

# Symmetric Formulas for Products of Permutations

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## Abstract

We study the formula complexity of the word problem  $\text{WORD}_{S_n, k} : \{0, 1\}^{kn^2} \rightarrow \{0, 1\}$ : given  $n$ -by- $n$  permutation matrices  $M_1, \dots, M_k$ , compute the  $(1, 1)$ -entry of the matrix product  $M_1 \cdots M_k$ . An important feature of this function is that it is invariant under action of  $S_n^{k-1}$  given by

$$(\pi_1, \dots, \pi_{k-1})(M_1, \dots, M_k) = (M_1 \pi_1^{-1}, \pi_1 M_2 \pi_2^{-1}, \dots, \pi_{k-2} M_{k-1} \pi_{k-1}^{-1}, \pi_{k-1} M_k).$$

This symmetry is also exhibited in the smallest known unbounded fan-in  $\{\text{AND}, \text{OR}, \text{NOT}\}$ -formulas for  $\text{WORD}_{S_n, k}$ , which have size  $n^{O(\log k)}$ .

In this paper we prove a matching  $n^{\Omega(\log k)}$  lower bound for  $S_n^{k-1}$ -invariant formulas computing  $\text{WORD}_{S_n, k}$ . This result is motivated by the fact that a similar lower bound for unrestricted (non-invariant) formulas would separate complexity classes  $\text{NC}^1$  and  $\text{Logspace}$ .

Our more general main theorem gives a nearly tight  $n^{d(k^{1/d}-1)}$  lower bound on the  $G^{k-1}$ -invariant depth- $d$   $\{\text{MAJ}, \text{AND}, \text{OR}, \text{NOT}\}$ -formula size of  $\text{WORD}_{G, k}$  for any finite simple group  $G$  whose minimum permutation representation has degree  $n$ . We also give nearly tight lower bounds on the  $G^{k-1}$ -invariant depth- $d$   $\{\text{AND}, \text{OR}, \text{NOT}\}$ -formula size in the case where  $G$  is an abelian group.

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## 1 Introduction

### 1.1 $P$ -invariant complexity

Let  $P$  be a permutation group on  $m$  elements (i.e., a subgroup of the symmetric group  $S_m$ ). There is a natural action of  $P$  on the set of Boolean function  $f : \{0, 1\}^m \rightarrow \{0, 1\}$  given by  $(\pi f)(x_1, \dots, x_m) = f(x_{\pi(1)}, \dots, x_{\pi(m)})$ . We say that  $f$  is  $P$ -invariant if  $\pi f = f$  for all  $\pi \in P$  (i.e.,  $f$  is symmetric under the action of  $P$ ). Many important Boolean functions are invariant under group actions on coordinates  $1, \dots, m$ : *symmetric functions* such as  $\text{PARITY}$  and  $\text{MAJORITY}$  are fully  $S_m$ -invariant, while *graph properties* such as  $\text{CONNECTIVITY}$  and  $\text{3-COLORABILITY}$  are invariant under  $P = S_n$  acting on  $m = \binom{n}{2}$  edge-indicator variables.

Any permutation group  $P$  also acts on combinatorial devices that compute functions on  $\{0, 1\}^m$ , such as  $m$ -variable circuits, formulas, branching programs, etc. In this paper, we focus on *formulas* in both the  $\text{AC}$  basis (unbounded fan-in  $\text{AND}$ ,  $\text{OR}$  gates) and the  $\text{TC}$  basis (unbounded fan-in  $\text{MAJ}$  gates) with negations on input literals. Here we view formulas as rooted trees in which leaves (“inputs”) are labeled by literals  $x_1, \bar{x}_1, \dots, x_m, \bar{x}_m$  or constants  $0, 1$ , and non-leaves (“gates”) are labeled by gate types. We treat the children of a gate as an unordered multiset, so that two formulas are identical if and only they are isomorphic as labeled rooted trees.



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$P$  acts on the set of formulas by relabeling literals  $x_i$  to  $x_{\pi(i)}$  and  $\bar{x}_i$  to  $\bar{x}_{\pi(i)}$ . A formula  $\Phi$  is said to be  $P$ -invariant if  $\pi\Phi = \Phi$  for all  $\pi \in P$ . Every  $P$ -invariant formula computes a  $P$ -invariant Boolean function. On the other hand, a  $P$ -invariant Boolean function  $f$  may be computed (often more efficiently) by a non- $P$ -invariant formula  $\Phi$ ; in this case, we would say that  $\Phi$  is *semantically  $P$ -invariant*, but not *syntactically  $P$ -invariant*.

The action of  $P$  on formulas preserves parameters such as *depth* (the maximum number of gates on a leaf-to-root branch) and *size* (the number of leaves labeled by literals). For general Boolean functions  $f$ , we write  $\mathcal{L}_{\text{AC}_d}(f)$  and  $\mathcal{L}_{\text{TC}_d}(f)$  for the minimum size of a depth- $d$  AC (respectively TC) formulas that computes  $f$ , and we write  $\mathcal{L}_{\text{AC}}(f)$  and  $\mathcal{L}_{\text{TC}}(f)$  allowing unrestricted depth. When  $f$  is  $P$ -invariant, we may consider the corresponding  $P$ -invariant complexity measures (indicated by superscript  $P$ ):

$$\begin{array}{ccc} \mathcal{L}_{\text{TC}_d}(f) \leq \mathcal{L}_{\text{AC}_d}(f) & \mathcal{L}_{\text{TC}}(f) \leq \mathcal{L}_{\text{AC}}(f) = \mathcal{L}_{\text{DeMorgan}}(f) \\ \uparrow \wedge & \uparrow \wedge & \uparrow \wedge \\ \mathcal{L}_{\text{TC}_d}^P(f) \leq \mathcal{L}_{\text{AC}_d}^P(f) & \mathcal{L}_{\text{TC}}^P(f) \leq \mathcal{L}_{\text{AC}}^P(f) \end{array}$$

Invariant circuit complexity has been previously studied in the context of descriptive complexity, an area concerned with characterizing complexity classes in terms of definability in different logics. Here the languages/Boolean functions considered are graph properties (or in general isomorphism-invariant properties of finite relational structures), and  $P$  is the symmetric group  $S_n$  acting on  $m = \binom{n}{2}$  edge-indicator variables (or  $m = \sum_{i=1}^t n^{r_i}$  indicator variables for properties of structures with relations of arity  $r_1, \dots, r_t$ ). Denenberg, Gurevich and Shelah [12] showed that  $P$ -invariant AC circuits of polynomial size and constant depth (subject to a certain uniformity condition) capture precisely the first-order definable graph properties. More recently, Anderson and Dawar [3] established a correspondence between polynomial-size  $P$ -invariant TC circuits and definability in fixed-point logic with counting. For both classes of formulas, *exponential gaps* are known between the  $P$ -invariant complexity and non-invariant complexity of explicit graph properties.

## 1.2 The word problem over a finite permutation group

Let  $G \leq S_n$  be a finite permutation group, which we assume to be transitive. Specifically, we will be interested in the case  $G = S_n$ , as well as the case where  $G$  is a simple group and  $G \leq S_n$  is a faithful permutation representation of minimum degree (e.g.,  $G$  is the alternating group  $A_n$ , or  $G$  is a cyclic group of prime order  $p$  and  $n = p$ ).

We write  $\bar{G}$  for the set of permutation matrices corresponding to elements of  $G$ . For  $k \in \mathbb{N}$ , we view  $\bar{G}^k$  as a subset of Hamming cube  $\{0, 1\}^{kn^2}$ . We write elements of  $\bar{G}^k$  as  $k$ -tuples of matrices  $(M_1, \dots, M_k)$ , and we identify coordinates of  $\{0, 1\}^{kn^2}$  with Boolean variables  $M_{i,a,b}$  for  $i \in [k]$  and  $a, b \in [n]$ .

► **Definition 1.** *The length- $k$  word problem for  $G$  is the Boolean function  $\text{WORD}_{G,k} : \bar{G}^k \rightarrow \{0, 1\}$  that outputs the  $(1, 1)$ -entry of the matrix product  $M_1 \cdots M_k$ .*

This is one natural version of the “word problem” for a finite permutation group. This should not be confused with the word problem for finitely presented groups (see Section 1.6.2). Note that the problem of determining whether  $M_1 \cdots M_k$  is the identity reduces to  $n$  instances of  $\text{WORD}_{G,k}$ .

We are interested in the invariant formula complexity of  $\text{WORD}_{G,k}$  with respect to the group  $G^{k-1}$  acting on  $\bar{G}^k$  by

$$(g_1, \dots, g_{k-1})(M_1, \dots, M_k) = (M_1 \bar{g}_1^{-1}, \bar{g}_1 M_2 \bar{g}_2^{-1}, \dots, \bar{g}_{k-2} M_{k-1} \bar{g}_{k-1}^{-1}, \bar{g}_{k-1} M_k)$$

where  $\overline{g_i}$  denotes the permutation matrix corresponding to  $g_i$ . Note that the action  $G^{k-1}$  on  $\overline{G}^k$  arises from an action on input variables  $M_{i,a,b}$  given by

$$(g_1, \dots, g_{k-1})M_{i,a,b} = M_{i, g_{i-1}(a), g_i(b)}$$

where  $g_0 = g_k = 1_G$ . Since this action is faithful, we may view  $G^{k-1}$  as a subgroup of  $S_{kn^2}$ .

### 1.3 The formula size of $\text{WORD}_{S_n, k}$

The functions  $\text{WORD}_{S_n, k}$  (also known as *iterated permutation matrix multiplication*) have an important place in complexity theory, since languages  $\{\text{WORD}_{S_5, k}\}_{k \in \mathbb{N}}$  and  $\{\text{WORD}_{S_n, n}\}_{n \in \mathbb{N}}$  are respectively complete for complexity classes  $\text{NC}^1$  and  $\text{Logspace}$  [5, 10]. Understanding the formula complexity of the word problem for  $S_n$  could eventually be key to separating these classes.

The smallest known DeMorgan formulas ( $\text{AND}_2, \text{OR}_2, \text{NOT}$  gates) for  $\text{WORD}_{S_n, k}$  have size  $n^{O(\log k)}$ . Allowing unbounded fan-in  $\text{AND}_\infty, \text{OR}_\infty, \text{NOT}$  gates (i.e.,  $\text{AC}$  formulas), we obtain the same size  $n^{O(\log k)}$  with depth  $O(\log k)$  and moreover by formulas that are syntactically  $S_n^{k-1}$ -invariant. In fact, the smallest known  $\text{AC}$  formulas of any given depth (via a natural divide-and-conquer construction that we describe in Corollary 8) happen to be  $S_n^{k-1}$ -invariant:

$$\mathcal{L}_{\text{AC}}^{S_n^{k-1}}(\text{WORD}_{S_n, k}) = n^{O(\log k)}, \quad \mathcal{L}_{\text{AC}_{d+1}}^{S_n^{k-1}}(\text{WORD}_{S_n, k}) = n^{O(d(k^{1/d}-1))}.$$

(Note that the size-depth tradeoff on the right implies the lower bound on the left, since  $\lim_{d \rightarrow \infty} d(k^{1/d} - 1) = \ln k$ .)

A matching lower bound  $\mathcal{L}_{\text{AC}}(\text{WORD}_{S_n, k}) = n^{\Omega(\log k)}$  for unrestricted (non-invariant) formulas would separate complexity classes  $\text{NC}^1$  and  $\text{Logspace}$ . This motivates the question of first showing matching lower bounds for  $S_n^{k-1}$ -invariant formulas.

### 1.4 Our results

Our first theorem shows that the above  $S_n^{k-1}$ -invariant upper bounds are nearly optimal, even for  $\text{TC}$  formula which include  $\text{MAJ}$  gates.

► **Theorem 2.** *For all  $n, k, d \geq 1$ ,*

$$\mathcal{L}_{\text{TC}}^{S_n^{k-1}}(\text{WORD}_{S_n, k}) \geq n^{\log_2 k}, \quad \mathcal{L}_{\text{TC}_d}^{S_n^{k-1}}(\text{WORD}_{S_n, k}) \geq n^{d(k^{1/d}-1)}.$$

(Note that our the righthand lower bound is off by 1 in depth compared to the upper bound.)

Note that  $\text{WORD}_{G, k}$  is simply the restriction of the function  $\text{WORD}_{S_n, k}$  to the subdomain  $\overline{G}^k \subseteq \overline{S_n}^k$ . The  $G^{k-1}$ -invariant complexity of  $\text{WORD}_{G, k}$  is moreover a nondecreasing function of  $G$  in the subgroup lattice of  $S_n$ .

Thus, Theorem 2 follows from the main result of this paper (Theorem 3 below), which lower bounds the  $G^{k-1}$ -invariant  $\text{TC}$  formula size of  $\text{WORD}_{G, k}$  for certain subgroups  $G \leq S_n$ . This is because the alternating subgroup  $A_n \leq S_n$  is simple for  $n \geq 5$  and has minimum faithful permutation representation of degree  $n$ .

This also motivates the question of strengthening Theorem 2 by finding more proper subgroups  $G \subseteq S_n$  with the same lower bound.

► **Theorem 3.** *Let  $G$  be a finite simple group and suppose that  $G \leq S_n$  is a faithful permutation representation of minimum degree. Then*

$$\mathcal{L}_{\text{TC}}^{G^{k-1}}(\text{WORD}_{G, k}) \geq n^{\log_2 k}, \quad \mathcal{L}_{\text{TC}_d}^{G^{k-1}}(\text{WORD}_{G, k}) \geq n^{d(k^{1/d}-1)}.$$

Our proof of Theorem 3 uses different arguments in the cases where  $G$  is nonabelian (Section 6.1) and where  $G$  is cyclic of order  $p$  (Section 6.2). However, both cases use the same framework developed in Section 5.

We remark that the *minimum degree* requirement is necessary in the nonabelian case, as the statement would be false for the regular representation  $G \leq S_{|G|}$ . Minimum degree ensures every proper subgroup  $H < G$  has index at least  $n$ ; this fact plays a role in our lower bound.

An additional result of this paper gives a lower bound on the  $C_q^{k-1}$ -invariant AC formula size of the word problem for cyclic groups  $C_q$  where  $q$  is a prime power. When  $q$  is not prime, we do not know whether the technique of Theorem 3 yields a stronger lower bound for TC formulas.

► **Theorem 4.** *Suppose that  $q$  is a prime power and  $C_q \leq S_q$  is cyclic of order  $q$ . Then*

$$\mathcal{L}_{AC}^{C_q^{k-1}}(\text{WORD}_{C_q,k}) \geq q^{\log_2 k}, \quad \mathcal{L}_{AC_d}^{C_q^{k-1}}(\text{WORD}_{C_q,k}) \geq q^{d(k^{1/d}-1)}.$$

► **Corollary 5.** *Let  $G$  be any finite group. Define*

$$n(G) = \max_{H \leq G: H \text{ is simple. } \pi: H \rightarrow S_m: \ker(\pi) = \{1_G\}} \min m, \quad q(G) = \max_{C_q \leq G: q \text{ is a prime power.}} q.$$

Then

$$\mathcal{L}_{TC_d}^{G^{k-1}}(\text{WORD}_{G,k}) \geq n^{d(k^{1/d}-1)}, \quad \mathcal{L}_{AC_d}^{G^{k-1}}(\text{WORD}_{G,k}) \geq q^{d(k^{1/d}-1)}.$$

Note that Corollary 5 is tight for abelian groups  $G$  up to a factor of  $|G|$ , since for abelian  $G$ , we can write  $G = C_{t_1} \times \dots \times C_{t_\ell}$  for some integers  $\{t_i\}_{i \in [\ell]}$ . This gives a divide and conquer  $G^{k-1}$ -invariant formula for  $\text{WORD}_{G,k}$  by solving  $\text{WORD}_{C_{t_i},k}$  for each  $i$ .

We remark that all of our lower bounds of the form  $n^{d(k^{1/d}-1)}$  for  $P$ -invariant depth- $d$  formulas imply lower bounds  $n^{k^{1/d}-1}/kn^2$  for  $P$ -invariant depth- $d$  circuits, since every ( $P$ -invariant) depth- $d$  circuit of size  $s$  unfolds to a ( $P$ -invariant) depth- $d$  formula of size at most  $s^{d-1}m$  where  $m$  is the number of variables.

The parameters defined in Corollary 5 have been studied before in the setting of all finite groups. Babai et al. [4] show that for any finite group  $G$ , the minimum degree of a faithful permutation representation of  $G$  upper bounds the size of every cyclic subgroup of prime power order. This shows that when  $G$  is a finite simple group, the formula size lower bound given by Theorem 3 is always stronger than the bound given by Theorem 4.

However, for some groups  $G$  the lower bounds on  $\mathcal{L}_{AC}^{G^{k-1}}(\text{WORD}_{G,k})$  implied by Theorem 3 and Theorem 4 are similar. In the full version of this paper we give examples where the gaps between the lower bounds may be large, conditional on some number-theoretic conjectures. Such examples justify the extra work to prove Theorem 3, especially in the case where  $G$  is nonabelian simple.

### 1.5 The lower bound technique

Let  $f : \{0, 1\}^m \rightarrow \{0, 1\}$  be a  $P$ -invariant partial function, where  $P \leq S_m$  and  $\Omega = \text{Dom}(f)$  is a  $P$ -invariant subset of  $\{0, 1\}^m$ . To prove Theorem 3, we introduce a general framework that lower bounds the  $P$ -invariant  $\text{TC}_d$  formula size of  $f$  in terms of a function  $\beta_{Q,\Omega,d}(P)$  where  $Q$  is any supergroup  $P \leq Q \leq S_m$  such that  $P$  is the  $Q$ -stabilizer of  $f$ . Functions  $\beta_{Q,\Omega,d}$  may be viewed as complexity measures on pairs of subgroups of  $Q$ , which are interesting in their own right. These complexity measures correspond to “formulas” that construct  $P$  in a certain manner starting from  $Q$ -stabilizers of points  $1, \dots, m$ .

This lower bound framework is described in Section 5. It is then applied to  $\text{WORD}_{G,k}$  in for nonabelian simple groups in Section 6.1, and for cyclic group of prime order in Section 6.2.

For a broad outline of our technique see Section 4

Our proof of Theorem 4 (in the full version of this paper) uses a different argument that generalizes a proof in the special case  $q = 2$  in previous work of the second author [22].

## 1.6 Related work

### 1.6.1 $\text{TC}_d$ and $\text{AC}_d$ formula lower bounds for $\text{WORD}_{S_n,k}$

For context, we state below the strongest known lower bounds for  $\text{WORD}_{S_n,k}$  with respect to bounded depth AC formulas (without the restriction of  $S_n^{k-1}$ -invariance). The first three results all use Switching Lemmas, which limits these lower bounds to depth  $o(\log n + \log k)$ . In contrast, our asymptotically tight lower bounds in the  $S_n^{k-1}$ -invariant setting extend to arbitrary depth (but stabilize at depth  $O(\log k)$ ).

- Rossman [21] gives a lower bound  $n^{\Omega(\log k)}$  on the AC formula size of  $\text{WORD}_{S_n,k}$  when  $k \leq \log \log n$ , however only up to depth  $d \leq \log n / (\log \log n)^{O(1)}$ .
- Beame, Impagliazzo and Pitassi [7] prove a size-depth tradeoff  $n^{\Omega(k^{1/\exp(d)})}$  for depth- $d$  AC formulas computing  $\text{WORD}_{S_n,k}$  when  $k \leq \log n$  and  $d \leq \log \log k$ .
- Chen, Oliveira, Servedio and Tan [9] give an improved tradeoff  $n^{\Omega(k^{1/2^d}/d)}$  when  $k \leq n^{1/5}$  and  $d \leq \log k / \log \log k$ , however not for the function  $\text{WORD}_{S_n,k}$  (a.k.a. iterated permutation matrix multiplication) but rather for the more general *iterated Boolean matrix multiplication* problem where input matrices  $M_1, \dots, M_k$  are not guaranteed to be permutation matrices.
- In the case  $n = 2$ , where  $\text{WORD}_{S_2,k}$  reduces to the  $\text{PARITY}_k$ , lower bound of [14, 17, 20] imply asymptotically tight tradeoffs:

$$\mathcal{L}_{\text{AC}}(\text{WORD}_{S_2,k}) = \Theta(k^2), \quad \mathcal{L}_{\text{AC}_{d+1}}(\text{WORD}_{S_2,k}) = 2^{\Theta(d(k^{1/d}-1))}.$$

Since these bounds are merely polynomial, they fail to separate  $\text{NC}^1$  from  $\text{Logspace}$ .

- Rossman [22] gives nearly tight bounds on the  $S_2^{k-1}$ -invariant formula size of  $\text{WORD}_{S_2,k}$ :

$$2^{d(k^{1/d}-1)} \leq \mathcal{L}_{\text{AC}_{d+1}}^{S_2^{k-1}}(\text{WORD}_{S_2,k}) \leq k2^{dk^{1/d}}.$$

- Impagliazzo, Paturi, and Saks [15] prove that a depth- $d$  TC circuit computing  $\text{WORD}_{S_2,k}$  must have size at least  $k^{1+1/(1+\sqrt{2})^d}$ . By hardness-magnification results of Chen and Tell [8] expanding on work by Allender and Koucký [2], there exists  $c > 1$  such that improving the above bound to  $k^{1+c^{-d}}$  would imply the separation  $\text{TC}^0 \neq \text{NC}^1$ .
- Using representation theory, Alexeev, Forbes, Tsimerman [1] prove upper bounds on the rank of the group tensor (the polynomial version of the word problem), hence proving upper bounds on arithmetic formula complexity of the word problem. They show that the depth- $d+1$  arithmetic formula size of the word problem polynomial  $W_{G,k}$  is at most  $\sum_{\rho \in \widehat{G}} \deg(\rho)^{dk^{1/d}}$ , where  $\widehat{G}$  is the set of irreducible representations of  $G$ . However, for many groups, their formulas are not  $G^{k-1}$ -invariant due to making choices of cosets of subgroups.

In this paper we improve on the result of [22] in two ways. First, in the setting of cyclic groups of prime order, we extend the lower bound from formulas in AC to formulas in TC as stated in Theorem 3. Second, we generalize the arguments used to cyclic groups of prime power order for the AC lower bound in Theorem 4.

### 1.6.2 The word problem for finitely presented groups

An important point to make is the distinction between our version of the word problem on groups and other more commonly studied versions. In the most commonly studied version of the word problem, one is given a set of generators for the group and the relations among the generators. Then the problem is to determine whether a sequence of these generators multiplies to equal the identity element. Often these problems deal with infinite groups. The complexity of this version of the word problem has been well-studied [11, 6, 19, 18]. On the other hand, we only deal with finite groups and do not worry about generators and relations, but are rather explicitly given the group elements serving as input to our problem.

## 2 Preliminaries

We fix the following notation throughout. For any  $k \in \mathbb{N}$ , let  $[k] = \{1, \dots, k\}$ .

$G$  shall always be a finite permutation group on  $n$  elements (i.e, a subgroup of  $S_n$ ), which we assume to be transitive.  $P$  and  $Q$  shall always be subgroups of  $S_m$ , where  $m = kn^2$  in our application.

We write  $1_G$  for the identity element in  $G$  and  $Z(G)$  for the center of  $G$ . If  $H$  is a subgroup of  $G$ , then we write  $H \leq G$ . If moreover  $H$  is a normal subgroup, then we write  $H \trianglelefteq G$ . If  $G$  has no nontrivial proper normal subgroup, then  $G$  is *simple*. We denote  $|G|$  to be the order of  $G$ .

We will often consider  $k$ -fold powers of a group  $G$ , denoted  $G^k$ . For  $S \subseteq [k]$ , let  $\pi_S : G^k \rightarrow G^S$  be the projection homomorphism to  $S$ . A *permutation representation* of  $G$  is a homomorphism  $G \rightarrow S_n$  for some positive integer  $n$ . We call  $n$  the *degree* of the representation, and if the homomorphism is injective, then the representation is called *faithful*. Define the subgroup  $\text{Diag}(G^2) < G^2$  to be the subgroup  $\{(g, g) : g \in G\}$ .

► **Definition 6.** A *labeled tree* is a finite unordered rooted tree in which each node is associated with a label.

An  $m$ -variable *AC formula* (respectively, *TC formula*) is a labeled tree in which each leaf (“input”) is labeled by a constant symbol or literal in the set  $\{0, 1, x_1, \bar{x}_1, \dots, x_m, \bar{x}_m\}$  and each non-input (“gate”) is labeled by AND or OR (respectively, labeled by MAJ).

The *depth* of a formula is the maximum number of gates on a leaf-to-root path. The *size* (a.k.a. *leaf-size*) of formula is the number of leaves that are labeled by literals.

For  $d \in \mathbb{N}$ , we write  $\text{AC}_d$  (resp.  $\text{TC}_d$ ) for the set of depth- $d$  AC formulas (resp. TC formulas) *up to isomorphism*. That is, we consider two formulas to be the same iff there exists an isomorphism between them as labeled graphs.

An alternative, concrete inductive definition of sets  $\text{AC}_d$  and  $\text{TC}_d$  is given by:

- Let  $\text{AC}_0 = \text{TC}_0 = \{0, 1, x_1, \bar{x}_1, \dots, x_m, \bar{x}_m\}$ .
- For  $d \geq 1$ ,  $\text{AC}_d$  (resp.  $\text{TC}_d$ ) is the set of pairs of form (AND,  $I$ ) or (OR,  $I$ ) (resp. of the form (MAJ,  $I$ )) where  $I$  is a multiset of formulas in  $\text{AC}_0 \cup \dots \cup \text{AC}_{d-1}$  (resp.  $\text{TC}_0 \cup \dots \cup \text{TC}_{d-1}$ ).

The symmetric group  $S_m$  acts on  $\text{AC}_d$  and  $\text{TC}_d$  by permuting indices on literals. For  $P \leq S_m$ , we say that a formula  $\Phi$  is *P-invariant* if  $\pi\Phi = \Phi$  for all  $\pi \in P$ .

Every formula  $\Phi$  computes a Boolean function denoted  $\llbracket \Phi \rrbracket : \{0, 1\}^m \rightarrow \{0, 1\}$  in the usual way. For a partial function  $f : \Omega \rightarrow \{0, 1\}$  where  $\Omega \subseteq \{0, 1\}^m$ , we say that  $\Phi$  *computes*  $f$  if  $\llbracket \Phi \rrbracket(x) = f(x)$  for all  $x \in \Omega$ .

Note that  $P$ -invariance of  $\Phi$  implies  $P$ -invariance of  $\llbracket \Phi \rrbracket$ , but the converse need not hold in general. Also note that MAJ gates can simulate both AND and OR gates (by padding by an appropriate number of zeros or ones). Any function computable by AC formulas of a given size and depth is therefore computable by a TC formula of the same size and depth.

### 3 $S_n^{k-1}$ -invariant formulas for $\text{WORD}_{S_n, k}$

► **Lemma 7.** For all  $n, k, d \geq 1$  such that  $k^{1/d}$  is an integer, the function  $\text{WORD}_{S_n, k}$  is computed by  $S_n^{k-1}$ -invariant  $\Sigma_{d+1}$  and  $\Pi_{d+1}$  formulas of size  $kn^{d(k^{1/d}-1)}$ .

**Proof.** For  $u_0, u_k \in [n]$ , let  $\text{WORD}_{S_n, k}^{(u_0, u_k)}(M_1, \dots, M_k)$  denote the  $(u_0, u_k)$ -entry of  $M_1 \cdots M_k$ . Thus,  $\text{WORD}_{S_n, k}$  is the function  $\text{WORD}_{S_n, k}^{(1, 1)}$ .

In the base case  $d = 1$ , we have  $\Sigma_2$  and  $\Pi_2$  formulas:

$$\begin{aligned} & \text{WORD}_{S_n, k}^{(u_0, u_k)}(M_1, \dots, M_k) \\ &= \bigvee_{u_1, u_2, \dots, u_{k-1} \in [n]} \left( M_{1, u_0, u_1} \wedge M_{2, u_1, u_2} \wedge \cdots \wedge M_{k-1, u_{k-2}, u_{k-1}} \wedge M_{k, u_{k-1}, u_k} \right), \\ & \text{WORD}_{S_n, k}^{(u_0, u_k)}(M_1, \dots, M_k) \\ &= \bigwedge_{u_1, u_2, \dots, u_{k-1} \in [n]} \left( \left( M_{1, u_0, u_1} \wedge M_{2, u_1, u_2} \wedge \cdots \wedge M_{k-1, u_{k-2}, u_{k-1}} \right) \Rightarrow M_{k, u_{k-1}, u_k} \right) \\ &= \bigwedge_{u_1, u_2, \dots, u_{k-1} \in [n]} \left( \bigvee_{j \in [k-1]} \neg M_{j, u_{j-1}, u_j} \vee M_{k, u_{k-1}, u_k} \right). \end{aligned}$$

Both formulas are clearly  $S_n^{k-1}$ -invariant and have size  $kn^{k-1}$ .

For the induction step, assume  $d \geq 2$  and let  $\ell = k^{(d-1)/d}$ . Then

$$\begin{aligned} & \text{WORD}_{S_n, k}^{(u_0, u_k)}(M_1, \dots, M_k) \\ &= \bigvee_{u_\ell, u_{2\ell}, \dots, u_{k-\ell} \in [n]} \bigwedge_{j \in [k^{1/d}]} \text{WORD}_{S_n, \ell}^{(u_{(j-1)\ell}, u_{j\ell})}(M_{(j-1)\ell+1}, \dots, M_{j\ell}). \end{aligned}$$

By induction, each  $\text{WORD}_{S_n, \ell}^{(\cdot, \cdot)}$  subformula has an  $S_n^{\ell-1}$ -invariant  $\Pi_{d-1}$  formulas of size  $\ell n^{(d-1)(\ell^{1/(d-1)}-1)}$ . Substituting these formulas above and collapsing the adjacent layers of AND below the output, we get an  $S_n^{k-1}$ -invariant  $\Sigma_d$  formula for  $\text{WORD}_{S_n, k}^{(u_0, u_k)}(M_1, \dots, M_k)$  of size  $n^{(k-\ell)/\ell} k^{1/d} \cdot \ell n^{(d-1)(\ell^{1/(d-1)}-1)}$ , which equals  $kn^{d(k^{1/d}-1)}$ . We similarly get an  $S_n^{k-1}$ -invariant  $\Pi_d$  formula from the observation that

$$\begin{aligned} & \text{WORD}_{S_n, k}^{(u_0, u_k)}(M_1, \dots, M_k) \\ &= \bigwedge_{u_\ell, u_{2\ell}, \dots, u_{k-\ell} \in [n]} \left( \bigvee_{j \in [k^{1/d}-1]} \neg \text{WORD}_{S_n, \ell}^{(u_{(j-1)\ell}, u_{j\ell})}(M_{(j-1)\ell+1}, \dots, M_{j\ell}) \right. \\ & \quad \left. \vee \text{WORD}_{S_n, \ell}^{(u_{k-\ell}, u_k)}(M_{k-\ell+1}, \dots, M_k) \right). \end{aligned}$$

Here we replace each  $\text{WORD}_{S_n, \ell}^{(\cdot, \cdot)}$  subformula under a negation with a  $\Pi_{d-1}$  formula, which we then convert to  $\Sigma_{d-1}$  formula using DeMorgan's law. ◀

When  $k^{1/d}$  is not necessary an integer, a similar divide-and-conquer construction gives the following upper bound:

► **Corollary 8.** For all  $n, k, d \geq 1$ , the function  $\text{WORD}_{S_n, k}$  is computed by  $S_n^{k-1}$ -invariant  $\Sigma_{d+1}$  and  $\Pi_{d+1}$  formulas of size  $n^{O(dk^{1/d})}$ . In particular,  $\text{WORD}_{S_n, k}$  has  $S_n^{k-1}$ -invariant AC formulas of size  $n^{O(\log k)}$  and depth  $O(\log k)$ .

The upper bound of Corollary 8 is the quantitatively strongest known (up to constants in the exponent), even for TC formulas that may include MAJ gates.



#### 4 Overall technique

The structure of the proof of Theorem 3 is quite similar to that of the upper bound. Both bounds are inductive on formula depth, and deal with dividing the product  $M_1 \cdots M_k$  into contiguous segments. While the upper bound keeps track of how many subproblems of permutation multiplication we have, the lower bound keeps track of the amount of symmetry that the subformulas of the formula must retain. This amount of symmetry is quantified by the length of the longest contiguous segment of the product (i.e.  $M_i \cdots M_j$  for some  $1 \leq i \leq j \leq k$ ) over which the formula is fully symmetric.

Before giving the proof of Theorem 3, we give a rough overview of our proof using the case in which  $G = A_n$ , the alternating group on  $n$  elements. However, we remark that there is no significant simplification in the proofs given in Section 6.1 by setting  $G = A_n$  rather than allowing  $G$  to be any nonabelian simple group. We merely focus on  $A_n$  here for concreteness of the permutation action on  $[n]$ .

Let  $\Phi$  be an  $A_n^{k-1}$ -invariant formula computing  $\text{WORD}_{A_n, k}$ . As a first step, we let  $A_n^{k-1}$  act on  $\Phi$  as a subgroup of a larger group action, given by letting  $Q = A_n^{2k}$  act on inputs  $(h_1, \dots, h_k)$ :

$$(g_1, \dots, g_{2k})(h_1, \dots, h_k) = (g_1 h_1 g_2^{-1}, \dots, g_{2k-1} h_k g_{2k}^{-1}).$$

Now  $A_n^{k-1}$  can be viewed as the subgroup of  $Q = A_n^{2k}$  given by the equations  $g_2 = g_3, \dots, g_{2k-2} = g_{2k-1}, g_{2k} = g_1 = 1_G$ .

We will define a complexity measure  $\mu$  on subgroups of  $Q$ . Roughly, if a formula is stabilized by a subgroup  $H$ , then  $\mu(H)$  provides a good lower bound on the size of that formula. In this overview, we will not say what exactly this complexity measure is, but we will say that it quantifies segments of the product  $M_1 \cdots M_k$  over which a formula is fully symmetric. What is more important here is the following two properties of  $\mu$ .

1. Let a set of formulas  $\{\Phi_j\}$  have  $Q$ -stabilizer  $H$ . If  $\{\Phi_j\} = O_1 \cup \cdots \cup O_t$  breaks this set into its orbits under  $H$ , then there is some orbit  $O_i$  with  $Q$ -stabilizer  $H'$  such that  $\mu(H') \geq \mu(H)$ .
2. Let  $\Phi_j$ , which has  $Q$ -stabilizer  $H_j$ , lie in the orbit  $O_i$  stabilized by  $H'$ . Then  $\mu(H_j) \geq \frac{\mu(H')}{1 + \log_n [H' : H' \cap H_j]}$ .

The idea for using these properties to get formula lower bounds is as follows. We can let  $Q$  act on a formula  $\Phi = (\text{MAJ}, \{\Phi_j\})$  and find its stabilizer  $H = \text{Stab}_Q(\Phi)$ . We then break  $\{\Phi_j\}$  into orbits under the action of  $H$ :  $\{\Phi_j\} = O_1 \cup \cdots \cup O_t$ . From the first property we know that one of the orbits  $O_i$  has a high complexity stabilizer  $H'$ .

We examine a representative  $\Phi_j \in O_i$  of that high complexity orbit. At this point we have roughly two things that can happen. First, if the  $Q$ -stabilizer  $H_j = \text{Stab}_Q(\Phi_j)$  of  $\Phi_j$  has  $[H' : H' \cap H_j]$  small then it must be that  $\mu(H_j)$  is large by the second property, and we apply induction to get a good lower bound on the size of  $\Phi_j$  in terms of  $\mu(H_j)$ . Each subformula in the orbit  $O_i$  has the same size as  $\Phi_j$  by transitivity, so this gives a good lower bound on the size of  $\Phi$ . Otherwise,  $[H' : H' \cap H_j]$  is large, and the orbit-stabilizer theorem tells us that there are many subformulas in  $O_i$ , which again provides a good lower bound on the size of  $\Phi$  in terms of  $\mu(H)$ .

Finally, we show that  $\mu(A_n^{k-1})$  is large in order to get a good lower bound on the size of a formula with  $Q$ -stabilizer equal to  $A_n^{k-1}$ . An  $A_n^{k-1}$ -invariant formula computing the word problem must have  $Q$ -stabilizer equal to  $A_n^{k-1}$ , so we have the lower bound we want.

This is an oversimplified overview of our actual analysis, and the real proof of the lower bound considers pairs of subgroups corresponding to formulas. One subgroup in the pair is the stabilizer of the formula, and the other is the stabilizer of the function computed by the



formula. This can be defined in a formal way. More care also needs to be put into the second property, especially in concerns such as the transition from  $H'$ -stabilizers to  $Q$ -stabilizers. Such considerations complicate the analysis, but the above ideas still lie at the core of the argument.

## 5 The lower bound framework

Let  $Q \leq S_m$ , and let  $\Omega$  be a  $Q$ -invariant subset of  $\{0, 1\}^m$ . We consider the action of  $Q$  on the set of functions with  $\Omega$  (and arbitrary codomain).

For an integer-valued function  $f : \Omega \rightarrow \mathbb{N}$ , let

$$\text{Stab}_Q(f) = \{\pi \in Q : f(x) = f(x_{\pi(1)}, \dots, x_{\pi(m)}) \text{ for all } x \in \Omega\}.$$

For a set of formulas  $\{\Phi_i\}$  with each  $\Phi_i$  taking inputs in  $\Omega$ , let

$$\text{Stab}_Q(\{\Phi_i\}) = \{\pi \in Q : \{\pi\Phi_i\} = \{\Phi_i\}\}.$$

To simplify notation, write  $\text{Stab}_Q(\Phi) = \text{Stab}_Q(\{\Phi\})$  when  $\{\Phi\}$  is a set containing just one formula.

Let  $\mathcal{B}_{Q,\Omega} \subseteq \mathcal{N}_{Q,\Omega}$  be the following sets of subgroups of  $Q$ :

$$\mathcal{B}_{Q,\Omega} = \{\text{Stab}_Q(f) : f \text{ is a Boolean function with domain } \Omega \text{ and codomain } \{0, 1\}\},$$

$$\mathcal{N}_{Q,\Omega} = \{\text{Stab}_Q(f) : f \text{ is a function with domain } \Omega \text{ and codomain } \mathbb{N}\}.$$

Note  $\mathcal{N}_{Q,\Omega}$  is the closure of  $\mathcal{B}_{Q,\Omega}$  under intersections.

Let  $\chi_1, \dots, \chi_m : \{0, 1\}^m \rightarrow \{0, 1\}$  be the coordinate functions  $\chi_i(x) = x_i$ . For a formula  $\Phi$ , we suppress notation and write  $\llbracket \Phi \rrbracket|_{\Omega} = \llbracket \Phi \rrbracket$  when  $\Omega$  is clear from context.

► **Definition 9.** For  $d = 0, 1, 2, \dots$ , we define a function  $\beta_{Q,\Omega,d} : \{(H, K) : H \leq K \leq Q, K \in \mathcal{B}_{Q,\Omega}\} \rightarrow \mathbb{N} \cup \{\infty\}$  inductively. For any  $H \leq K \leq Q$  with  $K \in \mathcal{B}_{Q,\Omega}$ , let

$$\beta_{Q,\Omega,0}(H, K) = \begin{cases} 0 & \text{if } H = Q, \\ 1 & \text{if } H < Q \text{ and } H = \text{Stab}_Q(\chi_i|_{\Omega}) \text{ for some } i \in [m], \\ \infty & \text{otherwise.} \end{cases}$$

Then, for  $d \geq 1$  define

$$\beta_{Q,\Omega,d}(H, K) = \min_{\substack{(H, K)\text{-good } r \in \mathbb{N}, \\ H_1, \dots, H_r \leq Q, \\ U_1, \dots, U_r \leq Q, \\ L_1, \dots, L_r \in \mathcal{N}_{Q,\Omega}, \\ V_1, \dots, V_r \in \mathcal{B}_{Q,\Omega}}} \max_{i \in [r]} [H_i : H_i \cap U_i] \beta_{Q,\Omega,d-1}(U_i, V_i),$$

where  $(r, (H_i)_{i \in [r]}, (U_i)_{i \in [r]}, (L_i)_{i \in [r]}, (V_i)_{i \in [r]})$  is  $(H, K)$ -good if and only if all of the following hold.

- (a) For all  $i \in [r]$ ,  $U_i \leq V_i$  and  $H_i \leq L_i$ .
- (b)  $H_1 \cap \dots \cap H_r = H$ .
- (c)  $L_1 \cap \dots \cap L_r \leq K$ .
- (d) For all  $i \in [r]$ ,  $\bigcap_{h \in H_i} h^{-1} U_i h \leq H_i$ .
- (e) For all  $i \in [r]$ ,  $\bigcap_{h \in H_i} h^{-1} V_i h \leq L_i$ .

Let  $\beta_{Q,\Omega}(H, K) = \lim_{d \rightarrow \infty} \beta_{Q,\Omega,d}(H, K)$ .

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Note that in the inductive definition of  $\beta_{Q,\Omega,d}$ , the quantity  $\beta_{Q,\Omega,d-1}(U_i, V_i)$  is defined because  $V_i \in \mathcal{B}_{Q,\Omega}$  and Item a. Also observe that monotonicity with respect to  $d$  is displayed:  $\beta_{Q,\Omega,0}(H, K) \geq \beta_{Q,\Omega,1}(H, K) \geq \beta_{Q,\Omega,2}(H, K) \geq \dots$ . Also, we have that  $\beta_{Q,\Omega,d}(H, Q) = 0$  for all  $d$  and  $H \leq Q$ .

Lemma 10 shows that given some  $Q$  acting on  $\Omega$ ,  $\beta_{Q,\Omega,d}(H, K)$  lower bounds the size of formulas  $\Phi$  with  $\text{Stab}_Q(\Phi) = H$  and  $\text{Stab}_Q(\llbracket \Phi \rrbracket) = K$ . This is helpful because it converts the problem of lower bounding formula size to a purely group-theoretic problem of lower bounding the inductively defined function  $\beta_{Q,\Omega,d}$  on pairs of subgroups.

Before stating and proving the result, we remark that it is important that we defined  $\beta_{Q,\Omega,d}$  on pairs of subgroups of  $Q$ , one of which is in  $\mathcal{B}_{Q,\Omega}$ . This is because the sets  $\mathcal{B}_{Q,\Omega}$  and  $\mathcal{N}_{Q,\Omega}$  may not contain  $\text{Stab}_Q(\Phi)$  for a formula  $\Phi$ . In general, formulas may carry more information about themselves than just the way they evaluate on inputs in  $\Omega$ .

However, it is often the case that  $\mathcal{N}_{Q,\Omega}$  is easier to study than the set of all subgroups of  $Q$ , which may contain badly behaved subgroups. Thus, in our applications, our complexity measures also may take into account  $\text{Stab}_Q(\llbracket \Phi \rrbracket)$ , which must lie in  $\mathcal{B}_{Q,\Omega}$ , since  $\llbracket \Phi \rrbracket$  is a Boolean function on  $\Omega$ . We then leverage the relationship  $\text{Stab}_Q(\Phi) \leq \text{Stab}_Q(\llbracket \Phi \rrbracket)$  frequently.

► **Lemma 10.** *Let  $\Phi$  be a  $\text{TC}_d$  formula. Then*

$$\text{size}(\Phi) \geq \beta_{Q,\Omega,d}(\text{Stab}_Q(\Phi), \text{Stab}_Q(\llbracket \Phi \rrbracket)).$$

**Proof.** Let  $H = \text{Stab}_Q(\Phi)$  and  $K = \text{Stab}_Q(\llbracket \Phi \rrbracket)$ . First note that  $\beta_{Q,\Omega,d}(H, K)$  is well-defined because  $H \leq K$  and  $K$  is the  $Q$ -stabilizer of the Boolean function  $\llbracket \Phi \rrbracket$  on  $\Omega$ . We prove by induction on  $d$  that  $\text{size}(\Phi) \geq \beta_{Q,\Omega,d}(H, K)$ . The base case  $d = 0$  follows from the definitions of  $\beta_{Q,\Omega,0}$  and  $\text{TC}_0$ .

Now assume that the result holds for all formulas up to depth  $d$ . Let  $\Phi$  be a  $\text{TC}_{d+1}$  formula. Assume that  $\Phi = (\text{MAJ}, I)$ , where  $I$  is a multiset of formulas in  $\text{TC}_0 \cup \dots \cup \text{TC}_d$ . The group  $H = \text{Stab}_Q(\Phi)$  then acts on  $I$  as a permutation group, since  $\Phi$  is  $H$ -invariant. Break  $I$  into its  $r$  orbits under the action of  $H$ .

For  $i \in [r]$ , let  $\{\Psi_{i,j} : j \in [t_i]\}$  be the  $i$ th orbit of  $I$  under the action of  $H$ . Let  $H_i = \text{Stab}_Q(\{\Psi_{i,j} : j \in [t_i]\})$ . Let  $U_i = \text{Stab}_Q(\Psi_{i,1})$  and let  $V_i = \text{Stab}_Q(\llbracket \Psi_{i,1} \rrbracket)$ . Finally, let  $L_i = \text{Stab}_Q(f_i)$ , where  $f_i$  is the integer-valued function  $f_i : \Omega \rightarrow \mathbb{N}$  defines by  $f_i(x) = |\{j : \Psi_{i,j}(x) = 1\}|$  for  $x \in \Omega$ . Note that this is the Hamming weight function on the wires from this orbit of subformulas.

We claim that the tuple  $(r, (H_i)_{i \in [r]}, (U_i)_{i \in [r]}, (L_i)_{i \in [r]}, (V_i)_{i \in [r]})$  is  $(H, K)$ -good. We prove that this tuple satisfies each one of the conditions stated in the definition below.

For all  $i \in [r]$ , we have  $U_i \leq V_i$  and  $H_i \leq L_i$  because for any formula  $\Psi$ , we have  $\text{Stab}_Q(\Psi) \leq \text{Stab}_Q(\llbracket \Psi \rrbracket)$ , so

$$U_i = \text{Stab}_Q(\Psi_{i,1}) \leq \text{Stab}_Q(\llbracket \Psi_{i,1} \rrbracket) = V_i.$$

Moreover, for each  $i \in [r]$ , we have

$$H_i = \text{Stab}_Q(\{\Psi_{i,j} : j \in [t_i]\}) \leq \text{Stab}_Q(f_i) = L_i.$$

That  $V_i$  is an element of  $\mathcal{B}_{Q,\Omega}$ , and  $L_i$  is an element of  $\mathcal{N}_{Q,\Omega}$  follows because  $V_i$  is the  $Q$ -stabilizer of the Boolean function  $\llbracket \Psi_{i,1} \rrbracket$  on  $\Omega$ , while  $L_i$  is the  $Q$ -stabilizer of the integer-valued function  $f_i$  on  $\Omega$ . This proves that Item a is satisfied.

Now we prove that  $H_1 \cap \dots \cap H_r = H$ . First, we have

$$\begin{aligned} \bigcap_{i \in [r]} H_i &= \bigcap_{i \in [r]} \text{Stab}_Q(\{\Psi_{i,j} : j \in [t_i]\}) \leq \text{Stab}_Q((\text{MAJ}, \bigcup_{i \in [r]} \{\Psi_{i,j} : j \in [t_i]\})) \\ &= \text{Stab}_Q(\Phi) = H. \end{aligned}$$

That  $H \leq H_1 \cap \dots \cap H_r$  follows because orbits of  $I$  under the action of  $H$  are  $H$ -invariant. This proves that Item b is satisfied.

If  $q \in Q$  is such that  $f_i(qx) = |\{j \in [t_i] : \Psi_{i,j}(qx) = 1\}| = |\{j \in [t_i] : \Psi_{i,j}(x) = 1\}| = f_i(x)$  for all  $x \in \Omega$  and  $i \in [r]$ , then  $[\Phi](qx) = [\Phi](x)$  for all  $x \in \Omega$ . Therefore, we have

$$\bigcap_{i \in [r]} L_i = \bigcap_{i \in [r]} \text{Stab}_Q(f_i) \leq \text{Stab}_Q([\Phi]) = K.$$

This proves that Item c is satisfied.

The  $Q$ -stabilizers of the subformulas  $\Psi_{i,j}$  in  $i$ th orbit are all  $H_i$ -conjugates of each other. To see this, note that for every  $\Psi_{i,j}$  there exists  $h \in H_i$  such that  $\Psi_{i,1}^h = \Psi_{i,j}$ . Then, for any  $s \in \text{Stab}_Q(\Psi_{i,1}) = U_i$ , we have

$$\Psi_{i,j}^{h^{-1}sh} = \Psi_{i,1}^{sh} = \Psi_{i,1}^h = \Psi_{i,j}.$$

This shows that  $h^{-1}\text{Stab}_Q(\Psi_{i,1})h \leq \text{Stab}_Q(\Psi_{i,j})$ . The symmetric argument shows the reverse inclusion. As a result, for each  $i \in [r]$ ,

$$\bigcap_{h \in H_i} h^{-1}U_i h = \bigcap_{j \in [t_i]} \text{Stab}_Q(\Psi_{i,j}) \leq \text{Stab}_Q(\{\Psi_{i,j} : j \in [t_i]\}) = H_i.$$

This proves that Item d is satisfied.

Item e follows similarly, since for each  $\Psi_{i,j}$  in the  $i$ th orbit, we have  $\text{Stab}_Q([\Psi_{i,j}]) = h^{-1}\text{Stab}_Q([\Psi_{i,1}])h$  for some  $h \in H_i$ . To see this, again note that for every  $\Psi_{i,j}$  there exists  $h \in H_i$  such that  $\Psi_{i,1}^h = \Psi_{i,j}$ . Then, for any  $s \in \text{Stab}_Q([\Psi_{i,1}]) = V_i$ , we have

$$[\Psi_{i,j}]^{h^{-1}sh} = [\Psi_{i,1}^{sh}] = [\Psi_{i,1}^h] = [\Psi_{i,j}].$$

This shows that  $h^{-1}\text{Stab}_Q([\Psi_{i,1}])h \leq \text{Stab}_Q([\Psi_{i,j}])$ . The symmetric argument shows the reverse inclusion.

If  $q \in Q$  is such that  $[\Psi_{i,j}](qx) = [\Psi_{i,j}](x)$  for all  $x \in \Omega$  and  $i \in [r]$ , then  $f_i(qx) = f_i(x)$  for all  $x \in \Omega$ . As a result, for each  $i \in [r]$ ,

$$\bigcap_{h \in H_i} h^{-1}V_i h = \bigcap_{j \in [t_i]} \text{Stab}_Q([\Psi_{i,j}]) \leq \text{Stab}_Q(f_i) = L_i.$$

We have finished verifying that our tuple of  $r$  and subgroups is  $(H, K)$ -good.

Recall that  $\{\Psi_{i,j} : j \in [t_i]\}$  is the  $i$ th orbit of this group action stabilized by  $H_i$ , and  $\text{Stab}_Q(\Psi_{i,1}) = U_i$ . Then by the orbit-stabilizer theorem applied to the action of  $H_i$  on the orbit, the size of the orbit is at least  $[H_i : H_i \cap U_i]$ . The size of each subformula in this orbit is equal to the size of  $\Psi_{i,1}$ , since for every  $\Psi_{i,j}$ , there exists  $h \in H$  such that  $\Psi_{i,j} = \Psi_{i,1}^h$ , and  $\text{size}(\Psi_{i,1}^h) = \text{size}(\Psi_{i,1})$  is clear. Therefore, for some  $(r, (H_i)_{i \in [r]}, (U_i)_{i \in [r]}, (L_i)_{i \in [r]}, (V_i)_{i \in [r]})$  that is  $(H, K)$ -good, we have

$$\begin{aligned} \text{size}(\Phi) &\geq \max_{i \in [r]} [H_i : H_i \cap U_i] \text{size}(\Psi_{i,1}) \\ &\geq \max_{i \in [r]} [H_i : H_i \cap U_i] \beta_{Q, \Omega, d}(U_i, V_i) \\ &\geq \beta_{Q, \Omega, d+1}(H, K). \end{aligned}$$

The second inequality follows by induction and monotonicity of  $\beta_{Q, \Omega, d}$  in  $d$ . The last inequality follows from the inductive definition of  $\beta_{Q, \Omega, d+1}$ .  $\blacktriangleleft$

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To use the lemma to lower bound the size of  $P$ -invariant formulas, we start off with some  $P$ -invariant function  $f : \Omega \rightarrow \{0, 1\}$  where  $\Omega$  is a  $P$ -invariant subset of  $\{0, 1\}^m$ . To apply the lemma, find a supergroup  $P \leq Q \leq S_m$  such that  $\Omega$  is  $Q$ -invariant and  $\text{Stab}_Q(f) = P$ . Then we get the lower bound  $\text{size}(\Phi) \geq \beta_{Q,\Omega,d}(P, P)$  for any  $\Phi \in \text{TC}_d$  with  $\text{Stab}_Q(\Phi) = \text{Stab}_Q(\llbracket \Phi \rrbracket) = P$ .

Note that choosing  $Q = P$  yields nothing, since  $\beta_{P,\Omega,d}(H, P) = 0$  for all  $d$  and  $H \leq P$ .

In our application,  $G \leq S_n$ ,  $m = kn^2$ ,  $P = G^{k-1}$ , and  $\Omega = \overline{G^k} = \{(M_1, \dots, M_k) \in \{0, 1\}^{kn^2}\} = \{0, 1\}^m$ , where the matrices are permutation matrices giving elements of  $G \leq S_n$ .

Thus, we are left with the problem of lower bounding  $\beta_{Q,\Omega,d}(P, P)$  for some choice of  $Q$ . This is a purely group-theoretic problem, and Lemma 11 provides a framework to solve it.

► **Lemma 11.** *Assume  $c \geq 1$  and that  $\mu : \{(H, L) : H \leq L \leq Q, L \in \mathcal{N}_{Q,\Omega}\} \rightarrow \mathbb{N}$  is such that for any  $H \leq K \leq Q$  with  $K \in \mathcal{B}_{Q,\Omega}$ ,*

- (i) *Let  $r \in \mathbb{N}$ ,  $H_1, \dots, H_r \leq Q$ ,  $L_1, \dots, L_r \in \mathcal{N}_{Q,\Omega}$  be such that  $H_i \leq L_i$  for all  $i \in [r]$ . Suppose that  $H = H_1 \cap \dots \cap H_r$  and  $K \geq L_1 \cap \dots \cap L_r$ . Then there is some  $i \in [r]$  such that  $\mu(H_i, L_i) \geq \mu(H, K)$ .*
- (ii) *Let  $U \leq V \leq Q$  with  $V \in \mathcal{B}_{Q,\Omega}$  and  $L \geq H$  with  $L \in \mathcal{N}_{Q,\Omega}$ . Suppose that  $\bigcap_{h \in H} h^{-1}Uh \leq H$  and  $\bigcap_{h \in H} h^{-1}Vh \leq L$ . Then we have  $\mu(U, V) \geq \mu(H \cap U, V) \geq \frac{\mu(H, L)}{1 + \log_c[\overline{H : H \cap U}]}$ .*
- (iii) *Let  $H = \text{Stab}_Q(\chi_i |_\Omega)$  for some coordinate function  $\chi_i$ . Then  $\mu(H, K) \leq 1$ .*
- (iv)  $\mu(Q, Q) = 0$ .

Then for all  $H \leq K \leq Q$  with  $K \in \mathcal{B}_{Q,\Omega}$ ,

$$\begin{aligned} \beta_{Q,\Omega,d}(H, K) &\geq c^{d(\mu(H,K)^{1/d}-1)} \text{ for all } d \geq 1, \\ \beta_{Q,\Omega}(H, K) &\geq c^{\log_2(\mu(H,K))}. \end{aligned}$$

**Proof.** We prove by induction on  $d$  and actually start at 0, interpreting for  $d = 0$

$$c^{d(\mu(H,K)^{1/d}-1)} = \begin{cases} 0 & \text{if } \mu(H, K) = 0, \\ 1 & \text{if } \mu(H, K) = 1, \\ \infty & \text{otherwise.} \end{cases}$$

Note that these interpretations fit into the inductive steps. The base case in which  $d = 0$  is clear by Item iii, Item iv, and the definition of  $\beta_{Q,\Omega,0}$ . Now assume that the result holds for  $\beta_{Q,\Omega,i}$  with  $i \leq d$ .

By definition, for some tuple  $(r, (H_i)_{i \in [r]}, (U_i)_{i \in [r]}, (L_i)_{i \in [r]}, (V_i)_{i \in [r]})$  that is  $(H, K)$ -good we have

$$\beta_{Q,\Omega,d+1}(H, K) \geq \max_{i \in [r]} [H_i : H_i \cap U_i] \beta_{Q,\Omega,d}(U_i, V_i).$$

Because the tuple is  $(H, K)$ -good, by Item a, Item b, and Item c, we have that the hypotheses of Item i are satisfied by the  $H_i$  and  $L_i$ , so for some  $i \in [r]$ , we have  $\mu(H_i, L_i) \geq \mu(H, K)$ .

Let  $m = 1 + \log_c[H_i : H_i \cap U_i]$ . By Item a, Item b, and Item e the hypotheses of Item ii are satisfied by  $U_i, H_i, V_i$ , and  $L_i$ , so we have  $\mu(U_i, V_i) \geq \mu(H_i \cap U_i, V_i) \geq \frac{\mu(H_i, L_i)}{m} \geq \frac{\mu(H, K)}{m}$ . Therefore,

$$\beta_{Q,\Omega,d+1}(H, K) \geq c^{m-1} \beta_{Q,\Omega,d}(U_i, V_i) \geq c^{m-1} c^{d((\frac{\mu(H,K)}{m})^{1/d}-1)} \geq c^{(d+1)(\mu(H,K)^{1/(d+1)}-1)}.$$

The last step follows from optimization over  $m$  using elementary calculus.

This lower bound approaches  $c^{\ln(\mu(H,K))}$  as  $d \rightarrow \infty$ , but since  $\mu$  takes integral values, when  $d$  becomes large we actually have a lower bound  $c^{\log_2(\mu(H,K))}$ . ◀

We can informally interpret the results of this section in the following way. Lemma 10 shows that the size of a formula  $\Phi$  with  $\text{Stab}_Q(\Phi) = H$  and  $\text{Stab}_Q(\llbracket \Phi \rrbracket) = K$  depends on a sequence of operations on the subgroup lattice  $\text{Sub}(Q)$  of  $Q$ . We begin with a pair  $(H, K) \in \text{Sub}(Q)^2$ . Then, to get to the pair stabilizing the orbits of subformulas of  $\Phi$ , we first “go up” in  $\text{Sub}(Q)^2$ .

That is, in the process of breaking the set of subformulas of  $\Phi$  into  $H$ -orbits, we find that the syntactic and semantic stabilizers of these orbits are supergroups of  $H$  and  $K$  that satisfy the intersection properties given by Item b and Item c stated.

The next step is to find the stabilizer of a single subformula in an orbit. Here we apply the orbit-stabilizer theorem, and are “going down” in the subgroup lattice. At this point there is a cost associated to how far down on the subgroup lattice we go by the orbit-stabilizer theorem.

The point of Lemma 11 is then to define a “lower-bound witness”  $\mu$  on a subset of  $\text{Sub}(Q)^2$  that accurately reflects the cost of these operations. A designer of a small invariant formula would then want to find sequences of subgroups (under the constraints stated in the definition of  $\beta_{Q,\Omega,d}$ ) in the subgroup lattice to get to the low-complexity subgroups (stabilizers of coordinate functions and  $Q$  itself) with as little cost as possible. Cost is accrued in the going down phase of the process. Lemma 11 states conditions sufficient for the formula designer to not be able to easily decrease the lower-bound witness  $\mu$ .

## 6 Applying the framework

In this section we prove Theorem 3. Section 6.1 takes care of the nonabelian case and Section 6.2 takes care of the abelian case.

### 6.1 TC lower bounds for nonabelian simple groups

Throughout this section let  $G \leq S_n$  be a finite nonabelian simple permutation group, where  $n$  is the minimum degree of a faithful representation of  $G$ . We want to use the framework set up in Section 5 to prove a lower bound on the formula size of  $G^{k-1}$ -invariant  $\text{TC}_d$  formulas computing the function  $\text{WORD}_{G,k} : \overline{G}^k \rightarrow \{0, 1\}$ .

To do so, we consider a larger group  $G^{k-1} \leq Q \leq S_{kn^2}$  such that  $\overline{G}^k$  is  $Q$ -invariant and  $\text{Stab}_Q(\text{WORD}_{G,k}) = G^{k-1}$ . We then get the bound  $\mathcal{L}_{\text{TC}_d}^{G^{k-1}}(\text{WORD}_{G,k}) \geq \beta_{Q,\Omega,d}(G^{k-1}, G^{k-1})$  by Lemma 10, so for our purposes it suffices to lower bound  $\beta_{Q,\Omega,d}(G^{k-1}, G^{k-1})$ . We do so by constructing a suitable  $\mu$  satisfying the hypotheses of Lemma 11 with  $c = n$ .

Recall that  $n$  is the minimum degree of a faithful permutation representation of  $G$ . The following lemma characterizes this  $n$  nicely.

► **Lemma 12.**  $n = \min(\sqrt{|G|}, \min_{A < G}[G : A])$ .

**Proof.** That  $\min(\sqrt{|G|}, \min_{A < G}[G : A]) = \min_{A < G}[G : A]$  follows from the classification of finite simple groups [23]. Then,  $\min_{A < G}[G : A] = n$  follows from [16]. ◀

We first consider the “left-right” action of  $G^{2k} \leq S_{kn^2}$  on  $\overline{G}^k$

$$(g_1, \dots, g_{2k})(M_1, \dots, M_k) = (\overline{g_1}M_1\overline{g_2}^{-1}, \dots, \overline{g_{2k-1}}M_k\overline{g_{2k}}^{-1}).$$

Note that odd coordinates in  $[2k]$  act on matrices on the left, while even coordinates act on the right. That is, the elements  $g_{2i-1}$  and  $g_{2i}$  act respectively on the left and right of the matrix  $M_i$ .

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The embedding of  $G^{k-1}$  in  $G^{2k}$  (as subgroups of  $S_{kn^2}$ ) is given by the shifted diagonal action.

$$(g_1, \dots, g_{k-1}) \mapsto (1_G, g_1^{-1}, g_1, g_2^{-1}, g_2, \dots, g_{k-1}^{-1}, g_{k-1}, 1_G). \quad (1)$$

However, the  $G^{2k}$ -stabilizer of  $\text{WORD}_{G,k}$  is larger than  $G^{k-1}$ , since it contains all elements of the form  $(g_1, 1_G, \dots, 1_G)$  with  $g_1 \in \text{Stab}_G(1)$  and  $(1_G, \dots, 1_G, g_{2k})$ , where  $g_{2k} \in \text{Stab}_G(1)$ .

For this reason, we take  $Q$  to be the subgroup of  $G^{2k}$  given by  $Q = \{(g_1, \dots, g_{2k}) \in G^{2k} : g_1 = g_{2k} = 1_G\}$ . Now we have  $G^{k-1} \leq Q \leq S_{kn^2}$  with  $\text{Stab}_Q(\text{WORD}_{G,k}) = G^{k-1}$ , as required.

Observe that the coordinate functions  $\chi_1, \dots, \chi_{kn^2} : \{0, 1\}^{kn^2} \rightarrow \{0, 1\}$  are given by the entries of the permutation matrices  $\{M_1, \dots, M_k\}$ .

We now define the  $\mu$  that we will use in conjunction with Lemma 11 to prove a lower bound on  $\beta_{Q,\Omega,d}(G^{k-1}, G^{k-1})$ .

► **Definition 13.** For  $H \leq Q$  let

$$E(H) = \{\{i, i+1\} : i \in [k-1] \text{ such that } H \upharpoonright_{\{2i, 2i+1\}} = \text{Diag}(G^{\{2i, 2i+1\}})\}.$$

We can view  $E(H)$  as the edge set of an undirected graph with vertex set  $[k]$ , which we denote  $([k], E(H))$ . Note that this graph is a spanning subgraph of the path graph with edge set  $\{\{1, 2\}, \{2, 3\}, \dots, \{k-1, k\}\}$ .

For  $H \leq K \leq Q$  with  $K \in \mathcal{N}_{Q,\Omega}$ , we define  $\mu(H, K)$  to be the number of vertices in the largest connected component of  $([k], E(H) \cap E(K))$ . Define  $\mu(H, Q) = 0$  for any  $H \leq Q$ .

We remark that this definition of  $\mu$  is highly dependent on both  $H$  and  $L$ . In fact, if  $([k], E(H))$  has a large connected component, it still may be possible for  $\mu(H, K)$  to be small for some choice of  $L$ .

The consequence for our lower bound is the following. Suppose  $\Phi$  is a formula with  $\text{Stab}_Q(\Phi) = G^{k-1}$ . Then  $\text{Stab}_Q(\Phi) = \text{Stab}_Q(\text{WORD}_{G,k})$ . However, we can say nothing about the size of  $\Phi$  unless we also have some information about  $\text{Stab}_Q(\llbracket \Phi \rrbracket)$ .

If  $\text{Stab}_Q(\llbracket \Phi \rrbracket) = Q$ , for example, then  $\mu(\text{Stab}_Q(\Phi), \text{Stab}_Q(\llbracket \Phi \rrbracket)) = 0$  by definition. Then Lemma 11 gives no meaningful lower bound on  $\beta_{Q,\Omega,d}(\text{Stab}_Q(\Phi), \text{Stab}_Q(\llbracket \Phi \rrbracket))$ , and hence we have no meaningful lower bound on  $\text{size}(\Phi)$ .

Such a  $\Phi$  can be realized in the following way. Let  $\Psi$  be such that  $\llbracket \Psi \rrbracket = \text{WORD}_{G,k}$  and  $\text{Stab}_Q(\Psi) = G^{k-1}$ . Let  $\Phi = (\text{OR}, \{1, \Psi\})$ . Then  $\llbracket \Phi \rrbracket$  is constant, so  $\text{Stab}_Q(\llbracket \Phi \rrbracket) = Q$ . However,  $\text{Stab}_Q(\Phi) = \text{Stab}_Q(\Psi) = G^{k-1}$ . Thus, even though  $\text{Stab}_Q(\Phi) = G^{k-1}$  is a subgroup for which we expect nontrivial lower bounds, we have no lower bound on its size. Fortunately, such a formula  $\Phi$  does not arise in an optimal construction, due to  $Q$  being useless to a formula designer in the “going up” phase of formula design.

In the rest of this section, we show that once we take into account semantic stabilizers (as we have done in Definition 13), we do get nontrivial lower bounds.

### 6.1.1 Structural results

Since  $Q$  can be thought of as a  $2k - 2$ -fold power of the group  $G$ , it is helpful to understand the structure of subgroups of direct powers of  $G$ . The following lemma of Goursat helps characterize these subgroups.

► **Lemma 14** ([13]). *Let  $A$  and  $B$  be groups. Let  $K$  be a subgroup of  $A \times B$  such that the projections  $\pi_A : K \rightarrow A$  and  $\pi_B : K \rightarrow B$  are surjective. Let  $M = \{a \in A : (a, 1) \in K\} \trianglelefteq A$  and  $N = \{b \in B : (1, b) \in K\} \trianglelefteq B$ . Then there exists an isomorphism  $\theta : A/M \rightarrow B/N$  such that  $K = \{(a, b) \in A \times B : \theta(aM) = bN\}$ .*

This lemma is especially helpful in the special case where  $A$  and  $B$  are both nonabelian simple groups, since the possibilities for  $M$  and  $N$  are then restricted to either being the full group  $A$  or  $B$  respectively, or trivial.

► **Corollary 15.** *Let  $\text{Diag}(G^2) < H \leq G^2$ . Then  $H = G^2$ .*

**Proof.** Since  $H > \text{Diag}(G^2)$  there exists  $(g_1, g_2) \in H$  such that  $g_1 g_2^{-1} \neq 1_G$ . Then, multiplying by  $(g_2^{-1}, g_2^{-1}) \in \text{Diag}(G^2) < H$ , we find that  $(g_1 g_2^{-1}, 1_G) \in H$ .

Note that by Lemma 14,  $\{1_G\}^{\{1\}} < H \upharpoonright_{\{1\}} \trianglelefteq \pi_{\{1\}}(H) = G^{\{1\}}$ . Because  $G$  is nonabelian simple, we must have  $H \upharpoonright_{\{1\}} = G^{\{1\}}$ . This also implies that  $H \upharpoonright_{\{2\}} = G^{\{2\}}$ . ◀

Another point at which Lemma 14 becomes useful is in our proof that  $\mu$  satisfies Item ii. Note that for  $\ell \geq 2$ , if  $H \leq Q$  is such that  $E(H)$  induces a connected component of size  $\ell$  in  $([k], E(H))$ , then  $H$  has some shifted diagonal subgroup isomorphic to  $G^\ell$ . Thus, the following definition is very natural.

► **Definition 16.** For a subgroup  $H \leq G^k$ , define a *support* for  $H$  to be a subset  $T \subseteq [k]$  such that for all  $i \notin T$ ,  $H \upharpoonright_{\{i\}} = G^{\{i\}}$ .

► **Lemma 17.** *For any  $\ell \in \mathbb{N}$ , let  $N \triangleleft G^\ell$  be such such that  $\frac{G^\ell}{N} \cong G$ . Then there is exactly one  $j \in [\ell]$  such that  $N \upharpoonright_{\{j\}} < G^{\{j\}}$ .*

**Proof.** For all  $j \in [\ell]$  we have that  $\pi_{\{j\}}(N) \trianglelefteq G^{\{j\}}$  by normality of  $N$  in  $G^\ell$ . Therefore, this projection must be either the full group  $G^{\{j\}}$  or trivial.

There is at most one  $j$  such that  $\pi_{\{j\}}(N) = \{1_G\}^{\{j\}}$  since otherwise we would have  $\pi_{\{j_1, j_2\}}(N) = \{1_G\}^{\{j_1, j_2\}}$ , and

$$[G^\ell : N] \geq [\pi_{\{j_1, j_2\}}(G^\ell) : \pi_{\{j_1, j_2\}}(N)] \geq |G|^2.$$

The first inequality follows from the elementary group theory fact that for any group homomorphism  $\varphi : A \rightarrow C$ , we have  $[\varphi(A) : \varphi(B)]$  for any  $B \leq A$ .

Assume that there is no  $j$  such that  $\pi_{\{j\}}(N) = \{1_G\}^{\{j\}}$ . Then there is at least one  $j \in [\ell]$  such that  $N \upharpoonright_{\{j\}} = \{1_G\}^{\{j\}}$ , since otherwise we would have  $N \upharpoonright_{\{j\}} = G^{\{j\}}$  for all  $j \in [\ell]$ . This follows from Lemma 14 implying that  $N \upharpoonright_{\{j\}} \trianglelefteq \pi_{\{j\}}(N)$  and simplicity of  $G$ . This would imply  $N = G^\ell$ , a contradiction. Without loss of generality assume that  $j = \ell$ .

Using Lemma 14, write  $N = \text{Graph}(\pi_{\{\ell\}}(N) \cong \frac{\pi_{\{\ell-1\}}(N)}{N \upharpoonright_{\{\ell-1\}}})$ . This shows that there exists some isomorphism  $\varphi : \frac{\pi_{\{\ell-1\}}(N)}{N \upharpoonright_{\{\ell-1\}}} \rightarrow G^{\{\ell\}}$  such that we can write  $N = \{(g_1, \dots, g_{\ell-1}, g_\ell) : \varphi(g_1, \dots, g_{\ell-1}) N \upharpoonright_{\{\ell-1\}} = g_\ell\}$ . But such a subgroup cannot be normal in  $G^\ell$ , since for any pair of non-identity group elements  $g \in G$ , there exists  $a \in G$  such that  $a^{-1} g a \neq g$  by  $G$  being nonabelian.

Thus, there is exactly one  $j$  such that  $\pi_{\{j\}}(N) = \{1_G\}^{\{j\}}$ . Assume without loss of generality that  $j = \ell$ . Then  $N = \text{Graph}(\{1_G\} \cong \frac{\pi_{\{\ell-1\}}(N)}{N \upharpoonright_{\{\ell-1\}}})$ . Since  $\pi_{\{\ell-1\}}(N) = G^{\{\ell-1\}}$ , it must also be the case that  $N \upharpoonright_{\{\ell-1\}} = G^{\{\ell-1\}}$ . Thus, this  $j$  is the only such that  $N \upharpoonright_j < G^{\{j\}}$ . ◀

► **Corollary 18.** *Let  $N \triangleleft H \leq G^k$  be such that  $\frac{H}{N} \cong G$ . Let  $T$  be a support for  $H$ . Then  $N$  has a support of size exactly  $|T| + 1$ .*

**Proof.** Without loss of generality assume that  $T = \{k - |T| + 1, \dots, k\}$ . We claim that  $\frac{H \upharpoonright_{[k-|T|]}}{N \upharpoonright_{[k-|T|]}}$  is isomorphic to a normal subgroup of  $\frac{H}{N}$ . To see this, first note that  $N \upharpoonright_{[k-|T|]} \trianglelefteq H \upharpoonright_{[k-|T|]}$ , since the intersection of a subgroup with a normal subgroup is normal.



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Now let  $(g_1, \dots, g_{k-|T|})N \upharpoonright_{[k-|T|]}$  and  $(h_1, \dots, h_{k-|T|})N \upharpoonright_{[k-|T|]}$  be two distinct elements of  $\frac{H \upharpoonright_{[k-|T|]}}{N \upharpoonright_{[k-|T|]}}$ . We claim that  $(g_1, \dots, g_{k-|T|}, 1_G^{|T| \text{ times}})N$  and  $(h_1, \dots, h_{k-|T|}, 1_G^{|T| \text{ times}})N$  must be distinct elements of  $\frac{H}{N}$ .

Assume otherwise. Then  $(g_1 h_1, \dots, g_{k-|T|} h_{k-|T|}, 1_G^{|T| \text{ times}})$  is an element of  $N$ . This implies that  $(g_1 h_1, \dots, g_{k-|T|} h_{k-|T|})$  is an element of  $N \upharpoonright_{[k-|T|]}$ . We have now contradicted that  $(g_1, \dots, g_{k-|T|})N \upharpoonright_{[k-|T|]}$  and  $(h_1, \dots, h_{k-|T|})N \upharpoonright_{[k-|T|]}$  are distinct.

This gives an injection of  $\frac{H \upharpoonright_{[k-|T|]}}{N \upharpoonright_{[k-|T|]}}$  into  $\frac{H}{N}$ . Normality of the image of this injection in  $\frac{H}{N}$  is clear by construction.

By simplicity of  $\frac{H}{N} \cong G$ , either  $\frac{H \upharpoonright_{[k-|T|]}}{N \upharpoonright_{[k-|T|]}} \cong \{1_G\}$ , in which case we are done immediately, or  $\frac{H \upharpoonright_{[k-|T|]}}{N \upharpoonright_{[k-|T|]}} \cong G$ , in which case we are done by Lemma 17, since  $H \upharpoonright_{[k-|T|]} = G^{\{k-|T|\}}$ . ◀

► **Corollary 19.** *If  $[G^k : H] < n^m$ , then  $H$  has a support of size at most  $m$ .*

**Proof.** We prove by induction on  $k$ . In the base case,  $k = 1$  and this is true since  $H$  has empty support unless  $H < G$ , in which case  $[G : H] \geq n$  by Lemma 12. Now assume that the result holds for subgroups of  $G^k$ . Let  $H \leq G^{k+1}$ , and use Lemma 14 to write  $H = \text{Graph}(\frac{G_1}{N_1} \cong \frac{G_2}{N_2})$ , where  $G_1 = \pi_{[k]}(H)$ ,  $N_1 = H \upharpoonright_{[k]}$ ,  $G_2 = \pi_{\{k+1\}}(H)$ , and  $N_2 = H \upharpoonright_{\{k+1\}}$ .

First consider if  $G_2 = G$ . If  $N_2 = G$ , then we have  $H = G_1 \times G$ , and there exists a support  $T$  for  $G_1$  of size at most  $\log_n([G^k : G_1]) = \log_n([G^{k+1} : H])$  that is also a support for  $H$ .

Otherwise we have  $N_2 = \{1_G\}$ , since  $G$  is simple. Let  $T$  be a minimum size support for  $G_1$ . Since  $\frac{G_1}{N_1} \cong \frac{G_2}{N_2} \cong G$ , Corollary 18 implies that  $N_1$  has a support given by  $T \cup \{j\}$  for some  $j \in [k]$ .

Therefore,  $T \cup \{j\}$  is a support for  $N_1$  of size at most

$$\begin{aligned} |T \cup \{j\}| &= |T| + 1 \leq \log_n([G^k : G_1]) + 1 = \log_n\left(\frac{|G^k|}{|G_1|}\right) + 1 \log_n\left(\frac{|G^{k+1}|}{|G||H|}\right) + 1 \\ &= \log_n([G^{k+1} : H]) + 1 - \log_n(|G|) \leq \log_n([G^{k+1} : H]) - 1. \end{aligned}$$

The first inequality follows by induction. The equality  $\log_n\left(\frac{|G^k|}{|G_1|}\right) = \log_n\left(\frac{|G^{k+1}|}{|G||H|}\right)$  follows from  $|H| = |G_1|$ , since  $H = \text{Graph}(\frac{G_1}{N_1} \cong G)$  and hence  $|G||N_1| = |G_1|$ . The last inequality uses that  $n \leq \sqrt{|G|}$ , a fact proved in Lemma 12. Now, we claim that  $T \cup \{j, k+1\}$  serves as a support for  $H$  of size at most  $\log_n([G^{k+1} : H])$ . Let  $i \in [k] \setminus (T \cup \{j\})$ . Then  $H \upharpoonright_{\{i\}} \geq N_1 \upharpoonright_{\{i\}} = G^{\{i\}}$ .

Now we consider when  $G_2 < G$ . In this case, let  $T$  be a minimum size support for  $N_1$ . Then by induction

$$\begin{aligned} |T| &\leq \log_n([G^k : N_1]) = \log_n\left(\frac{|G^k|}{|N_1|}\right) \\ &= \log_n\left(\frac{|G^{k+1}||G_2|}{|G||H|}\right) = \log_n([G^{k+1} : H]) - \log_n([G : G_2]) \leq \log_n([G^{k+1} : H]) - 1. \end{aligned}$$

The first inequality follows by induction. The equality  $|N_1| = \frac{|H|}{|G_2|}$  used in the third step follows because  $H$  is the graph of an isomorphism  $\frac{G_1}{N_1} \cong \frac{G_2}{N_2}$ . This implies that each coset of  $N_2$  in  $G_2$  yields  $|N_1||N_2|$  group elements of  $H$ . There are  $\frac{|G_2|}{|N_2|}$  such cosets, so  $|H| = \frac{|N_1||N_2||G_2|}{|N_2|} = |N_1||G_2|$ . The last inequality follows from the fact that  $n \leq \min_{L < G} [G : L]$ , proved in Lemma 12.

Then  $T \cup \{k+1\}$  has size at most  $\log_n([G^{k+1} : H])$ . We show that now  $T \cup \{k+1\}$  serves as a support for  $H$ . For any  $i \in [k+1] \setminus (T \cup \{k+1\}) = [k] \setminus T$ , we have  $H \upharpoonright_{\{i\}} \geq N_1 \upharpoonright_{\{i\}} = G^{\{i\}}$ . ◀

### 6.1.2 The properties of $\mu$

Now we begin proving that  $\mu$  satisfies the hypotheses of Lemma 11. Item iv is clearly satisfied by definition of  $\mu$ . We verify that  $\mu$  satisfies Item iii.

► **Lemma 20.** *Let  $H = \text{Stab}_Q(\chi_i|\Omega)$  for some  $i \in [kn^2]$ . Then  $\mu(H, K) \leq 1$  for any  $K \geq H$ .*

**Proof.** In this case  $H$  is the  $Q$ -stabilizer of a function on  $\overline{G}^k$  depending only on  $M_{i^*}$  for some  $i^* \in [k]$ . Therefore, we have that for any  $j \in \{2, \dots, 2k-1\} \setminus \{2i^*-1, 2i^*\}$ , we have  $H \upharpoonright_{\{j\}} = G^{\{j\}}$ . As a result,  $E(H) \cap E(K)$  is empty and the largest connected component in  $([k], E(H) \cap E(K))$  is a single vertex. ◀

The following lemma proves that  $\mu$  satisfies Item i.

► **Lemma 21.** *Let  $H \leq K \leq Q$ . Let  $S \subseteq \{\{i, i+1\} : i \in [k]\}$  be the edge set of a connected component in  $([k], E(H) \cap E(K))$ . Let  $r \in \mathbb{N}$  and for each  $i \in [r]$ , fix  $H_i \leq L_i \leq Q$  such that  $L_i$  is an element of  $\mathcal{N}_{Q, \Omega}$ . Suppose that  $H_1 \cap \dots \cap H_r = H$  and  $L_1 \cap \dots \cap L_r \leq K$ . Then there exists  $i \in [r]$  such that  $S$  is a subset of edges in  $([k], E(H_i) \cap E(L_i))$ .*

**Proof.** Assume without loss of generality that  $S = \{\{j, j+1\} : j \in [k-1]\}$ , since in all other cases  $S$  is still a contiguous path on  $[k]$  and the proof is the same. Notice that  $L_i \geq H_i \geq H$  for all  $i \in [r]$ , so  $L_i \upharpoonright_{\{2i, 2i+1\}} \geq H \upharpoonright_{\{2i, 2i+1\}} = \text{Diag}(G^{\{2i, 2i+1\}})$  for all  $i \in [r]$ .

For any  $i \in [r]$ , assume that there is some  $j \in [k-1]$  such that  $L_i \upharpoonright_{\{2j, 2j+1\}} > \text{Diag}(G^{\{2j, 2j+1\}})$ . By Corollary 15 we have  $L_i \upharpoonright_{\{2j+1\}} = G^{\{2j+1\}}$ .

Since  $L_i$  is an element of  $\mathcal{N}_{Q, \Omega}$ , there exists a function  $f$  on  $\Omega$  for which  $L_i$  is the  $Q$ -stabilizer. Let  $g_0 \in G$  be any group element. For any tuple of matrices  $(M_1, \dots, M_k) \in \overline{G}^k$ ,

$$\begin{aligned} & (1_G, \dots, 1_G, g_0, 1_G) f(M_1, \dots, M_k) \\ &= f(M_1, \dots, M_{k-1}, \overline{g_0} M_k) \\ &= f(M_1, \dots, M_{k-1} \overline{g_0}^{-1}, M_k) \text{ (since } L_i \upharpoonright_{\{2k-2, 2k-1\}} \geq \text{Diag}(G^{\{2k-2, 2k-1\}})) \\ &= f(M_1, \dots, \overline{g_1} M_{k-1}, M_k) \text{ for some } g_1 \in G \\ & \quad \vdots \\ &= f(M_1, \dots, \overline{g_{k-j+1}} M_{j+1}, \dots, M_k) \text{ for some } g_{k-j+1} \in G \\ &= f(M_1, \dots, M_k). \end{aligned}$$

The last equality follows from  $f$  being  $L_i$  invariant and  $L_i \upharpoonright_{\{2j+1\}} = G^{\{2j+1\}}$ . This shows that  $(1_G, \dots, 1_G, g_0, 1_G)$  is an element of  $L_i$ , since  $L_i$  is the  $Q$ -stabilizer of  $f$ .

If for all  $i \in [r]$  there exists  $j \in [k-1]$  such that  $L_i \upharpoonright_{\{2j, 2j+1\}} > \text{Diag}(G^{\{2j, 2j+1\}})$ , then we have shown that  $(1_G, \dots, 1_G, g_0, 1_G)$  is an element of  $L_1 \cap \dots \cap L_r \leq K$  for any  $g_0 \in G$ , a contradiction, since  $\{k-1, k\} \in S \subseteq E(K)$  implies that  $K \upharpoonright_{\{2k-2, 2k-1\}} = \text{Diag}(G^{\{2k-2, 2k-1\}})$ .

Therefore, there is some  $L_i$  such that  $L_i \upharpoonright_{\{2j, 2j+1\}} = \text{Diag}(G^{\{2j, 2j+1\}})$  for all  $j \in [k-1]$ . Now  $H_i \upharpoonright_{\{2j, 2j+1\}} = \text{Diag}(G^{\{2j, 2j+1\}})$  for all  $j \in [k-1]$  because  $H \leq H_i \leq L_i$  implies that for all  $j \in [k-1]$ ,

$$\text{Diag}(G^{\{2j, 2j+1\}}) = H \upharpoonright_{\{2j, 2j+1\}} \leq H_i \upharpoonright_{\{2j, 2j+1\}} \leq L_i \upharpoonright_{\{2j, 2j+1\}} = \text{Diag}(G^{\{2j, 2j+1\}}).$$

Thus,  $S$  is a subset of edges in  $([k], E(H_i) \cap E(L_i))$ . ◀

It remains to prove that  $\mu$  satisfies Item ii with  $c = n$ . A designer attempting to build a small formula would want to decrease  $\mu$ , and hence the size of the largest connected component in  $([k], E(H) \cap E(K))$  quickly, so we would like to characterize the cost of removing edges

(which is stated using indices of subgroups). Because a group  $H$  with  $([k], E(H))$  having some connected component with  $\ell \geq 2$  vertices has a shifted diagonal subgroup isomorphic to  $G^\ell$ , and the designer gets to delete edge when the restrictions  $H \upharpoonright_{\{2i, 2i+1\}}$  are no longer equal to  $\text{Diag}(G^{\{2i, 2i+1\}}) \cong G$ , the notion of support (Definition 16) is useful.

We use Corollary 19 to prove that  $\mu$  satisfies Item ii with  $c = n$ .

► **Lemma 22.** *Let  $U \leq V \leq Q$  with  $V \in \mathcal{N}_{Q, \Omega}$ . Let  $H \leq L \leq Q$  with  $L \in \mathcal{N}_{Q, \Omega}$ . Suppose that  $\bigcap_{h \in H} h^{-1}Uh \leq H$  and  $\bigcap_{h \in H} h^{-1}Vh \leq L$ . Let  $[H : U] \leq n^{m-1}$ . Then*

$$\mu(U, V) \geq \mu(U \cap H, V) \geq \frac{\mu(H, L)}{m}.$$

**Proof.** First we prove that  $\mu(U, V) \geq \mu(U \cap H, V)$ . To do this, consider any edge  $\{i, i+1\} \in E(U \cap H) \cap E(V)$ . Then  $(U \cap H) \upharpoonright_{\{2i, 2i+1\}} = \text{Diag}(G^{\{2i, 2i+1\}})$ , so that  $U \upharpoonright_{\{2i, 2i+1\}} \geq \text{Diag}(G^{\{2i, 2i+1\}})$ .

Assume for sake of contradiction that  $U \upharpoonright_{\{2i, 2i+1\}} > \text{Diag}(G^{\{2i, 2i+1\}})$ , so that  $U \upharpoonright_{\{2i, 2i+1\}} = G^{\{2i, 2i+1\}}$  by Corollary 15.

In the first case  $H \upharpoonright_{\{2i, 2i+1\}} \leq \text{Diag}(G^{\{2i, 2i+1\}})$ . Then, since  $(\bigcap_{h \in H} h^{-1}Uh) \upharpoonright_{\{2i, 2i+1\}} = G^{\{2i, 2i+1\}} > H \upharpoonright_{\{2i, 2i+1\}}$ , we have  $\bigcap_{h \in H} h^{-1}Uh \not\leq H$ . This is a contradiction. Otherwise, we have  $H \upharpoonright_{\{2i, 2i+1\}} = G^{\{2i, 2i+1\}}$ . Then,  $(U \cap H) \upharpoonright_{\{2i, 2i+1\}} = G^{\{2i, 2i+1\}}$ , giving a contradiction with  $(U \cap H) \upharpoonright_{\{2i, 2i+1\}} = \text{Diag}(G^{\{2i, 2i+1\}})$ .

Therefore, it must be that  $\{i, i+1\} \in E(U) \cap E(V)$ , proving that  $E(U \cap H) \cap E(V) \subseteq E(U) \cap E(V)$ . This implies that  $\mu(U, V) \geq \mu(U \cap H, V)$ .

Now we prove the second statement  $\mu(U \cap H, V) \geq \frac{\mu(H, L)}{m}$ . Let  $\{\{i, i+1\} : i \in R\}$  be the edge set of any connected component of the graph  $([k], E(H) \cap E(L))$ , where  $R \subseteq [k-1]$ . Without loss of generality assume that  $R = \{1, \dots, r\}$ , where  $r = |R|$ . First, note that

$$[H \upharpoonright_{\bigcup_{i \in R} \{2i, 2i+1\}} : (H \cap U) \upharpoonright_{\bigcup_{i \in R} \{2i, 2i+1\}}] \leq [H : H \cap U] \leq n^{m-1}. \quad (2)$$

This follows from the standard group theory fact that if  $A \leq B$  and  $D$  is any other group, then  $[B \cap D : A \cap D] \leq [B : A]$ , and that the restriction  $\upharpoonright$  is an intersection with a subgroup of  $Q$ .

Then  $\varphi : H \upharpoonright_{\bigcup_{i \in R} \{2i, 2i+1\}} \rightarrow G^r$  is a surjective homomorphism given by

$$\varphi(g_2, \dots, g_{2r+1}) = (g_2, g_4, \dots, g_{2r})$$

Let  $T$  be a support for  $\varphi((U \cap H) \upharpoonright_{\bigcup_{i \in R} \{2i, 2i+1\}})$  such that  $|T| \leq m-1$ . Existence of such a  $T$  is guaranteed by Corollary 19, surjectivity of  $\varphi$ , and (2).

For any  $i \in R \setminus T$ , we have that  $(U \cap H) \upharpoonright_{\{2i, 2i+1\}} = \text{Diag}(G^{\{2i, 2i+1\}})$ . Therefore, for every  $i \in R \setminus T$ , the edge  $\{i, i+1\}$  is present in  $E(U \cap H)$ .

We claim that  $\{i, i+1\}$  is an edge in  $E(V)$  as well. Assume otherwise. Then since  $V \geq U \geq U \cap H$ , we have that  $V \upharpoonright_{\{2i, 2i+1\}} > \text{Diag}(G^{\{2i, 2i+1\}})$ . By Corollary 15,  $V \upharpoonright_{\{2i, 2i+1\}} = G^{\{2i, 2i+1\}}$ . But since  $i$  is an element of  $R$ , we have that  $\{i, i+1\}$  is an element of  $E(H) \cap E(L) \subseteq E(L)$ , so that  $L \upharpoonright_{\{2i, 2i+1\}} = \text{Diag}(G^{\{2i, 2i+1\}})$ . Now it is impossible to have  $\bigcap_{h \in H} h^{-1}Vh \leq L$ . We have our contradiction.

This shows that we obtain the graph  $([k], E(U \cap H) \cap E(V))$  from the graph  $([k], E(H) \cap E(L))$  by removing  $|T| \leq m-1$  edges. For any connected component in  $([k], E(H) \cap E(L))$  with  $t$  vertices, it must be that there exists a connected component in  $([k], E(U \cap H) \cap E(V))$  with at least  $\frac{t}{m}$  vertices, since  $([k], E(H) \cap E(L))$  is a subgraph of a path graph. ◀

We have now achieved our goal of lower bounding  $\beta_{Q, \Omega, d}(G^{k-1}, G^{k-1})$ .

► **Theorem 23.**  $\beta_{Q,\Omega,d}(G^{k-1}, G^{k-1}) \geq n^{d(k^{1/d}-1)}$  and  $\beta_{Q,\Omega}(G^{k-1}, G^{k-1}) \geq n^{\log_2 k}$ .

**Proof.** By definition of the embedding  $G^{k-1} < Q$  in (1), we have  $\mu(G^{k-1}, G^{k-1}) = k$ , since  $E(G^{k-1}) = \{\{i, i+1\} : i \in [k-1]\}$ . This section shows that  $\mu$  satisfies the hypotheses of Lemma 11 with  $c = n$ , so simply apply that result. ◀

## 6.2 TC lower bounds for cyclic groups of prime order

In this section we prove Theorem 3 in the case where the group  $G$  is abelian using the framework set up in Section 5. Note that if a finite simple group is abelian, then it must be a cyclic group of prime order, denoted  $C_p$  for some prime  $p$ .

We frequently work with  $C_p^k$  as the additive group of the vector space  $\mathbb{F}_p^k$ . We fix the following notation. Given a vector  $x = (x_1, \dots, x_k) \in \mathbb{F}_p^k$ , define  $\text{Supp}(x) = \{i \in [k] : x_i \neq 0\}$ , and define the Hamming weight  $|x| = |\text{Supp}(x)|$ . For  $x_1, \dots, x_\ell \in \mathbb{F}_p^k$ , let  $\langle x_1, \dots, x_\ell \rangle$  be the subspace generated by  $x_1, \dots, x_\ell$ .

We want to use Lemma 10 to prove a lower bound on the formula size of  $C_p^{k-1}$ -invariant  $\text{TC}_d$  formulas computing the function  $\text{WORD}_{C_p,k}$ . As we did in Section 6.1, we prove a lower bound on  $\beta_{Q,\Omega,d}(C_p^{k-1}, C_p^{k-1})$  for some choice of  $Q$  using Lemma 11 by constructing a lower-bound witness  $\mu$  that satisfies the hypotheses of that theorem.

We choose  $Q = C_p^k$ , viewed as a supergroup  $C_p^{k-1} < Q \leq S_{kp^2}$ , which acts on  $\Omega = \overline{C_p^k}^k \subseteq \{0, 1\}^{kp^2}$  via

$$(x_1, \dots, x_k)(M_1, \dots, M_k) = (\overline{x_1}M_1, \dots, \overline{x_k}M_k)$$

for  $(x_1, \dots, x_k) \in C_p^k$ .

The embedding  $C_p^{k-1} < C_p^k = Q$  is given by

$$(x_1, \dots, x_{k-1}) \mapsto (x_1, x_2 \dots, x_{k-1}, \prod_{i \in [k-1]} x_i^{-1}). \quad (3)$$

Observe that the coordinate functions  $\chi_1, \dots, \chi_m : \{0, 1\}^{kp^2} \rightarrow \{0, 1\}$  are given by the entries of the permutation matrices  $\{M_1, \dots, M_k\}$ .

Another important point to note is that  $\mathcal{B}_{Q,\Omega}$  (and hence also  $\mathcal{N}_{Q,\Omega}$ ) consists of all subgroups of  $C_p^k$ , since each  $H \leq C_p^k$  is the  $Q$ -stabilizer of the Boolean function on  $\Omega = \overline{C_p^k}^k$  that evaluates to 1 on  $(M_1, \dots, M_k)$  if and only if  $(M_1, \dots, M_k) \in H$ , where we view each  $M_i$  as an element of  $C_p$ .

We now define the lower-bound witness  $\mu$  and prove that it satisfies the hypotheses of Lemma 11 with  $c = p$ . We regard subgroups as the additive groups of subspaces of  $\mathbb{F}_p^k$ .

► **Definition 24.** For  $H < \mathbb{F}_p^k$  and any  $K \geq H$ , define

$$\mu(H, K) = \max_{W > V : \dim(W) = \dim(V) + 1} \min_{v \in V^\perp \setminus W^\perp} |v|.$$

Define  $\mu(\mathbb{F}_p^k, \mathbb{F}_p^k) = 0$ . The value  $\mu(H, K)$  only depends on  $H$ . Thus, we will suppress notation and write  $\mu(H)$  to denote  $\mu(H, K)$  for any  $K \leq \mathbb{F}_p^k$ .

### 6.2.1 The properties of $\mu$

First note that Item iv is satisfied by  $\mu$  by definition. We check that  $\mu$  satisfies Item iii.

► **Lemma 25.** Let  $H = \text{Stab}_Q(\chi_i)$  for some  $i \in [kp^2]$ . Then  $\mu(H) \leq 1$ .

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**Proof.** Assume that  $H$  is the  $Q$ -stabilizer of a function  $f$  depending on a single  $M_{i^*}$  for some  $i^* \in [k]$ . Since  $f$  only depends on  $i^*$ , we have  $H \upharpoonright_{[k] \setminus \{i^*\}} = \mathbb{F}_p^{[k] \setminus \{i^*\}}$ . Therefore, any  $x \in H^\perp$  has its support  $\text{Supp}(x)$  contained in  $\{i^*\}$ .  $\blacktriangleleft$

The following lemmas prove that  $\mu$  satisfies Item ii with  $c = p$ .

► **Lemma 26.** *Let  $U, H \leq \mathbb{F}_p^k$  be such that  $\bigcap_{h \in H} h^{-1}Uh \leq H$ . Then  $\mu(U) \geq \mu(H \cap U)$ .*

**Proof.** Since  $\mathbb{F}_p^k$  is abelian as an additive group,  $U = \bigcap_{h \in H} h^{-1}Uh \leq H$ . Therefore,  $U = U \cap H$ .  $\blacktriangleleft$

► **Lemma 27** ([22], Lemma 3.5). *Let  $H < W \leq \mathbb{F}_p^k$  with  $\dim(W) = \dim(H) + 1$ . Let  $U \leq H$ . Then there exists  $T > U$  with  $\dim(T) = \dim(U) + 1$  and*

$$\min_{u \in U^\perp \setminus T^\perp} |u| \geq \frac{1}{\dim(H) - \dim(U) + 1} \min_{h \in H^\perp \setminus W^\perp} |h|.$$

Moreover,  $H + T = W$ .

Finally, we prove that  $\mu$  satisfies Item i.

► **Lemma 28.** *Let  $H \leq \mathbb{F}_p^k$  and assume that  $H = H_1 \cap \dots \cap H_r$  where each  $H_i \leq \mathbb{F}_p^k$ . Then there is some  $i \in [r]$  such that  $\mu(H_i) \geq \mu(H)$ .*

**Proof.** We can assume that  $r = 2$ , since all other cases follow by induction. Let  $y \in H^\perp$  be the vector with  $|y| = \mu(H)$  and  $y = \arg \min_{x \in H^\perp \setminus W^\perp} |x|$ , where  $\dim(W) = \dim(H) + 1$ . Let  $H = H_1 \cap H_2$ . This means that  $H^\perp = H_1^\perp + H_2^\perp$ .

There exist  $y_1 \in H_1^\perp$  and  $y_2 \in H_2^\perp$  such that  $y = y_1 + y_2$ . First assume that  $y_1$  and  $y_2$  are linearly independent. There exists some  $a_1, a_2 \in \mathbb{F}_p$  such that  $a_1 y_1 + a_2 y_2 \in W^\perp$  and one of the  $a_i$  is nonzero. Assume otherwise. Then, for any such pair  $y_1, y_2$  we would have  $W^\perp \cap \langle y_1, y_2 \rangle = \{0\}$ , and

$$\begin{aligned} \dim(H^\perp) &= \dim(W^\perp) + \dim(\langle y_1, y_2 \rangle) - \dim(W^\perp \cap \langle y_1, y_2 \rangle) \\ &= \dim(W^\perp) + 2. \end{aligned}$$

This contradicts that  $\dim(W) = \dim(H) + 1$ , so assume that  $a_1 y_1 + a_2 y_2$  is an element of  $W^\perp$ . Assume that  $a_1 \neq 0$ . The case where  $a_2 \neq 0$  follows similarly. Then  $y_2$  cannot be an element of  $W^\perp$ , since otherwise  $y = y_1 + y_2 = \frac{1}{a_1}(a_1 y_1 + a_2 y_2) - (\frac{a_2}{a_1} + 1)y_2$ , implying that  $y \in W^\perp$ . As a result,  $\mu(H) \geq |y_2|$ . Since  $|y|$  is minimal in  $H^\perp \setminus W^\perp$ , we have  $|y_2| \geq |y|$ . We also have  $|y_2| \leq |y|$  since otherwise  $\mu(H) \geq |y_2| > |y|$ .

Now consider the space  $W + H_2$ . We have that  $H \leq W \cap H_2$  so  $\dim(W \cap H_2) \geq \dim(H)$ . Thus,

$$\begin{aligned} \dim(W + H_2) &= \dim(W) + \dim(H_2) - \dim(W \cap H_2) \\ &\leq \dim(H) + 1 + \dim(H_2) - \dim(H) \\ &= \dim(H_2) + 1. \end{aligned}$$

If  $\dim(W + H_2) = \dim(H_2)$ , then we would have  $W = H_2$ . Then  $y_2$  would be an element of  $W^\perp$ , a contradiction. Therefore,  $\dim(W + H_2) = \dim(H_2) + 1$ . Finally, we have that  $y_2$  is an element of  $H_2^\perp \setminus (W + H_2)^\perp$ , and we claim that  $|y_2|$  is minimal among all  $v \in H_2^\perp \setminus (W + H_2)^\perp$ .

Assume otherwise that there exists another  $v \in H_2^\perp \setminus (W + H_2)^\perp$  with  $|v| < |y_2|$ . Since  $H_2^\perp \leq H^\perp$ ,  $v$  is an element of  $H^\perp$ . Then it must be that  $v$  is an element of  $W^\perp$ , since otherwise we would contradict minimality of  $|y|$  in  $H^\perp \setminus W^\perp$ . But now  $v$  is an element of  $W^\perp \cap H_2^\perp = v \in (W + H_2)^\perp$ , a contradiction. Therefore,  $y_2 = \arg \min_{x \in H_2^\perp \setminus (W + H_2)^\perp} |x|$ . We have proved in this case that  $\mu(H_2) \geq |y_2| \geq |y| = \mu(H)$ .

Finally, if  $y_1$  and  $y_2$  are not linearly independent, then at least one is a nonzero scalar multiple of  $y$ . Assume it is  $y_2$ . By the same argument,  $y_2 = \arg \min_{x \in H_2^\perp \setminus (W+H_2)^\perp} |x|$ , and we have  $\mu(H_2) \geq |y_2| = |y| = \mu(H)$ . ◀

Having proved that  $\mu$  satisfies all the hypotheses of Lemma 11, we have the following result.

► **Theorem 29.**  $\beta_{Q,\Omega,d}(C_p^{k-1}, C_p^{k-1}) \geq p^{d(k^{1/d}-1)}$  and  $\beta_{Q,\Omega}(C_p^{k-1}, C_p^{k-1}) \geq p^{\log_2 k}$ .

**Proof.** By the embedding (3),  $C_p^{k-1}$  is regarded as the additive group of the subspace  $V = \{(x_1, \dots, x_k) \in \mathbb{F}_p^k : \sum_{i \in [k]} x_i = 0\}$ . Thus,  $\mu(C_p^{k-1}, C_p^{k-1}) = k$  as witnessed by  $\min_{v \in V \setminus \{0\}} |v| = k$ . This section shows that  $\mu$  satisfies the hypotheses of Lemma 11 with  $c = p$ , so we have our result. ◀

### 6.3 Proof of Theorem 3

Having covered both nonabelian and abelian simple groups, we can now prove our main theorem, which we restate here.

► **Theorem 3.** *Let  $G$  be a finite simple group and suppose that  $G \leq S_n$  is a faithful permutation representation of minimum degree. Then*

$$\mathcal{L}_{\text{TC}}^{G^{k-1}}(\text{WORD}_{G,k}) \geq n^{\log_2 k}, \quad \mathcal{L}_{\text{TC}_d}^{G^{k-1}}(\text{WORD}_{G,k}) \geq n^{d(k^{1/d}-1)}.$$

**Proof.** In the first case  $G$  is nonabelian. In this case Theorem 23 shows that for the choice  $Q = \{(g_1, \dots, g_{2k}) \in G^{2k} : g-1 = g_{2k} = 1_G\}$  with the embedding  $G^{k-1} \rightarrow Q$  given by (1), we have  $\beta_{Q,\Omega,d}(G^{k-1}, G^{k-1}) \geq n^{d(k^{1/d}-1)}$  and  $\beta_{Q,\Omega}(G^{k-1}, G^{k-1}) \geq n^{\log_2 k}$ .

Any  $\Phi \in \text{TC}_d$  such that  $\llbracket \Phi \rrbracket = \text{WORD}_{G,k}$  must have  $\text{Stab}_Q(\llbracket \Phi \rrbracket) = \text{Stab}_Q(\text{WORD}_{G,k}) = G^{k-1}$ . If  $\text{Stab}_Q(\Phi) \geq G^{k-1}$ , then we must have  $\text{Stab}_Q(\Phi) = G^{k-1}$ , since  $\text{Stab}_Q(\Phi) \leq \text{Stab}_Q(\llbracket \Phi \rrbracket)$ . Therefore,  $\text{size}(\Phi) \geq \beta_{Q,\Omega,d}(G^{k-1}, G^{k-1})$  by Lemma 10. Similarly argue for unbounded-depth formulas, and we have

$$\begin{aligned} \mathcal{L}_{\text{TC}_d}^{G^{k-1}}(\text{WORD}_{G,k}) &\geq \beta_{Q,\Omega,d}(G^{k-1}, G^{k-1}) \geq n^{d(k^{1/d}-1)}, \\ \mathcal{L}_{\text{TC}}^{G^{k-1}}(\text{WORD}_{G,k}) &\geq \beta_{Q,\Omega}(G^{k-1}, G^{k-1}) \geq n^{\log_2 k}. \end{aligned}$$

Otherwise,  $G$  is abelian, and is therefore  $C_p$  for some prime  $p$ . Note that then  $C_p \rightarrow S_p$  is the minimum degree faithful permutation representation. For the choice  $Q = C_p^k$  and embedding  $C_p^{k-1} \rightarrow C_p^k$  given by (3),  $\beta_{Q,\Omega,d}(C_p^{k-1}, C_p^{k-1}) \geq n^{d(k^{1/d}-1)} = p^{d(k^{1/d}-1)}$  and  $\beta_{Q,\Omega}(C_p^{k-1}, C_p^{k-1}) \geq n^{\log_2 k} = p^{\log_2(k)}$  by Theorem 29.

Any  $\Phi \in \text{TC}_d$  such that  $\llbracket \Phi \rrbracket = \text{WORD}_{C_p,k}$  must have  $\text{Stab}_Q(\llbracket \Phi \rrbracket) = \text{Stab}_Q(\text{WORD}_{C_p,k}) = C_p^{k-1}$ . If  $\text{Stab}_Q(\Phi) \geq C_p^{k-1}$ , then we must have  $\text{Stab}_Q(\Phi) = C_p^{k-1}$ , since  $\text{Stab}_Q(\Phi) \leq \text{Stab}_Q(\llbracket \Phi \rrbracket)$ . Therefore,  $\text{size}(\Phi) \geq \beta_{Q,\Omega,d}(C_p^{k-1}, C_p^{k-1})$  by Lemma 10. Similarly argue for unbounded-depth formulas, and we have

$$\begin{aligned} \mathcal{L}_{\text{TC}_d}^{C_p^{k-1}}(\text{WORD}_{C_p,k}) &\geq \beta_{Q,\Omega,d}(C_p^{k-1}, C_p^{k-1}) \geq p^{d(k^{1/d}-1)}, \\ \mathcal{L}_{\text{TC}}^{C_p^{k-1}}(\text{WORD}_{C_p,k}) &\geq \beta_{Q,\Omega}(C_p^{k-1}, C_p^{k-1}) \geq p^{\log_2(k)}. \end{aligned}$$

This completes the proof. ◀

## 7 Future directions

In this paper we have proved tight bounds on the size of invariant TC formulas for the word problem on all finite simple groups (Theorem 3). We also prove a tight lower bound on the size of invariant AC formulas for the word problem of cyclic groups of prime power order (Theorem 4). One direction for future work would be to improve this AC lower bound to a TC lower bound.

► **Question 30.** *Do we have  $\mathcal{L}_{\text{TC}_d}^{C_q^{k-1}}(\text{WORD}_{C_q,k}) \geq q^{d(k^{1/d}-1)}$ ?*

Another direction for progress is characterizing the invariant TC or AC formula size for all finite groups. With Corollary 5 we have answered the following question is true for simple groups and abelian groups.

► **Question 31.** *What is  $\mathcal{L}_{\text{AC}_d}^{G^{k-1}}(\text{WORD}_{G,k})$  for a finite group  $G$ ?*

Finally, it is natural to ask whether these invariant formulas are optimal even without the restriction of invariance. Currently Corollary 8 gives the best known upper bounds on the formula size for  $\text{WORD}_{S_n,k}$ .

► **Question 32.** *Do we have  $\mathcal{L}_{\text{TC}_d}(\text{WORD}_{S_n,k}) \geq n^{\Omega(d(k^{1/d}-1))}$ ?*

An affirmative answer to this question, even for  $k = \log(n)$ , would prove that  $\text{Logspace} \neq \text{NC}^1$ .

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