set s in the set system a set of l assignments $\mathscr{F}_1^s, \cdots, \mathscr{F}_l^s$. For ease of presentation, allow us also to assume that for every two sets in the set system s, s' there exists a unique permutation $\pi_{s,s'}$ such that $\mathscr{F}_i^s|_{s \cap s'} = \mathscr{F}_{\pi_{s,s'}(i)}^{s'}|_{s \cap s'}$. Consider the following definition:

▶ **Definition 4** (Coboundary structure). *Let $A \subseteq \binom{S}{2}$ and let G be a group. We say that a function $f : A \to G$ has coboundary structure over A with coefficients in G if there exists $g : S \to G$ such that $f(a,b) = g(b) \left(g(a)\right)^{-1}$.*

We will show that, regardless of the set A, $\mathscr{F}$ exhibits list agreement iff $f(s, s') = \pi_{s,s'}$ has a coboundary structure with coefficients in S_l (the symmetric group with l elements). Recall $\mathscr{F}$ exhibits list agreement if there exists l global functions $\mathcal{F}_1, \cdots, \mathcal{F}_l$ and, for every set s there exists a permutation π_s such that $\mathscr{F}_i^s = \mathcal{F}_{\pi_s(i)}|_s$. It is easy to see that in this case $\pi_{s,s'} = \pi_{s'} \left(\pi_s\right)^{-1}$. In addition, if $f(s, s')$ has a coboundary structure then there exists g such that $f(a,b) = g(b) \left(g(a)\right)^{-1}$. In that case, the global functions $\mathcal{F}_i(v)$ can be calculated by picking any set s that contains v and setting $\mathcal{F}_i(v) = \mathscr{F}_{g(s)(i)}^s(v)$. We are therefore interested in finding out if the permutations $\pi_{s,s'}$ exhibit a coboundary structure.

Alas the fact that two local lists of assignments agree with each other under some permutation does not necessarily mean that the permutation remains the same when correcting f so that it exhibits coboundary structure (even if there is only one permutation that causes agreement). It is therefore natural to ask "*how far are $\pi_{s,s'}$ from exhibiting a coboundary structure?*". This is *exactly* what coboundary expansion with coefficients in S_l measures.

Then, after we have corrected the function f so that it exhibits a coboundary structure, we might have ruined some of the agreement we started with. We will note, however, that now we have l *independent* instances of an agreement problem (One for each $\left\{ \mathscr{F}^s_{g(s)(i)} \right\}_{s \in S}$, where $g : S \to S_l$ such that $f(s, s') = g(s')\left(g(s)\right)^{-1}$). We use the agreement expansion to resolve those.

To conclude, we use spectral expansion in order to derive the agreement and topological expansion in order to derive this additional structure.

1.3   Proof Layout

We start by assuming that the complex is a 1-up agreement expander and therefore the 1-up test is a good agreement test. We then model the choices done by the 1-up agreement test as a simplicial complex which we dub "the representation complex". The modeling is done in the following way: The vertices of the new complex will correspond to the k-dimensional faces of the original complex. The edges of the new complex will correspond to the choices done by the 1-up agreement test (i.e. if a pair of faces σ_1, σ_2 are chosen with some probability then there is an edge between σ_1 and σ_2). We will construct the higher dimensional faces of the representation complex so that the weight of the an edge (σ_1, σ_2) in the representation complex will correspond with the probability that the pair σ_1, σ_2 is chosen by the 1-up agreement tester (see Definition 42 for a formal definition). Note that the edges of the representation complex can be thought of as the set A from Definition 4. We will therefore be interested in examining the expansion properties of the second dimension of the representation complex. Alas, the representation complex is not a coboundary expander but it does have a structure that facilitates bounding the distance of a cochain from being a coboundary using a local property (the exact details of which are covered in the full version).

Before we move on we note that the function f from Definition 4 can be thought of as a description of a near cover of the representation complex in the following way: Every vertex v is covered by $|G|$ elements - $\{[v, 1], \cdots, [v, |G|]\}$. The edges are described by the function f in the following way: If v and u are connected in the original complex then $[v, i]$ and $[u, f(\{v, u\})(i)]$ are connected in the near cover. In order to simplify the presentation of the rest of the proof we are going to use the language of near covers. It is important to note that it is equivalent to the presentation in the proof strategy.

We start by showing how to relate any l-assignment whose local assignments are sufficiently differing to an l-near-cover of the representation complex. Specifically, every one of the local assignments to σ in the l-assignment will correspond to a vertex that covers σ and two vertices in the cover are connected if the assignments they correspond to agree on their intersection (and, of course, if the edge that they cover exists in the representation complex). Unlike the assumption in the proof strategy, we cannot assume that every pair of lists agree with each other. We therefore cover every edge (v_1, v_2) whose assignments do not agree with each other using some fixed matching between the vertices that cover σ_1 and σ_2. Full details of this construction can be found in Section 5.

Now that we have constructed the near cover we want it to have the additional structure that we presented in the proof strategy. We note that a near cover exhibits the additional structure we are interested in iff it corresponds to a coboundary. We also note that the near cover corresponds to a coboundary iff it is a genuine cover that is comprised of l distinct copies of the original complex. We use the connection between cochains and near covering spaces as well as the testability of coboundaries in the representation complex to show that it is possible to bound the distance of the near cover we constructed from being comprised of

l distinct copies of the original complex. Consider now a corrected version in the near cover. While now we have the additional structure we are interested in, we still have not gotten the agreement we are interested in.

Assume for now that we can query this genuine cover directly (which we cannot as we only have access to a near cover that is close to it). In the genuine cover, each of the copies of the original cover has an associated assignment to its k-faces (since every vertex in the cover is associated with a single local assignment to the face it represents). Consider running the 1-up agreement test on each of the copies of the representation complex with its associated assignments. The test picks a $(k + 1)$-face σ and two of its k-sub-faces σ_1, σ_2, it then queries the local assignment of these k-faces and accepts iff they agree on their intersection. Another way of looking at the test is that it samples an edge e in the representation complex and accepts iff the faces represented by the vertices in the edge agree on their intersection. We can therefore bound the distance of the set of assignments from agreeing using the agreement expansion of the original complex.

Alas we cannot query the genuine cover directly. We can, however, bound from above the probability that the agreement test would reject had we ran it on that genuine cover. Recall the construction of the near cover and note that if e was covered by edges that represent agreement in the original near cover and, in addition, the cover of e was not changed then the assignments to both sides of e agree on their intersection. Therefore the agreement test would only reject if it picked an edge that had no agreement in the first place or an edge that was changed when the near cover was corrected. Note that the norm of the edges that satisfy these properties can be derived locally by querying the lists of assignments - the former by observing whether there is an agreement and the latter due to the testability of coboundaries in the representation complex. We use this in order to bound the distance of any l-assignment from being an agreeing l-assignment.

Before we move on we wish to emphasise how the covering spaces allow us to decouple dependencies. Consider the near covering space of the representation complex. In an ideal case, the near cover induced by the l-assignment is a genuine cover that corresponds to a coboundary. In such cases the cover is comprised of l independent copies of the original complex and we can run the agreement test l times independently and get the distance of the l-assignment from agreeing. In most cases, however, the near cover induced by the l-assignment is not a genuine cover that corresponds to a coboundary. In these cases the copies of the complex are dependent on each other in the sense that what would have been copies of the original complex are now connected via various edges. A crucial step in the test we propose involves "decoupling" these dependencies. We will show that, despite the fact that the representation complex's cohomology does not vanish, the coboundaries of the representation complex are still locally testable. Using this test we can bound the distance of the l-assignment from inducing independent instances of agreement testing. We believe that the technique of modeling objects of interest as near covers of a coboundary expander could be a useful measure of dependency as not only does it bound the distance of the object from being independent but it is also locally testable.

1.4   Related Work

Agreement testing is an inherit part of most PCP constructions as well as various other applications and were extensively studied in the past few years (for examples, see [9, 3, 7, 10, 1, 19, 6]). In recent years a connection between high dimensional expanders and agreement testing was established: In [9] Dinur and Kaufman defined the notion of agreement expanders described above. Later Kaufman and Mass presented in [17] a new method for constructing

agreement expanders. In this work we generalize the notion of agreement expanders to the case where each face has l local assignments. The goal in this new setting is to check whether there exists l global functions that match all the local assignments (for formal definition see 23).

In order to provide said generalization we use the connection between cocycles and cover spaces of simplicial complexes. The connection between cocycles and cover spaces of a simplicial complex was first introduced in [21]. Then in [11] Dinur and Meshulam defined the notion of cover stable complexes which are complexes where if a near cover satisfies most local conditions then it is close to a genuine cover. They then show that a simplicial complex is cover stable if and only if it is an expanding with respect to non-abelian cohomology. We use the connection between covers of high dimensional expanders and their cocycles in a new way: Specifically, we use the structure of covers that correspond with coboundaries in order to untangle the near cover into disjoint copies of the original complex.

Direct sums are natural construction in hardness of approximation. A function $F : \binom{[n]}{k} \to \{0, 1\}$ is a k-direct-sum of $f : [n] \to \{0, 1\}$ if for every $\sigma \in \binom{[n]}{k}$ it holds that $F(\sigma) = \sum_{v \in \sigma} v$. Testing direct sums is a problem that was extensively studied in recent years. The first connections between testing direct sums and high dimensional expanders was presented in [16] in which Kaufman and Lubotzky presented a test for the case of $k = 2$. Later in [2] (and generalized to high dimensional expanders in [9]) a test for constant even values of k was found. This test, however, could not deal with the case where k is odd due to inherit limitations. After that, another test was proposed by Gotlib and Kaufman in [12]. Their test could deal with the case where k is an odd constant, however there seem to be no simple way of extending their setting to the case where k is even. In this work we use list agreement expanders in order to present the first test for direct sums for constant values of k regardless of its parity.

1.5 Paper Organisation

We will start by defining the notions of high dimensional expansion we use throughout the paper in Section 2. In addition to that Section 2 includes a formal definition of list agreement expansion and the distance measure we are going to use. Then, in section 3, we present the representation complex and discuss some of its properties. In section 4 we present properties of covers that correspond to coboundaries. We then move on to show how assignments to the original complex imply a near cover for the representation complex in section 5. After that we show how to use the cover stability of the representation complex in order to show that the complex is a list agreement expander in section 6.

In the full version of this paper we show how to use list agreement expanders in order to provide a test for k-direct-sum that is independent of the oddity of k. In addition, we discuss the 2-differing assumption further and show that without the 2-differing assumption there are some complexes of interest (for example, the spherical buildings) in which there are no tests for list agreement that perform a constant number to queries. Finally, we show a lower bound to the number of queries required for list agreement.

2 Preliminaries

2.1 Simplicial Complexes

As we stated before, the generalization of graphs we will be using are simplicial complexes, which we will now present more thoroughly:

▶ **Definition 5** (Simplicial complex)**.** *A simplicial complex X is set of sets such that if $\sigma \in X$ then every $\sigma' \subseteq \sigma$ is also in X. Each of the sets in X are termed the faces of X.*

▶ Note. The faces of a simplicial complex have an orientation. The orientation must be consistent in the sense that there is an ordering of the 0-dimensional faces that determines the orientation of all the faces in higher dimensions. In the vast majority of this paper the orientation is irrelevant to the arguments and therefore we will ignore it for the most part. When we are forced to consider the orientation of a face $\sigma = \{v_0, \ldots, v_i\}$ we will denote the face with round brackets (i.e. $\sigma = (v_0, \ldots, v_i)$).

Let us also define the dimension of a face and the dimension of the complex

▶ **Definition 6** (Dimensions). *Let X be a simplicial complex. Define the dimension of a face $\sigma \in X$ to be $\dim(\sigma) = |\sigma| - 1$. Also, denote the set of faces of dimension i by $X(i)$ In addition, define the dimension of the simplicial complex X to be $\dim(X) = \max_{\sigma \in X}(\dim(\sigma))$.*

▶ **Definition 7** (Pure simplicial complex). *A simplicial complex is called pure if all of its maximal faces are of the same dimension.*

From this point onward we limit our discussion to pure simplicial complexes (and whenever we refer to a simplicial complex we will actually refer to a pure simplicial complex). A standard weight function is defined over the various faces of a simplicial complex which we will present below:

▶ **Definition 8** (Weight function). *Let X be a d dimensional simplicial complex. Define the weight of a face $\sigma \in X$ as the fraction of maximal faces in X that contain σ. Formally:*

$$\mathrm{w}_X(\sigma) = \frac{|\{\tau \in X(d) | \sigma \subseteq \tau\}|}{\binom{d+1}{|\sigma|} |X(d)|}$$

When the complex is clear from context we will omit it from the notation.

Note that this weight function can be thought of as a probability distribution over the faces of each dimension (and indeed we will think of it as such). Moreover there is a way to sample a face with the probability distribution defined by the weight. We can also use this weight function to define the following norm over sets of faces:

▶ **Definition 9** (Norm on faces). *Let X be a d-dimensional simplicial complex, let $-1 \le i \le d$ and also let $S \subseteq X(i)$. Define the norm of S in X to be $\|S\|_X = \sum_{\sigma \in S} \mathrm{w}_X(\sigma)$. When the complex is clear from context we will omit it from the notation.*

We will now introduce the notions of cochains, cocycles and coboundaries which are natural spaces of functions over any given simplicial complex:

▶ **Definition 10** (Cochains). *Let X be a simplicial complex and let G be a group. Define the set of 0-cochains with coefficients in G to be the set of functions from the vertices of X to G. In addition, denote the set of 0-cochains with coefficients in G by $C^0(X; G)$. Also, define the set of 1-cochains with coefficients in G as the following set:*

$$C^1(X; G) = \left\{ F : X(1) \to G \middle| F(v, u) = (F(u, v))^{-1} \right\}$$

We also define the following operators:

▶ **Definition 11** (Coboundary operators). *Define the following three operators*
- *Define the operator $d_{-1} : C^{-1}(X; G) \to C^0(X; G)$ to be $d_{-1}F(v) = F(\emptyset)$.*
- *Define the operator $d_0 : C^0(X; G) \to C^1(X; G)$ to be $d_0F(u, v) = F(u)(F(v))^{-1}$.*
- *Define the operator d_1 such that for every cochain $F \in C^1(X; G)$ and every $(u, v, w) \in X(2)$: $d_1F(u, v, w) = F(u, v)F(v, w)F(w, u)$.*

These operators define the following spaces over the first three dimensions of the simplicial complex:

▶ **Definition 12** (Cocycles and coboundaries). *Let X be a simplicial complex and G a group, define the following spaces:*

- *For $i \in \{0, 1, 2\}$ define the i dimensional coboundaries to be*
  $B^i(X; G) = \{d_{i-1}F | F \in C^{i-1}(X; G)\}.$
- *For $i \in \{0, 1\}$ define the i dimensional cocycles as $Z^i(X; G) = \{F \in C^i(X; G) | d_i F = \mathbb{1}\}.$*

And, as with the cochains, when $G = \mathbb{F}_2$ we omit G from the notation.

▶ Note. We only define these spaces in the first two dimensions because we are working with a general group G rather then an abelian group. If we assume that G is an abelian group one can generalize the definition of cochains, cocycles and coboundaries as well as the coboundary operators to higher dimensions.

Consider also the following:

▶ **Fact 13.** *For every simplicial complex X and group G: $B^i(X; G) \subseteq Z^i(X; G) \subseteq C^i(X; G)$*

Of particular interest to our case are coboundaries with coefficients in the symmetric group with l elements that we denote S_l.

We also extend the norm defined in Definition 9 to a norm over the cochains:

▶ **Definition 14** (Norm of a cochain). *Let X be a simplicial complex. For any $F \in C^i(X; G)$, define the following norm: $\|F\|_X = \|\{\sigma \in X(i) | F(\sigma) \neq 1\}\|$. When the complex is clear from context we will omit it from the notation.*

This norm also defines a natural distance function between any two cochains:

▶ **Definition 15** (Distance between cochains). *Let F_1, F_2 be two cochains in some simplicial complex X. Define the distance between F_1 and F_2 to be $\text{dist}(F_1, F_2) = \left\| F_1 (F_2)^{-1} \right\|.$*

It is also natural, given a simplicial complex, to describe the simplicial complex that is constructed using only faces whose dimension is at most some i. This is called the *skeleton* of the simplicial complex and is formally defined below:

▶ **Definition 16** (Skeleton). *Let X be a simplicial complex. Define its i-th skeleton to be the following simplicial complex: $X^{(i)} = \{\sigma \in X | \sigma \leq i\}.$*

It would also be useful to define local views of faces in a simplicial complex. We call these local views *links* and think of them as the faces seen by a certain face.

▶ **Definition 17** (Link). *Let X be a d-dimensional simplicial complex and let $\sigma \in X(i)$. Define the link of σ in X as the following $(d-i)$-dimensional simplicial complex: $X_\sigma = \{\tau \setminus \sigma | \sigma \subseteq \tau$ and $\sigma \in X\}.$*

Note that the weight function of the faces in the link of a face σ is strongly connected to the weight of σ and the weight function of the original complex:

▶ **Definition 18.** *Let X be a d-dimensional simplicial complex and let σ be an i-dimensional face. The weight of a j-dimensional face in the link of σ is $\text{w}_{X_\sigma}(\tau) = \frac{\text{w}_X(\tau \cup \sigma)}{\binom{|\sigma|+|\tau|}{|\tau|} \text{w}_X(\sigma)} = \frac{\text{w}_X(\tau \cup \sigma)}{\binom{i+j+2}{j+1} \text{w}_X(\sigma)}.$*

We finish our presentation of simplicial complexes by defining isomorphic simplicial complexes:

▶ **Definition 19** (Isomorphic simplicial complexes). *Let X, Y be two simplicial complexes. We say that X is isomorphic to Y and denote $X \cong Y$ if there exists an invertible function $f : Y \to X$ such that of every $\sigma, \tau \in Y$ it holds that $\sigma \subseteq \tau \Leftrightarrow f(\sigma) \subseteq f(\tau)$.*

2.2   On Assignments and l-Assignments

We will now present the notions of assignments and l-assignments we will then move on to define list agreement expanders.

We will start by defining assignments and agreeing assignments. Assignments are essentially a set of local functions from every k-dimensional face of the complex to $\{0,1\}$, while agreeing assignments can be thought of as "snippets" of some global function. We will end the discussion of assignments by defining the distance between two assignments as the number of faces on which they differ.

▶ **Definition 20** (Assignment). *Define an assignment to the k-faces of a simplicial complex X to be $\mathcal{F} = \{\mathcal{F}^\sigma\}_{\sigma \in X(k)}$ such that $\mathcal{F}^\sigma : \sigma \to \{0,1\}$. We also denote the set of assignments by $\boldsymbol{\mathcal{S}}$.*

▶ **Definition 21** (Agreeing assignment). *Define the set of agreeing assignments to the k-faces to be $\boldsymbol{\mathcal{A}} = \left\{ \mathcal{F} \middle| \exists F \in C^0(X) \, \forall \sigma \in X(k) : F|_\sigma = \mathcal{F}^\sigma \right\}$. We also say that $\mathcal{F}$ agrees with F.*

▶ **Definition 22** (Distance function for assignments). *Define the distance between two k-assignments as $\operatorname{dist}(\mathcal{F}, \mathcal{G}) = \|\{\sigma \in X(k) | \mathcal{F}^\sigma \neq \mathcal{G}^\sigma\}\|$. In addition, given a set of assignments $\boldsymbol{\mathcal{S}}$ define the distance of an assignment $\mathcal{F}$ from $\boldsymbol{\mathcal{S}}$ to be $\operatorname{dist}(\mathcal{F}, \boldsymbol{\mathcal{S}}) = \min_{\mathcal{S} \in \boldsymbol{\mathcal{S}}} \{\operatorname{dist}(\mathcal{F}, \mathcal{S})\}$.*

We are now ready to define l-assignments and agreeing l-assignments. l-assignments are l parallel assignments, i.e. every face in the complex has l local functions associated with it.

▶ **Definition 23** (l-assignments). *Given a simplicial complex X define a k-dimensional l-assignment to be $\mathscr{F} = \{\mathscr{F}_i^\sigma\}_{\sigma \in X(k), i \in [l]}$ such that $\mathscr{F}_i^\sigma : \sigma \to \{0,1\}$ i.e. $\mathscr{F}$ is an assignment of l local function to each k-face of X.*

We define the distance between two l-assignments as the number of local functions on which they differ (normalised by the weights of the faces and the length of the list).

▶ **Definition 24** (Distance between l-assignments). *Define the distance between two k-dimensional l-assignments as $\operatorname{dist}(\mathscr{F}, \mathscr{G}) = \sum_{\sigma \in X(k)} \operatorname{w}(\sigma) \frac{|\{i \in [l] | \mathscr{F}_i^\sigma \neq \mathscr{G}_i^\sigma\}|}{l}$. In addition, given a set of l-assignments $\boldsymbol{\mathscr{S}}$ define the distance of an l-assignment $\mathscr{F}$ from $\boldsymbol{\mathscr{S}}$ to be $\operatorname{dist}(\mathscr{F}, \boldsymbol{\mathscr{S}}) = \min_{\mathscr{S} \in \boldsymbol{\mathscr{S}}} \{\operatorname{dist}(\mathscr{F}, \mathscr{S})\}$.*

The notion of agreement in this case is more complicated than in the non-list case. In regular assignments an agreeing assignment is an assignment that is consistent with some global function. In the l-assignment agreement case we are interested whether there are l global functions such that the local assignments of each vertex are a list of "snippets" of the l global functions.

▶ **Definition 25** (Agreeing l-assignment). *Define an agreeing k-dimensional l-assignments to be a k-dimensional l-assignment $\mathscr{F}$ such that there are l cochains $F_1, \ldots, F_l \in C^0(X)$ and for every face $\sigma \in X(k)$ there exists a permutation π_σ such that the assignment $\mathcal{F}_i = \left\{ \mathscr{F}_{\pi_\sigma(i)}^\sigma \right\}_{\sigma \in X(k)}$ agrees with F_i. Denote the set of agreeing assignments as $\boldsymbol{\mathscr{A}}$.*

We are now ready to define list agreement expansion:

▶ **Definition 26** (List agreement expander). *Let X be a d-dimensional pure simplicial complex. We say that X is a (β, l)-agreement-expander if there is a probabilistic algorithm A such that for every dimension k and every k-dimensional l-assignment $\mathscr{F}$ it holds that $\Pr\left[A^{\mathscr{F}} \text{ rejects}\right] \geq \beta \operatorname{dist}(\mathscr{F}, \boldsymbol{\mathscr{A}})$.*

In this paper we will present a weaker notion of list agreement expander. Specifically we are going to assume that the l-assignments are 2-locally-different defined below:

▶ **Definition 27** (Locally differing l-assignment)**.** *Let $\mathscr{F}$ be an l-assignment. We say that $\mathscr{F}$ is 2-locally-differing if for every $i \neq j$, every $\sigma \in X(k)$ there exists $x_1^{\sigma,i,j} \neq x_2^{\sigma,i,j}$ such that $\mathscr{F}_i^\sigma(x_1^{\sigma,i,j}) \neq \mathscr{F}_j^\sigma(x_1^{\sigma,i,j})$ and $\mathscr{F}_i^\sigma(x_2^{\sigma,i,j}) \neq \mathscr{F}_j^\sigma(x_2^{\sigma,i,j})$.*

We note that the 2-differing assumption is inherent for list agreement testing with a constant number of queries as we show in the full version of this paper. Before we conclude this section we state the following properties of the distance between an l-assignment and the agreeing l-assignments which we prove in the full version:

▶ **Lemma 28.** *For every set of permutations $\{\pi_\sigma\}_{\sigma \in X(k)}$ it holds that:*

$$\mathrm{dist}\,(\mathscr{F}, \boldsymbol{\mathscr{A}}) \leq \frac{1}{l} \sum_{i=1}^{l} \mathrm{dist}\left(\left\{\mathscr{F}_{\pi_\sigma(i)}^\sigma\right\}_{\sigma \in X(k)}, \boldsymbol{\mathscr{A}}\right)$$

▶ **Corollary 29.** *Let $\mathscr{F}$ be an l-assignment and let $\tilde{\mathscr{F}}$ be an agreeing l-assignment such that $\mathrm{dist}\,(\mathscr{F}, \boldsymbol{\mathscr{A}}) = \mathrm{dist}\,(\mathscr{F}, \tilde{\mathscr{F}})$. In addition let $\{\pi_\sigma\}_{\sigma \in X(k)}$ be a permutation such that for every i: $\left\{\tilde{\mathscr{F}}_{\pi_\sigma(i)}^\sigma\right\}_{\sigma \in X}$ is an agreeing assignment then $\mathrm{dist}\,(\mathscr{F}, \boldsymbol{\mathscr{A}}) = \frac{1}{l} \sum_{i=1}^{l} \mathrm{dist}\left(\left\{\mathscr{F}_{\pi_\sigma(i)}^\sigma\right\}_{\sigma \in X(k)}, \boldsymbol{\mathscr{A}}\right)$.*

2.3 Coboundary Expansion

The first form of high dimensional expansion we will present is *coboundary expansion*. Coboundary expansion was first defined by Linial and Meshulam [20] and independently by Gomov [14]. This form of expansion generalizes the notion of 1-dimensional expansion naturally. Before we define the coboundary expansion to higher dimensions, we will re-define the 0-dimensional expansion in the language that will allow for easier generalization. Consider the standard definition of Cheeger's constant:

▶ **Definition 30** (Cheeger's constant)**.** *Let $X = (X, E)$ be a graph, define the Cheeger constant of a graph to be $\min_{A \neq V, \emptyset} \left\{ \frac{|E(A, V \setminus A)|}{\min |A|, |V \setminus A|} \right\}$*

In our redefinition, instead of subsets of the vertices we will consider 0-dimensional cochains with coefficients in $\mathbb{F}_2$. We will represent a set of vertices $A \subseteq V$ be the following cochain:

$$F_A(v) = \begin{cases} 1 & v \in A \\ 0 & v \notin A \end{cases}$$

Consider how, in our new setting, we can describe an edge between A and $\bar{A}$ and note that the set of edges $\{v, u\}$ that are between A and $\bar{A}$ are exactly the edges for which $d_0 F_A(v, u) \neq 0$. Lastly, consider the following reformulation of the Cheeger's constant:

$$\min_{F \in C^0(X) \setminus B^0(X)} \left\{ \frac{\|d_0 F\|}{\mathrm{dist}\,(F, B^0\,(X))} \right\}$$

Note that $B^0\,(X)$ contains the cochains that correspond to V and $\emptyset$. Now, consider the following generalization of that reformulation of Cheeger's constant:

▶ **Definition 31** (i-dimensional expansion). *Define the ith dimensional Cheeger constant to be:*

$$h_i(X; G) = \min_{F \in C^i(X;G) \backslash B^i(X;G)} \left\{ \frac{\|d_i F\|}{\text{dist}\,(F, B^i\,(X;G))} \right\}$$

If $h_i(X; G) \geq \epsilon$ we say that the ith dimension of X ϵ-expands with G-coefficients.

And the following generalization of expansion:

▶ **Definition 32** (Coboundary expansion). *Let X be a d-dimensional simplicial complex. We say that X is an ϵ-coboundary expander with coefficients in G if for every face $\sigma \in X$ with dimension smaller than $d - 2$ it holds that $h_0(X_\sigma; G) \geq \epsilon$ and $h_1(X_\sigma; G) \geq \epsilon$.*

In [18] Kaufman and Mass showed how to construct coboundary expanders independently of the underlying group.

2.4 Covers of Simplicial Complexes

We will now move on to formally present the concept of covers of simplicial complexes and their connection to the cocycles of the simplicial complex:

▶ **Definition 33** (Cover space). *Let X, Y be two d-dimensional simplicial complexes. We say that (Y, f_Y) l-fold evenly covers X if f_Y is a surjective map from Y to X such that:*
- *f_Y is a translation function: for every $\sigma \in Y$ it holds that $|\sigma| = |f_Y(\sigma)|$ and $\sigma \subseteq \tau \Leftrightarrow f_Y(\sigma) \subseteq f_Y(\tau)$.*
- *Locally X and Y look the same: for every face $\sigma \in Y$, f_Y is an isomorphism between $\{\tau \in Y | \sigma \subseteq \tau\}$ and $\{\tau \in X | f_Y(\sigma) \subseteq \tau\}$.*
- *Every non-empty face of X is covered by exactly l-faces from Y: $\forall \sigma \in X \backslash \{\emptyset\} : |f_Y^{-1}(\sigma)| = l$.*

We will refer to f_Y as the covering map from Y to X. In addition we are going to say that Y is an l-cover of X if there exists a covering map from Y to X. Lastly we would sometimes refer to a cover as defined here as a genuine *cover (as opposed to a near cover, defined below).*

▶ **Definition 34** (Lift). *Let X be a simplicial complex, Y be a cover of X with the covering map f_Y. Also let $\gamma = (\gamma_1, \cdots, \gamma_n)$ be a path in X. A lift of γ to Y is a path $\gamma' = (\gamma'_1, \cdots, \gamma'_n)$ in Y such that for all i it holds that $f_Y(\gamma'_i) = \gamma_i$.*

We also adopt the definition of near covers defined by [11]:

▶ **Definition 35** (Near cover). *Let X be a simplicial complex and let Y be another simplicial complex. We say that Y is an l near cover of X if there exists a function $f_Y : Y \to X$ such that:*
- *For each vertex $v \in X(0)$ it holds that f_Y^{-1} can be identified with $[l]$. We will therefore use the notation $[u, i]$ to denote the vertex in $f_Y^{-1}[u]$ that corresponds to i.*
- *For every edge $(v, u) \in X(1)$ there exists $\pi \in S_l$ such that if $f_Y([v, i], [u, j]) = (v, u)$ then $i = \pi(j)$.*

Where S_l is the symmetric group of order l.

Note that the difference between near covers and genuine covers is that in near covers the faces of dimension larger or equal to 2 might not be properly covered. Consider, for example, a complex that contains a single triangle $T = \{\{v_1, v_2, v_3\}\}$ and consider the near cover of T whose 1-dimensional faces are (where $G = \mathbb{F}_2$):

$$Y(1) = \left\{ \begin{array}{l} \{[v_1,0],[v_2,0]\}, \{[v_2,0],[v_3,1]\}, \{[v_3,1],[v_1,1]\}, \\ \{[v_1,1],[v_2,1]\}, \{[v_2,1],[v_3,0]\}, \{[v_3,0],[v_1,0]\} \end{array} \right\}$$

Note that it is indeed a near cover since every vertex v_i is covered by $[v_1, 0], [v_1, 1]$ which can be identified with 0 and 1 respectively. In addition, since $\mathbb{F}_2$ is the symmetric group the second condition holds trivially. Also note that Y is not a cover of T since the second and third conditions of being a cover fail: The first condition fails for any vertex in the complex (since every vertex is a member of the triangle therefore $\{\tau \in X | f_Y(\sigma) \subseteq \tau\}$ contains a set of size 3 while $\{\tau \in Y | \sigma \subseteq \tau\}$ contains no such sets[9]). Let us now note that every 1-dimensional cochain in X implies a near cover by the following:

▶ **Definition 36.** *Let G be a group acting on the left of a set S and let $F \in C^1(X; G)$. Define the complex Y_F as the complex whose 0-dimensional faces are*
$Y_F(0) = \{[v, s] | v \in X \text{ and } s \in S\}$ *and its higher dimensional faces are:*

$$Y_F(i) = \{\{[v_0, s_0], \ldots [v_i, s_i]\} | \{v_0, \ldots, v_i\} \in X \text{ and } \forall i, j : s_i = F(v_i, v_j)s_j\}$$

Surowski showed in [21, Proposition 3.2] a characterization of when Y_F is a genuine cover of X. Specifically:

▶ **Lemma 37** (Proposition 3.2 in [21], rephrased). *Y_F is a genuine cover of X with the covering map $f_{Y_F}(\{[v_0, s_0], \ldots [v_i, s_i]\}) = \{v_0, \ldots, v_i\}$ iff F is a cocycle.*

2.5 Agreement Expansion

Another type of high dimensional expansion is agreement expansion. This type of expansion is concerned with the relation between agreement of local assignments and the distance of an assignment from agreeing. It was first introduced by Dinur and Kaufman in [9]. In this subsection we will introduce the concept of agreement testing which is a crucial part of our construction:

▶ **Definition 38** (Agreement expansion in the i-th dimension). *Let X be a d-dimensional simplicial complex and let $\mathcal{D}$ be a distribution over pairs of i dimensional faces that intersect each other on ξ vertices[10]. Define: $a_{i,\xi}(X, \mathcal{D}) = \min_{\mathcal{A} \in \mathcal{S}} \left\{ \frac{\Pr_{(\sigma,\tau) \sim \mathcal{D}}[\mathcal{A}^\sigma |_{\sigma \cap \tau} = \mathcal{A}^\tau |_{\sigma \cap \tau}]}{\text{dist}(\mathcal{A}, \bar{\mathcal{A}})} \right\}.$*

In addition let us define an agreement expander:

▶ **Definition 39** (Agreement expander). *Let X be a d-dimensional simplicial complex and let $\mathcal{D}$ be a distribution. We say that X is an α-agreement-expander if $\forall 0 \leq i \leq d : a_{i,\xi}(X, \mathcal{D}) \geq \alpha \left(1 - \frac{\xi}{i}\right)$.*

Note that the dependency on $\left(1 - \frac{\xi}{i}\right)$ stems from the second eigenvalue of the random walk that walks between two i dimensional faces via a $i + \xi$ dimensional face.

In this paper we will be interested in a special case of agreement expander, specifically 1-up agreement expander.

▶ **Definition 40** (1-up agreement expander). *Define the distribution $\mathcal{D}_\uparrow$ as the distribution that, in order to sample two i-dimensional faces, samples an $(i + 1)$-dimensional face and then sample two of its i-dimensional sub-faces. Note that this distribution guarantees an intersection of size $i - 1$. We say that a simplicial complex is a 1-up agreement expansion if it is an agreement expander with the distribution $\mathcal{D}_\uparrow$.*

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[9] Y's 1-dimensional faces form a cycle of length 6 and therefore no triangle can be formed using the edges of Y.

[10] p may be dependent on i.

3 The Representation Complex

In this section we are going to present a new complex that represents the agreement test over the complex we are interested in. We will do that by constructing an edge for each of the possible choices of the 1-up agreement test. We will add higher dimensional faces so that the norm of the faces in the representation complex would correspond with the weight of the faces they represent.

▶ **Definition 41** (Representation Function). *Define the representation function of a set of k-faces to be $R : P(X(k)) \to P(X(0))$ to be $R(s) = \bigcup_{s' \in s} s'$.*

▶ **Definition 42** (Representation Complex). *Given a simplicial complex X define the representation complex of X to be $\hat{R}^k(X)$ such that:*
- *$\hat{R}^k(X)(-1) = \{\emptyset\}$*
- *$\hat{R}^k(X)(0) = \{\{\sigma\} | \sigma \in X(k)\}$*
- *$\forall 1 \le i \le d-k : \hat{R}^k(X)(i) = \left\{ \sigma \in \binom{X(k)}{i+1} \middle| R(\sigma) \in X(i+k) \text{ and } \left| \bigcap_{v \in \sigma} v \right| = k \right\}$*

▶ Note. For clarity of notation we will treat $\hat{R}^k(X)(0)$ as if it equals $X(k)$ (i.e. we will consider $\{\sigma\}$ and σ to be equivalent when discussing the representation complex's 0-dimensional faces). Moreover, we will sometimes treat R as a function whose origin is the faces in $\hat{R}^k(X)$ (which are subsets of $X(k)$) and range is the faces of X.

▶ **Definition 43.** *We say that a face $\sigma \in X$ is represented by $r_\sigma \in \hat{R}^k(X)$ if $R(r_\sigma) = \sigma$. Conversely we say that r_σ represents σ.*
The full version of this paper includes a through discussion on the exact structure of the representation complex. In it we prove that the representation complex is indeed a simplicial complex, and that every face whose dimension is larger then 1 has sunflower-like structure i.e. there is a core that is a subset of each of the vertices in the face. And, in addition, the core is the intersection of any two vertices in the face. We therefore define the following:

▶ **Definition 44** (Core of a face). *Define the core of a face $\sigma \in \hat{R}^k(X)$ to be: $core(\sigma) = \bigcap_{v \in \sigma} v$.*

The analysis from the full version of the paper yields the following three Lemmas that are crucial for our result:

▶ **Lemma 45.** *For every face $\sigma \in X(k+i)$ and every representation of the face r_σ it holds that $w_{\hat{R}^k(X)}(r_\sigma) = \frac{1}{|R^{-1}(\sigma)|} w_X(\sigma)$.*

▶ **Lemma 46.** *One can sample from the representation complex according to its norm.*

▶ **Lemma 47** (The Coboundaries of the Representation Complex are Testable). *If X is a γ-coboundary-expander then there exists a tester T that queries exactly 3 edges[11] and a constant $\eta = \eta(k, \gamma)$ such that: $\text{dist}\left(F, B^1\left(\hat{R}^k(X)\right)\right) \le \eta \cdot \Pr[T \text{ rejects } F]$*

These Lemmas show that the weight function of the representation complex given the correct weights to the edges, that we can sample a face from the representation complex (meaning that we can run tests on it) and that the coboundaries of the representation complex are testable.

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[11] We can claim something even stronger. Specifically that T picks three vertices and queries the edges between them.

4 On the Covers That Correspond to Coboundaries

Recall the connection between cocycles and cover spaces of a simplicial complex. Specifically, recall that Y_Z is a cover space of X iff Z is a cocycle. In this section we will be interested in the properties of cover spaces that corresponds with coboundaries. We start by showing that, if Z is a coboundary then the lift of any cycle C in X is a cycle in Y_Z:

▶ **Lemma 48** (Lifts of cycles are cycles). *Let $\phi \in B^1(X;G)$ and let Y_ϕ be a cover of X with the simplicial map $f : Y_\phi \to X$. Also let $C = (v_0, \dots, v_i, v_0)$ be a vertex-edge cycle in X. Then every lift of C is a vertex-edge cycle in Y_ϕ that starts and ends on a lift of v_0.*

Proof. Consider the lift of the cycle to the cover Y_ϕ and note that it is comprised of the following edges (for some $s \in G$):

$$\{[v_0, s], [v_1, \phi(v_1, v_0)s]\}, \{[v_1, \phi(v_1, v_0)s], [v_2, \phi(v_0, v_1)\phi(v_1, v_0)s]\},$$

$$\dots, \left\{ [v_i, \prod_{j=0}^{i-1} \phi(v_{j+1}, v_j)s], [v_0, \phi(v_0, v_i) \prod_{j=0}^{i-1} \phi(v_{j+1}, v_j)s] \right\}$$

Because ϕ is a coboundary there exists a $\psi \in C^0(X)$ such that $\phi(\sigma_1, \sigma_2) = \psi(\sigma_1)\psi(\sigma_1)^{-1}$ therefore:

$$\phi(v_0, v_i) \prod_{j=0}^{i-1} \phi(v_{j+1}, v_j) = \psi(v_0)\psi(v_i)^{-1} \prod_{j=0}^{i-1} \psi(v_{j+1})\psi(v_j)^{-1} = 1$$

Therefore the path starts and ends on the same vertex. ◀

We can show something even stronger, namely that the cover is actually comprised of l distinct instances of the original complex.

▶ **Lemma 49** (Cover decomposition). *Let $\phi \in B^1(X;G)$ and let Y_ϕ be a cover of X then there exists $Y_1, \dots, Y_l$ such that $Y_\phi = \bigcup_{i=1}^{l} Y_i$ and $Y_1 \cong \dots \cong Y_l \cong X$.*

Proof. Y_ϕ is a cover of X. ϕ is a coboundary therefore there exists $\psi \in C^0(X;G)$ such that $\phi = d_1\psi$. We will show that for any dimension i and every face $\sigma \in Y_\phi(i)$ is of the form $\sigma = \{[v_1, \psi(v_1)s], \dots, [v_i, \psi(v_i)s]\}$ such that $\{v_1, \dots, v_i\} \in X(i)$. Note that if $[v_1, \psi(v_1)s] \in \sigma$ then the rest of the vertices in the face are of the form $[v_j, \phi(v_j, v_1)\psi(v_1)s] = [v_j, \psi(v_j)\psi(v_1)^{-1}\psi(v_1)s] = [v_j, \psi(v_j)s]$. Consider the sub-complexes of Y_ϕ defined by $Y_j(i) = \{\{[v_1, \psi(v_1)i], \dots, [v_i, \psi(v_i)i]\}|\{v_1, \dots, v_i\} \in X(i)\}$. It is easy to see that these are all simplicial complexes. We will now show that they are distinct, that their union is Y_ϕ and that they are indeed isomorphic to X.

Assume that $Y_{j_1} \cap Y_{j_2} \neq \{\emptyset\}$. Let $[v, k]$ be a vertex in $Y_{j_1} \cap Y_{j_2}$ (there must be one due to the closure property of both complexes). Therefore $\psi(v)j_1 = k = \psi(v)j_2$ and therefore $j_1 = j_2$ (due to the fact that $\psi(v) \in G$ and therefore it has an inverse).

For every dimension i let $\{[v_1, \psi(v_1)s], \dots, [v_i, \psi(v_i)s]\} \in Y_\phi(i)$. Note that $s \in S$ and $\{v_1, \dots, v_i\} \in X(i)$ and therefore $\{[v_1, \psi(v_1)s], \dots, [v_i, \psi(v_i)s]\} \in Y_j(i)$. It is easy to see that $Y_j \subseteq Y_\phi$ for every j.

For every Y_j consider $f : X \to Y_j$ to be $f(\{v_1, \dots, v_i\}) = \{[v_1, \psi(\sigma)j], \dots, [v_i, \psi(\sigma)j]\}$. Notice that this is a bijection (with the inverse function being $f^{-1}(\{[v_1, \psi(\sigma)j], \dots, [v_i, \psi(\sigma)j]\}) = \{v_1, \dots, v_i\}$). In addition notice that if $\sigma_1 \subseteq \sigma_2$ then $f(\sigma_1) \subseteq f(\sigma_2)$. ◀

5   Local Assignments in the Original Complex Imply a Near Cover in the Representation Complex

In this section we will show how to derive a near cover for the representation complex from local assignments in the original complex. More specifically, we will provide an algorithm that, given some edge, returns the edges that cover it in the near cover. This cover will not only be helpful in separating the complex to somewhat-consistent components but will also allow us to bound how close each component is to being fully consistent.

Consider the following algorithm for querying the edges of the near cover:

Algorithm 1 query cover edge ($\{\sigma_1, \sigma_2\}$).

---

1 Query the list functions on σ_1: $\mathscr{F}_1^{\sigma_1}, \ldots, \mathscr{F}_l^{\sigma_1}$.

2 Query the list functions on σ_2: $\mathscr{F}_1^{\sigma_2}, \ldots, \mathscr{F}_l^{\sigma_2}$.

3 **if** *there exists π such that for every $v \in \sigma_1 \cap \sigma_2$ and every i it holds that*
  $\mathscr{F}_i^{\sigma_1}(v) = \mathscr{F}_{\pi(i)}^{\sigma_2}(v)$ **then**

4 $\quad$ **return** $\left\{ \left\{ \mathscr{F}_i^{\sigma_1}, \mathscr{F}_{\pi(i)}^{\sigma_2} \right\} \middle| i \in [l] \right\}$

5 **else**

6 $\quad$ **return** $\{\{\mathscr{F}_i^{\sigma_1}, \mathscr{F}_i^{\sigma_2}\} | i \in [l]\}$

7 **end**

Before we move on consider the following intuition to what Algorithm 1 does: Effectively Algorithm 1 tries to find a matching between the lifts of σ_1 and σ_2 that could be extended into a coboundary.

We consider the near cover whose vertices are the assignments associated with each k-face, its edges are the result of Algorithm 1 and we also complete the cover upwards (i.e. if a higher dimensional face can be covered it will be).

It will also be useful to have a way to distinguish between edges that satisfy the condition in step 3 and those that are not.

▶ **Definition 50.** *An adequately covered edge is an edge for which there exists a permutation π such that for every $v \in \sigma_1 \cap \sigma_2$ and every i it holds that $\mathscr{F}_i^{\sigma_1}(v) = \mathscr{F}_{\pi(i)}^{\sigma_2}(v)$.*

We can also check whether an edge is adequately covered:

Algorithm 2 is adequately covered ($\{\sigma_1, \sigma_2\}$).

---

1 Query the list functions on σ_1: $\mathscr{F}_1^{\sigma_1}, \ldots, \mathscr{F}_l^{\sigma_1}$.

2 Query the list functions on σ_2: $\mathscr{F}_1^{\sigma_2}, \ldots, \mathscr{F}_l^{\sigma_2}$.

3 **return** *whether there exists π such that for every $v \in \sigma_1 \cap \sigma_2$ and every i it holds*
  that $\mathscr{F}_i^{\sigma_1}(v) = \mathscr{F}_{\pi(i)}^{\sigma_2}(v)$

Now that we have defined the near cover implied by the functions on the k-faces of the complex it is time to move on to present the test.

6   Presenting a Test for List Agreement

Now we are finally ready to present our test:

▶ Note. We have shown in Lemma 46 that one can indeed sample from the representation complex.

◼ **Algorithm 3** test for list agreement.

1	**pick** *with probability 0.5*
2	Run the test guaranteed by Lemma 47. Whenever the test performs a query run Algorithm 1 and provide the test with its answer.
3	Sample an edge in the representation complex and check whether it is adequately covered (using Algorithm 2).

We will spend the rest of this section proving that this algorithm is indeed a test for list agreement.

▶ **Theorem 51** (Main Theorem). *Algorithm 3 is a test for list agreement when the l-assignment is 2-locally-differing over a complex that is a 1-up agreement expander and whose every link is a coboundary expander over the symmetric group with l elements* [12].

Before we prove the main theorem let us reiterate the proof's strategy: We are first going to use the connection between cochains and near covers in order to claim that the near cover implied by the functions in the lists of the k-faces is close to a genuine cover that correspond to a coboundary. Consider the genuine cover that is close to that near cover - this cover can be thought of as l independent copies of the representation complex. Note that because our near cover was made using the functions in the lists of the different vertices each one of the copies in the cover can be thought of as assigning each vertex of the representation complex with a single local function. Also note that the vertices of the representation complex corresponds to the k-faces of the original complex. Using this fact one can think of each of the copies of the representation complex in the cover as an assignment of a singular function to each of the k-faces of the original complex. We would then proceed to estimate the distance of each of these assignments of functions from being agreeing assignments. In an ideal setting we would be able to run the 1-up agreement expander test described in [9] on each of the copies and measure their rejection probability. Alas we do not have access to query the genuine cover (and therefore the correct local assignments for each vertices). What we can do, however, is bound from above the rejection probability of these tests without running them. We do so by first noting that the random choice done in that test corresponds to picking an edge in the representation complex. Then consider when the agreement test rejects an edge in the representation complex, this can happen in one of two cases - Either the edge is not adequately covered [13] or the edge was adequately covered in the near cover but the near cover and the cover differ on that edge. Using this fact we can bound from above the probability that the 1-up agreement test rejects when given access to each one of the copies of the representation complex. Therefore the distance of each copy from having a global function can be estimated. This is also a bound on the distance of the lists from having l global functions that agree with them.

▶ **Notation.** *Denote by Y the near cover generated by Algorithm 1 and by $\tilde{Y}$ a genuine cover that is close to Y and represents a coboundary (Specifically the one guaranteed by Lemma 47). Also denote by $f_Y : Y \to \hat{R}(X)$ the near covering map implied by Algorithm 1 and by $f_{\tilde{Y}} : \tilde{Y} \to \hat{R}(X)$ the covering map between $\tilde{Y}$ and $\hat{R}(X)$. Lastly denote by $\tilde{Y}_1, \ldots, \tilde{Y}_l$ the l copies* [14] *of $\hat{R}(X)$ that make up $\tilde{Y}$ and by $f_{\tilde{Y}_i} : \tilde{Y}_i \to \hat{R}(X)$ the restriction of the covering map to the copy i.*

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[12] This includes the entire complex as it is the link of $\emptyset$.

[13] Note that if the edge is not adequately covered in the near cover then it is not adequately covered in the cover.

[14] Note that this is indeed the structure of a cover that corresponds to a coboundary due to Lemma 49.

▶ **Definition 52.** *Let $\mathscr{F}$ be a k-dimensional l-assignment of X and let $\tilde{Y}$ be the cover of X described above. Define $\mathcal{F}_1, \ldots, \mathcal{F}_l$ the sub-assignments of $\mathscr{F}$ implied by Y to be assignments such that: $\forall i \in [l] : \tilde{Y}_i(0) = \mathcal{F}_i$.*

▶ **Notation.** *Denote the set of inadequately covered edges of $\hat{R}(X)$ as $I_{\hat{R}^k(X)}$.*

▶ **Notation.** *Denote by A (stands for adjustment) the set of edges that are covered differently between Y and $\tilde{Y}$, formally: $A = \left\{ \sigma \in \hat{R}^k(X)(1) \middle| f_Y^{-1}(\sigma) \neq f_{\tilde{Y}}^{-1}(\sigma) \right\}$.*

▶ **Definition 53.** *Define D_i to be the set of disagreeing edges of $\tilde{Y}_i$, formally: $D_i = \left\{ \{r_1, r_2\} \in \hat{R}^k(X)(1) \middle| \mathcal{F}_i^{r_1}|_{core(\{r_1, r_2\})} \neq \mathcal{F}_i^{r_2}|_{core(\{r_1, r_2\})} \right\}$.*

Consider what happens when we run the 1-up agreement test on X with the k-dimensional assignments of $\mathcal{F}_i$. The 1-up agreement test picks a $(k + 1)$-dimensional face and then two of its k-subsets and check whether the local assignment of the sub-faces from $\mathcal{F}_i$ agree on their intersection. Note that this is completely equivalent to picking an edge in the representation complex (since this is exactly the process of sampling an edge in $\hat{R}(X)$, more details on that can be found in the full paper) and check whether the functions associated with its vertices agree on the core of the edge. Therefore, when viewing the 1-up agreement test through the lens of the representation complex the set of edges that the 1-up agreement test rejects is exactly D_i. The following lemma is a formal proof of this notion:

▶ **Lemma 54.** *For every $\tilde{Y}_i$ if the 1-up agreement test rejects the edge $\sigma \in \hat{R}(X)$ then either σ is inadequately covered or σ is covered differently by $\tilde{Y}$ then it is covered by Y, formally: $D_i \subseteq A \cup I_{\hat{R}^k(X)}$.*

Proof. Let $\{\sigma_1, \sigma_2\} \in D_i$. Assuming that $\{\sigma_1, \sigma_2\} \notin A \cup I_{\hat{R}^k(X)}$ then, because $\{\sigma_1, \sigma_2\} \notin A$ it holds that $f_Y^{-1}(\{\sigma_1, \sigma_2\}) = f_{\tilde{Y}}^{-1}(\{\sigma_1, \sigma_2\})$. In addition, because $\{\sigma_1, \sigma_2\} \notin I_{\hat{R}^k(X)}$ then σ is adequately covered i.e. $\forall \left\{ \mathcal{F}_i^{\sigma_1}, \mathcal{F}_j^{\sigma_2} \right\} \in f_Y^{-1}(\sigma) : \mathcal{F}_i^{\sigma_1}|_{core(\{\sigma_1, \sigma_2\})} = \mathcal{F}_j^{\sigma_2}|_{core(\{\sigma_1, \sigma_2\})}$. Therefore $\forall \left\{ \mathcal{F}_i^{\sigma_1}, \mathcal{F}_j^{\sigma_2} \right\} \in f_{\tilde{Y}}^{-1}(\sigma) : \mathcal{F}_i^{\sigma_1}|_{core(\{\sigma_1, \sigma_2\})} = \mathcal{F}_j^{\sigma_2}|_{core(\{\sigma_1, \sigma_2\})}$ which contradicts the fact that $\sigma \in D_i$. ◀

▶ **Lemma 55.** *Let X be a γ-coboundary-expander and α-agreement-expander. Also let $\eta = \eta(k, \gamma)$ be the constant from Lemma 47 then for every l-assignment $\mathscr{F}$ it holds that:*

$$\frac{\eta}{2\eta + 2} \frac{\alpha}{k} \cdot \operatorname{dist}(\mathscr{F}, \mathscr{A}) \leq \Pr[\text{Algorithm 3 rejects}]$$

Proof. First consider:

$$\operatorname{dist}(\mathscr{F}, \mathscr{A}) \leq \frac{1}{l} \sum_{i=1}^{l} \operatorname{dist}(\mathcal{F}_i, \mathscr{A}) \leq \frac{1}{l} \sum_{i=1}^{l} \frac{k}{\alpha} \|D_i\| \leq \frac{1}{l} \sum_{i=1}^{l} \frac{k}{\alpha} \left\| A \cup I_{\hat{R}^k(X)} \right\| \leq \frac{k}{\alpha} \left\| A \cup I_{\hat{R}^k(X)} \right\|$$

Note that step 3 of Algorithm 3 picks an edge in $\hat{R}^k(X)$ and checks whether it is in A therefore it holds that $\left\| I_{\hat{R}^k(X)} \right\| = \Pr[\text{Step 3 of Algorithm 3 rejects}]$. In addition due to Lemma 47 it holds that $\eta \|A\| \leq \Pr[\text{Step 2 of Algorithm 3 rejects}]$. If algorithm 3 rejects with probability ϵ then both $\Pr[\text{Step 2 of algorithm 3 rejects}] \leq 2\epsilon$ and $\Pr[\text{Step 3 of algorithm 3 rejects}] \leq 2\epsilon$ and therefore:

$$\left\| A \cup I_{\hat{R}^k(X)} \right\| \leq \|A\| + \left\| I_{\hat{R}^k(X)} \right\| \leq \frac{1}{\eta} \Pr[\text{Step 2 of algorithm 3 rejects}] +$$

$$+ \Pr[\text{Step 3 of algorithm 3 rejects}] \leq 2 \left(\frac{1}{\eta} + 1 \right) \epsilon$$

Hence $\operatorname{dist}(\mathscr{F}, \mathscr{A}) \leq 2 \frac{k}{\alpha} \left(\frac{1}{\eta} + 1 \right) \epsilon = \left(\frac{2\eta + 2}{\eta \alpha} \right) \epsilon$. And therefore $\frac{\eta}{2\eta + 2} \frac{\alpha}{k} \operatorname{dist}(\mathscr{F}, \mathscr{A}) \leq \epsilon$. ◀

We will now move on to prove that Algorithm 3 accepts with probability 1 any agreeing 2-locally-differing l-assignment. Before we do that, however, let us start by examining the results of Algorithm 1:

▶ **Lemma 56** (π found in Algorithm 1 is singular). *Let $\mathscr{F}$ be a 2-locally-differing l-assignment. If Algorithm 1 finds a permutation π in step 3 when given access to $\mathscr{F}$ then it is singular.*

Proof. We will use the counter positive argument - assume that there exists two permutations π_1 and π_2 ($\pi_1 \neq \pi_2$) such that for every $v \in \sigma_1 \cap \sigma_2$ and every i it holds that $\mathscr{F}_i^{\sigma_1}(v) = \mathscr{F}_{\pi_1(i)}^{\sigma_2}(v)$ and $\mathscr{F}_i^{\sigma_1}(v) = \mathscr{F}_{\pi_2(i)}^{\sigma_2}(v)$. Let i be an integer such that $\pi_1(i) \neq \pi_2(i)$ and note that for every vertex $v \in \sigma_1 \cap \sigma_2$ it holds that $\mathscr{F}_i^{\sigma_1}(v) = \mathscr{F}_{\pi_1(i)}^{\sigma_2}(v) = \mathscr{F}_{\pi_2^{-1}(\pi_1(i))}^{\sigma_1}(v)$. Note that because $|\sigma_1 \cap \sigma_2| = |\sigma_1| - 1$ it holds that there exists at most one vertex v' such that $\mathscr{F}_i^{\sigma_1}(v') \neq \mathscr{F}_{\pi_2^{-1}(\pi_1(i))}^{\sigma_1}(v')$ which contradicts the fact that $\mathscr{F}$ is locally differing. ◀

We are now ready to prove the Lemma:

▶ **Lemma 57.** *If $\mathscr{F}$ is an agreeing and 2-locally-differing k-dimensional l-assignment then it passes the test with probability 1.*

Proof. In order to show that the l-assignment passes the test with probability 1 we will first show that every edge is adequately covered and find the nature of the permutation π found by Algorithm 1. Then we will use this in order to show that the test always accepts.

$\mathscr{F}$ is an agreeing assignment therefore there exists $\mathcal{F}_1, \cdots, \mathcal{F}_l$ such that for every face $\sigma \in X(k)$ there exists a permutation π_σ such that for every i it holds that $\mathcal{F}_i|_\sigma = \mathscr{F}_{\pi_\sigma(i)}^\sigma$. Therefore at step 3 Algorithm 1 finds $\pi = \pi_{\sigma_2} \pi_{\sigma_1}^{-1}$ due to the fact that π is singular and for every i it holds that $\forall v \in \sigma_1 \cap \sigma_2 : \mathscr{F}_i^{\sigma_1}(v) = \mathcal{F}_{\pi_{\sigma_1}^{-1}(i)}(v) = \mathscr{F}_{\pi_{\sigma_2}(\pi_{\sigma_1}^{-1}(i))}^{\sigma_2}(v)$.

This immediately implies that every edge is adequately covered and that the cochain created by Algorithm 1 is a coboundary. Therefore, if Algorithm 3 checks if an edge is adequately covered then it accepts since every edge is adequately covered. Otherwise Algorithm 3 checks whether the cochain generated by Algorithm 1 is a coboundary and therefore it accepts with probability 1. ◀

We are now finally ready to prove the main theorem:

Proof of Theorem 51. In order to calculate the cover over each edge in the representation complex all the local assignments of both vertices are queried. The empty triangle test queries a triangle (either empty or proper) and therefore it queries 3 edges using Algorithm 1. Each run of Algorithm 1 requires $2l$ queries. Some vertices, however, are queried by Algorithm 1 twice. Therefore we can reduce the query complexity of this step to $3l$. Checking whether an edge is adequately covered takes $2l$ queries (using the same argument). Therefore algorithm 3 queries $\mathscr{F}$ at most $3l$ times. We finish the proof by noting that the fact that algorithm 3 is indeed a test for whether the l-assignment is agreeing stems directly from Lemma 55 and Lemma 57. ◀

We therefore conclude that Algorithm 3 is a test for list agreement expansion in the 2-differing case. We note that the test's distance improves as the complex is a better agreement expander and coboundary expander as well as when the dimension of the local assignments decrease.

Note that, in fact, we could have proven a more general statement than the main theorem which we will now present. Before we do that, however, we have to define the test graph of an agreement tester:

▶ **Definition 58** (Test Graph). *Let X be a simplicial complex and let T be an agreement test on the kth dimensional faces of X. Given a set of local functions $\{f_\sigma : \sigma \to \{0,1\}\}_{\sigma \in X(d)}$ the test picks two faces $\{\sigma_1, \sigma_2\} \sim \mathcal{D}$ and checks whether $f_{\sigma_1}|_{\sigma_1 \cap \sigma_2} = f_{\sigma_2}|_{\sigma_1 \cap \sigma_2}$. Define the test graph of the test T to be the weighted graph $G = (V, E, w_T)$ such that:*

- $V = X(k)$
- $E = \left\{ \{\sigma_1, \sigma_2\} \middle| \Pr_{\{\tau_1, \tau_2\} \sim \mathcal{D}} \left[\tau_1 = \sigma_1 \text{ and } \tau_2 = \sigma_2 \right] > 0 \right\}$
- $w_T(\{\sigma_1, \sigma_2\})) = \Pr_{\{\tau_1, \tau_2\} \sim \mathcal{D}} \left[\tau_1 = \sigma_1 \text{ and } \tau_2 = \sigma_2 \right]$

▶ **Theorem 59** (Main Theorem, Generalized). *Let X be a simplicial complex and let T be an agreement test on the k-th dimensional faces of X. Then if the test graph of T is a 1-skeleton of a coboundary expander with respect to S_l (denoted X') such that $w_T(\{\sigma_1, \sigma_2\}) = w_{X'}(\{\sigma_1, \sigma_2\})$ and there exists an algorithm that queries the test graph then Algorithm 3 is a test for l-agreement over $X(k)$.*

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