Geometric Embeddability of Complexes Is $\exists\mathbb{R}$-Complete

Mikkel Abrahamsen
University of Copenhagen, Denmark

Linda Kleist
Technische Universität Braunschweig, Germany

Tillmann Miltzow
Utrecht University, The Netherlands

Abstract

We show that the decision problem of determining whether a given (abstract simplicial) k-complex has a geometric embedding in $\mathbb{R}^d$ is complete for the Existential Theory of the Reals for all $d \geq 3$ and $k \in \{d-1, d\}$. Consequently, the problem is polynomial time equivalent to determining whether a polynomial equation system has a real solution and other important problems from various fields related to packing, Nash equilibria, minimum convex covers, the Art Gallery Problem, continuous constraint satisfaction problems, and training neural networks. Moreover, this implies NP-hardness and constitutes the first hardness result for the algorithmic problem of geometric embedding (abstract simplicial) complexes. This complements recent breakthroughs for the computational complexity of piece-wise linear embeddability.

2012 ACM Subject Classification Theory of computation → Computational geometry

Keywords and phrases simplicial complex, geometric embedding, linear embedding, hypergraph, recognition, existential theory of the reals

Digital Object Identifier 10.4230/LIPIcs.SoCG.2023.1

Related Version *Full details can be found in our ArXiv paper*: `https://arxiv.org/abs/2108.02585`

Funding *Mikkel Abrahamsen*: Supported by Starting Grant 1054-00032B from the Independent Research Fund Denmark under the Sapere Aude research career programme and part of Basic Algorithms Research Copenhagen (BARC), supported by the VILLUM Foundation grant 16582.
Linda Kleist: Partially supported by a postdoc fellowship of the German Academic Exchange Service (DAAD).
Tillmann Miltzow: Generously supported by the Netherlands Organisation for Scientific Research (NWO) under project no. 016.Veni.192.250.

Acknowledgements We thank Arkadiy Skopenkov for his kind and swift help with acquiring literature and Martin Tancer for pointing out a mistake in a previous version of this manuscript.

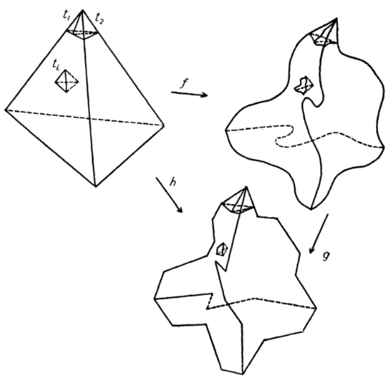

Figure 1 Illustration of different embeddings of a complex; figure taken from Bing [8, Annals of Mathematics 1959].

39th International Symposium on Computational Geometry (SoCG 2023).
Editors: Erin W. Chambers and Joachim Gudmundsson; Article No. 1; pp. 1:1–1:19
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

1 Introduction

For now almost 100 years, much attention has been devoted to studying embeddings of complexes [8, 21, 30, 31, 42, 54, 66, 67]. Typical types of embeddings include geometric (also referred to as linear), piecewise linear (PL), and topological embeddings, see also Figure 1. For formal definitions, we refer to Section 1.2; here we give an illustrative example. Embeddings of a 1-complex in the plane correspond to crossing-free drawings of a graph in the plane. In a topological embedding, each edge is represented by a Jordan arc, in a PL embedding it is a concatenation of a finite number of segments, and in a geometric embedding each edge is represented by a segment.

We are interested in the problem of deciding whether a given k-complex has a linear/-piecewise linear/topological embedding in $\mathbb{R}^d$. Several necessary and sufficient conditions are easy to identify and have been known for many decades. For instance, a k-simplex requires $k+1$ points in general position in $\mathbb{R}^d$ and, thus, $k \leq d$ is an obvious necessary condition. Moreover, it is straight-forward to verify that every set of n points in $\mathbb{R}^3$ in general position allows for a geometric embedding of any 1-complex on n vertices, i.e., the points are the vertices of a straight-line drawing of a (complete) graph. Indeed, this fact generalizes to higher dimensions: every k-complex embeds (even linearly) in $\mathbb{R}^{2k+1}$ [42]. Van Kampen and Flores [25, 57, 66] showed that this bound is tight by providing k-complexes that do not topologically embed into $\mathbb{R}^{2k}$. For some time, it was believed that the existence of a topological embedding also implies the existence of a geometric embedding, e.g., Grünbaum conjectured that if a k-complex topologically embeds in $\mathbb{R}^{2k}$, then it also geometrically embeds in $\mathbb{R}^{2k}$ [30]. In $\mathbb{R}^2$, this is in fact true: For 1-complexes this is commonly known as Fáry's theorem [35] but it also follows from Steinitz' earlier theorem [62]; for 2-complexes one needs a few additional arguments [32]. In higher dimensions, however, the conjecture was disproven. In particular, for every $k, d \geq 2$ with $k+1 \leq d \leq 2k$, there exist k-complexes that have a PL embedding in $\mathbb{R}^d$, but no geometric embedding in $\mathbb{R}^d$ [9, 10, 11]. In contrast, PL and topological embeddability coincides in many cases, e.g., if $d \leq 3$ [8, 48] or $d-k \geq 3$ [12]. Very recently, Frick, Hu, Scheel, and Simon [27] characterized when a complex on $d+3$ vertices embeds into the d-sphere, namely, if and only if its non-faces do not form an intersecting family. Additionally, they showed that if a complex on $d+3$ vertices embeds topologically into $\mathbb{R}^d$ then it also embeds linearly into $\mathbb{R}^d$. There are many further necessary and sufficient conditions known for geometric embeddings [6, 46, 47, 57, 63, 64] and PL or/and topological embeddings [20, 26, 49, 54, 65, 61].

In recent years, the **algorithmic complexity** of deciding whether or not a given complex is embeddable gained attention. In the absence of a complete characterization, an efficient algorithm is the best tool to decide embeddability. For instance, deciding whether a 1-complex embeds in the plane corresponds to testing graph planarity and is thus polynomial time decidable [33]. Similarly, Gross and Rosen [29] present a linear time planarity algorithm for 2-complexes in the plane. On the other hand, PL embeddability is sometimes even algorithmically undecidable. To give a concrete example, let $\text{E}\textsc{mbed}_{k\to d}$ denote the algorithmic problem of determining whether a given k-complex has a PL embedding in $\mathbb{R}^d$. Because $\text{E}\textsc{mbed}_{4\to 5}$ has been shown to be algorithmically undecidable [40], there is no algorithm to decide the problem (never mind an efficient one). This provides strong evidence that PL embeddability for these parameters does not allow a reasonable characterization.

More recently, there have been several breakthroughs concerning the **PL embeddability**. For an overview of the state of the art, consider Table 1. In dimensions $d \geq 4$, the decision problem $\text{E}\textsc{mbed}_{k\to d}$ is polynomial-time decidable for $k < 2/3 \cdot (d-1)$ [16, 13, 15, 36] and

Table 1 Overview of the complexity of $\textsc{Embed}_{k\to d}$.

k \ d	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	P	P	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
2	✗	P	D	?	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
3	✗	✗	D	?	?	P	✓	✓	✓	✓	✓	✓	✓	✓
4	✗	✗	✗	?	U	?	?	P	✓	✓	✓	✓	✓	✓
5	✗	✗	✗	✗	U	U	?	?	P	P	✓	✓	✓	✓
6	✗	✗	✗	✗	✗	U	U	?	?	?	P	P	✓	✓

✓ always yes
✗ always no
P polynomial-time
D decidable
U undecidable
NP-hard

NP-hard for all remaining non-trivial cases [40], i.e., for all k with $2/3 \cdot (d-1) \le k \le 2d$. For $d \ge 5$ and $k \in \{d-1, d\}$, $\textsc{Embed}_{k\to d}$ is even known to be undecidable [40]. For all other NP-hard cases and $d \ge 4$ decidability is unknown; we note that the proof for undecidability in the case of codimension > 1 in [24] has an error [58]. For the case $d = 3$, Matoušek, Sedgwick, Tancer, and Wagner proved decidability of $\textsc{Embed}_{2\to 3}$ and $\textsc{Embed}_{3\to 3}$ [39] and de Mesmay, Rieck, Sedgwick, and Tancer proved NP-hardness [43].

Building upon [40], Skopenkov and Tancer [60] proved NP-hardness for a relaxed notion called *almost (PL/topological) embeddability* where it is only required that disjoint sets are mapped to disjoint objects, i.e., two edges incident to a common vertex may cross in an interior point. More precisely, they showed that recognizing almost embeddability of k-complexes in $\mathbb{R}^d$ is NP-hard for all $d, k \ge 2$ with $d \pmod 3 = 1$ and $2/3 \cdot (d-1) \le k \le d$.

The analogous questions for **geometric embeddings** are wide open. Let $\text{GEM}_{k\to d}$ denote the algorithmic problem of determining whether a given k-complex has a geometric embedding in $\mathbb{R}^d$. In contrast to PL embeddability, however, it is easy to see that $\text{GEM}_{k\to d}$ is decidable for all k, d, since every instance can be expressed as a sentence in the first order theory of the reals, which is decidable; for more details see Section 1.1.

The question of whether $\text{GEM}_{k\to d}$ is complete for $\exists\mathbb{R}$ is a well-known open problem, mentioned for example by Cardinal [18, Section 4].

Our Results. In this work, we present the first results concerning open problem for any non-trivial entry with $d \ge 3$. More precisely, we establish the exact computational complexity of $\text{GEM}_{k\to d}$ for all values $d \ge 3$ and $k \in \{d-1, d\}$. This includes a complete understanding of the most intriguing entries with $d = 3$.

▶ **Theorem 1.** *For every $d \ge 3$ and each $k \in \{d-1, d\}$, the decision problem $\text{GEM}_{k\to d}$ is $\exists\mathbb{R}$-complete. Moreover, the statement remains true even if a PL embedding is given.*

Table 2 summarizes the current knowledge on the computational complexity of $\text{GEM}_{k\to d}$. Our proof implies that distinguishing between k-complexes with PL and geometric embeddings in $\mathbb{R}^d$ is complete for $\exists\mathbb{R}$. Because $\text{NP} \subseteq \exists\mathbb{R}$, our result yields NP-hardness for $d \ge 3$ and each $k \in \{d-1, d\}$. This confirms the conjecture by Skopenkov that $\text{GEM}_{k\to d}$ is NP-hard for all k, d with $\max\{3, k\} \le d \le 3/2 \cdot k + 1$ for the corresponding values of k and d [59, Conjecture 3.2.2]. Moreover, if $\text{NP} \neq \exists\mathbb{R}$, the problem $\text{GEM}_{k\to d}$ cannot be tackled with well developed tools for NP-complete problems such as SAT and ILP solvers. For more details, we refer to Section 1.1.

Table 2 Overview of the computational complexity of $\mathrm{GEM}_{k\to d}$.

k \ d	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	P	P	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
2	✗	P	∃ℝc	?	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
3	✗	✗	∃ℝc	∃ℝc	?	?	✓	✓	✓	✓	✓	✓	✓	✓
4	✗	✗	✗	∃ℝc	∃ℝc	?	?	?	✓	✓	✓	✓	✓	✓
5	✗	✗	✗	✗	∃ℝc	∃ℝc	?	?	?	?	✓	✓	✓	✓
6	✗	✗	✗	✗	✗	∃ℝc	∃ℝc	?	?	?	?	?	✓	✓

- ✓ always yes
- ✗ always no
- P polynomial-time
- ∃ℝc ∃ℝ-complete

The closely related question of polyhedral complexes (generalizing simplicial complexes because each simplex is a basic polyhedron), posed in the Handbook of Discrete and Computational Geometry, reads as follows: When is a given finite poset isomorphic to the face poset of some polyhedral complex in a given space $\mathbb{R}^d$? [53, Problem 20.1.1]. The recognition of polyhedral complexes (with triangles and quadrangles) in $\mathbb{R}^3$ has been claimed to be ∃ℝ-complete [18, Theorem 5]. Focussing on convex polytopes, Richter-Gebert proved that recognizing convex polytopes in $\mathbb{R}^4$ is ∃ℝ-complete [50, 51]. Our result settles the computational aspects of the question, even for the special case of simplicial complexes.

A geometric embedding of a complex can also be viewed as a *simplicial representation* of a hypergraph, i.e., a representation in which every hyperedge is represented by a simplex. Of particular interest is the case of uniform hypergraphs where all hyperedges have the same number of elements. Thus, in the language of hypergraphs, our result reads as follows.

▶ **Corollary 2.** *For all $d \geq 3$ and every $k \in \{d-1, d\}$, deciding whether a $(k+1)$-uniform hypergraph has a simplicial representation in $\mathbb{R}^d$ is ∃ℝ-complete.*

Outline and techniques. Our proof of Theorem 1 consists of three steps: Establishing ∃ℝ-membership, showing ∃ℝ-hardness in $\mathbb{R}^3$, i.e., of $\mathrm{GEM}_{2\to 3}$ and $\mathrm{GEM}_{3\to 3}$, and reducing $\mathrm{GEM}_{k\to d}$ to $\mathrm{GEM}_{k+1\to d+1}$. The core of the proof lies in establishing hardness of $\mathrm{GEM}_{2\to 3}$.

The main idea to prove hardness of $\mathrm{GEM}_{2\to 3}$ is to reduce from the problem STRETCHABILITY. In STRETCHABILITY, we are given an arrangement of pseudolines (curves) in the plane and we are asked to decide whether there exists a set of straight lines that has the same combinatorial pattern as the pseudoline arrangement, see Figure 2(a) for an illustration and Section 1.2 for a formal definition. Given a pseudoline arrangement L, we construct a 2-complex C which has a geometric embedding in $\mathbb{R}^3$ if and only if L is stretchable. On a high level, our construction of C goes along the following lines: We add a helper triangle that contains all intersections of the pseudolines, see Figure 2(b). We place each pseudoline in $\mathbb{R}^3$ and replace it by a *special* edge of the complex C; these will not be part of any triangle of C. We surround the special edges by so called *tunnels*, which are tubes formed by triangular sections, see Figure 2(c) and (d). One side of the tunnel defines its *bottom*, while the other two span its *roof*. For each crossing in L, we glue the corresponding tunnel sections together, see Figure 2(e). At last, we insert an apex u high above that is connected to all visible tunnel parts, see Figure 2(f) and we insert additional objects in order to ensure that the neighborhood of u is an essentially 3-connected graph, Figure 2(i). The objects incident to the apex will also ensure that the special edges actually lie inside the tunnel.

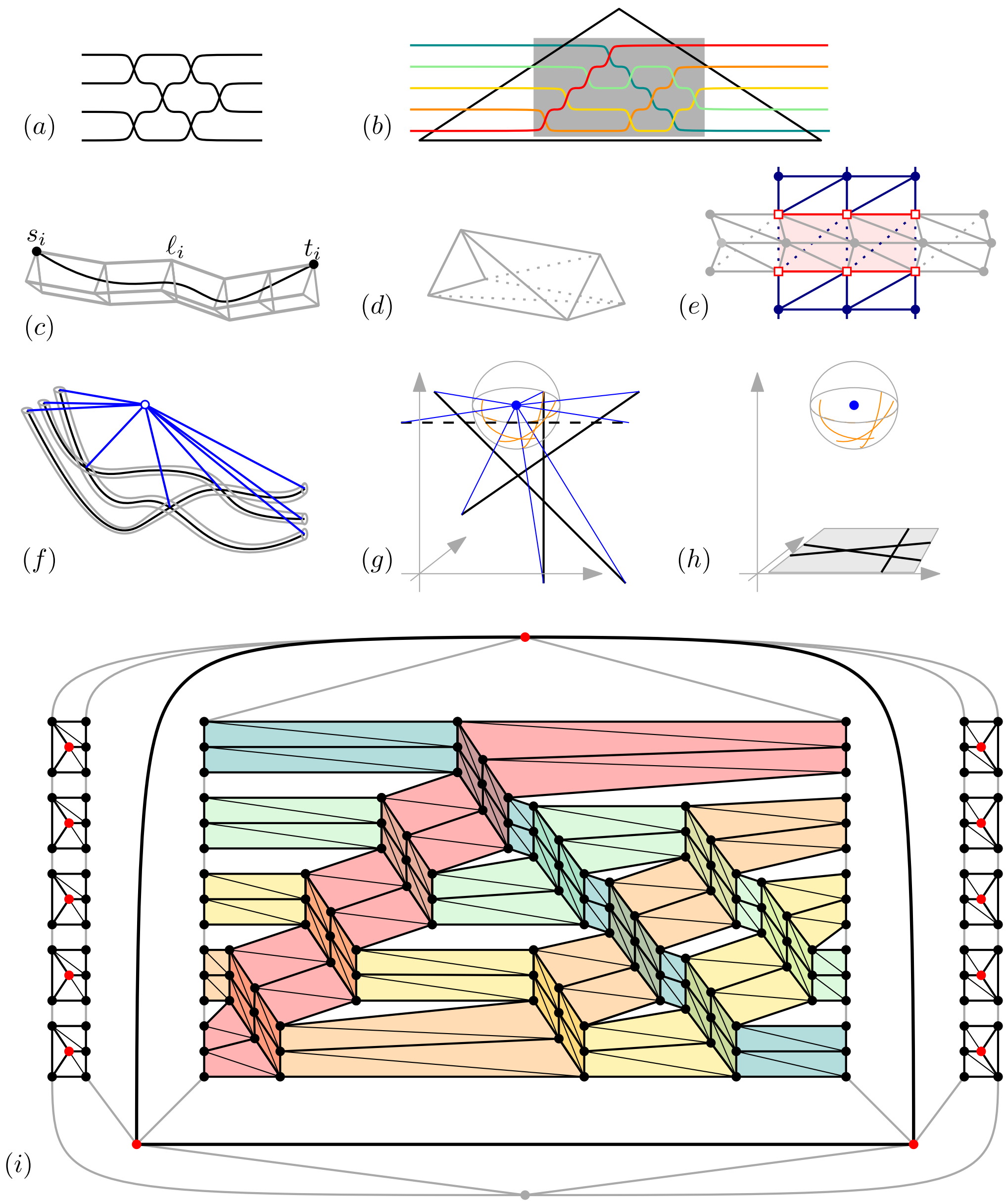

Figure 2 (a) We start with a pseudoline arrangement L. (b) We add three segments forming a triangle that contains all intersections of L. (c) Each pseudoline is represented by a special edge that is surrounded by a tunnel. (d) Each tunnel consists of tunnel sections. (e) For the crossings of the special edges, we identify parts of the tunnels. (f) We add an apex u and insert triangles to the visible parts of the construction; we enhance the neighborhood of the apex to an essentially 3-connected graph depicted in (i). (g) In the correctness proof, we use a small sphere around the apex and the projection of each special edge onto the sphere. (h) We argue that the combinatorics of the projected special edges on the sphere are equivalent to L and then project the special edges onto a plane. This will yield a stretched arrangement. (i) The neighborhood graph of the apex u.

It is relatively straightforward to verify that if L is stretchable, then the complex C embeds geometrically into $\mathbb{R}^3$. The other direction requires more care and work: We show that a geometric embedding of C induces a line arrangement with the same combinatorics as L. The idea of the proof is to consider a small sphere around the apex u and to project its neighborhood and the special edges onto the sphere, see Figure 2(g). Because the neighborhood graph of u is essentially 3-connected by construction, all its crossing-free drawings on the sphere are equivalent. This is a crucial property to show that each special edge lies in the projection of its tunnel roof (when restricting the attention to an interesting part within the helper triangle). We remark that our proof does not show this explicitly. Instead, we establish some even stronger properties. As a consequence, the projection of the tunnels have the intended combinatorics and thus also the special edges which represent the pseudolines. At last, we project the arcs from the sphere onto a plane, see Figure 2(h). In this way, we obtain a line arrangement with the same combinatorics as L.

In order to show hardness of $\text{GEM}_{3\to 3}$, we use a similar construction, in which we "fatten" each triangle to a tetrahedron, by adding extra vertices.

We finally present a dimension reduction, i.e., we reduce $\text{GEM}_{k\to d}$ to $\text{GEM}_{k+1\to d+1}$. Given a k-complex C, we create a $(k+1)$-complex C^+ that contains C and has two additional vertices a and b. Moreover, for each subset e of C, C^+ has the additional subsets $e \cup \{a\}$ and $e \cup \{b\}$. We prove that C geometrically embeds in $\mathbb{R}^d$ if and only if C^+ geometrically embeds in $\mathbb{R}^{d+1}$. In this way, we show that distinguishing PL embeddable and geometrically embeddable complexes is ∃ℝ-complete.

1.1 Existential Theory of the Reals

The class of the existential theory of the reals ∃ℝ (pronounced as is a complexity class which has gained a lot of interest in recent years, specifically in the computational geometry community. To define this class, we first consider the algorithmic problem *Existential Theory of the Reals (ETR)*. An instance of this problem consists of a sentence of the form

$$\exists x_1, \ldots, x_n \in \mathbb{R} : \Phi(x_1, \ldots, x_n),$$

where Φ is a well-formed quantifier-free formula in the variables and the alphabet $\{0, 1, +, \cdot, \geq, >, \wedge, \vee, \neg\}$, and the goal is to check whether this sentence is true. As an example of an ETR-instance, consider $\exists x, y \in \mathbb{R} \colon \Phi(x, y) = (x \cdot y^2 + x \geq 0) \wedge \neg(y < x)$, for which the goal is to determine whether there exist real numbers x and y satisfying the formula $\Phi(x, y)$.

The *complexity class* ∃ℝ is the family of all problems that admit a polynomial-time many-one reduction to ETR. It is known that NP $\subseteq$ ∃ℝ $\subseteq$ PSPACE. The first inclusion follows from the definition of ∃ℝ. Showing the second inclusion was first established by Canny in his seminal paper [17]. The complexity class ∃ℝ gains its significance because a number of well-studied problems from different areas of theoretical computer science have been shown to be complete for this class.

Famous examples from discrete geometry are the recognition of geometric structures, such as unit disk graphs [41], segment intersection graphs [38], Stretchability [45, 56], and order type realizability [38]. Other ∃ℝ-complete problems are related to graph drawing [37], Nash-Equilibria [7, 28], geometric packing [5], the art gallery problem [3], non-negative matrix factorization [55], polytopes [22, 51], geometric linkage constructions [1], training neural networks [4], visibility graphs [19], continuous constraint satisfaction problems [44], and convex covers [2]. The fascination for the complexity class stems not merely from the number of ∃ℝ-complete problems but from the large scope of seemingly unrelated ∃ℝ-complete problems. We refer the reader to the lecture notes by Matoušek [38] and surveys by Schaefer [52] and Cardinal [18] for more information on the complexity class ∃ℝ.

1.2 Definitions

Simplex. A k-simplex σ is a k-dimensional polytope which is the convex hull of its $k+1$ vertices V, which are not contained in the same $(k-1)$-dimensional hyperplane. Hence, a 0-simplex corresponds to a point, a 1-simplex to a segment, and a 2-simplex to a triangle etc. The convex hull of any nonempty proper subset of V is called a *face* of σ. A *simplicial complex* K is a set of simplices satisfying the following two conditions: (i) Every face of a simplex from K is also in K. (ii) For any two simplices $\sigma_1, \sigma_2 \in K$ with a non-empty intersection, the intersection $\sigma_1 \cap \sigma_2$ is a face of both simplices σ_1 and σ_2. The purely combinatorial counterpart to a simplicial complex is an abstract simplicial complex, which we refer to simply as a *complex*.

Complex. A *complex* $C = (V, E)$ is a finite set V together with a collection of subsets $E \subseteq 2^V$ which is closed under taking subsets, i.e., $e \in E$ and $e' \subseteq e$ imply that $e' \in E$. A *k-complex* is a complex where the largest subset contains exactly $k+1$ elements. We call a complex *pure* if all (inclusion-wise) maximal elements in E have the same cardinality.

For any vertex $v \in V$ in a k-complex $C = (V, E)$, the neighbourhood of v gives rise to a lower dimensional complex $C_v := (V', E')$, where $E' := \{e \setminus \{v\} \mid v \in e \in E\}$ and $V' := N(v) = \bigcup_{e \in E'} e$ are the *neighbors* of v. Complexes are in close relation to Hypergraphs.

Hypergraphs. Hypergraphs generalize graphs by allowing edges to contain any number of vertices. Formally, a *hypergraph* H is a pair $H = (V, E)$ where V is a set of vertices, and E is a set of non-empty subsets of V called *hyperedges* (or edges). A k-uniform hypergraph is a hypergraph such that all its hyperedges contain exactly k elements. Note that the maximal sets of a pure k-complex yield a $(k+1)$-uniform hypergraph and vice versa. Hence, $(k+1)$-uniform hypergraphs and pure k-complexes are in a straightforward one-to-one correspondence. A *simplicial representation* of a $(k+1)$-uniform hypergraph is a geometric embedding of the corresponding complex.

Geometric embeddings. A *geometric embedding* of a complex $C = (V, E)$ in $\mathbb{R}^d$ is a function $\varphi \colon V \to \mathbb{R}^d$ fulfilling the following two properties: (i) for every $e \in E$, $\overline{\varphi}(e) := \operatorname{conv}(\{\varphi(v) \colon v \in e\})$ is a simplex of dimension $|e|-1$ and (ii) for every pair $e, e' \in E$, it holds that

$$\overline{\varphi}(e) \cap \overline{\varphi}(e') = \overline{\varphi}(e \cap e').$$

Note that if φ is a geometric embedding, then $\{\overline{\varphi}(e) : e \in E\}$ is a simplicial complex. The problem $\mathrm{GEM}_{k \to d}$ asks whether a given k-complex has a geometric embedding in $\mathbb{R}^d$.

Topological and PL embeddings. Consider a complex $C = (V, E)$. In contrast to geometric embeddings, for PL or topological embeddings it is not sufficient to describe the mapping of the vertices V. Choose d' so large that C admits a geometric embedding $\varphi' : V \to \mathbb{R}^{d'}$, and define $S = \bigcup_{e \in E} \overline{\varphi'}(e)$. We then say that an injective and continuous function $\varphi : S \to \mathbb{R}^d$ is a *topological embedding* of C in $\mathbb{R}^d$. If furthermore for each $e \in E$, the image $\varphi(\overline{\varphi'}(e))$ is a finite union of connected subsets of $(|e|-1)$-dimensional hyperplanes, then φ is a *piecewise linear (PL) embedding*. The problem $\textsc{Embed}_{k \to d}$ asks whether a given k-complex has a PL embedding in $\mathbb{R}^d$.

Graph Drawings. A graph is a 1-complex. A graph is *planar* if there exists a crossing-free drawing in the plane, i.e., a (topological) embedding in $\mathbb{R}^2$. As mentioned above, a graph has a topological embedding in $\mathbb{R}^2$ if and only if it has a geometric embedding in $\mathbb{R}^2$. A *plane* graph is a planar graph together with a *rotation system*, i.e., a cyclic ordering of the incident edges around each vertex that comes from a crossing-free drawing. By means of stereographic projection, any graph that has a crossing-free drawing in the plane also has a crossing-free drawing on the sphere and vice versa. Two crossing-free drawings of a graph (in the plane or on the sphere) are *equivalent* if they can be transformed into one another by a homeomorphism (of the plane or the sphere); note that the homeomorphism could be orientation reversing. In particular, two equivalent drawings have the same rotation system; two equivalent drawings in the plane additionally have the same outer face. When talking about an arbitrary drawing D of a plane graph G, we mean a crossing-free drawing with the same rotation system.

Stretchability. A *pseudoline arrangement* is a family of curves that apart from "straightness" share similar properties with a line arrangement. More formally, a *(Euclidean) pseudoline arrangement* is a set of labeled x-monotone curves in the Euclidean plane such that any two meet in exactly one point. A curve in $\mathbb{R}^2$ is *x-monotone* if it is the image of a continuous function $f\colon \mathbb{R} \to \mathbb{R}$. In fact, each pseudoline arrangement can be encoded by a *wiring diagram*; see also Figure 4. A pseudoline arrangement is *stretchable* if it is combinatorially equivalent to an arrangement of straight-lines, i.e., if the arrangements can be transformed into one another by a homeomorphism of the plane. STRETCHABILITY denotes the algorithmic problem of deciding whether a given pseudoline arrangement is stretchable. In a seminal paper, Shor [56] proved that STRETCHABILITY is NP-hard. Shor points out that Mnëv's proof implies that stretchability is complete for the existential theory of the reals. For a stream-line exposition of this result see the expository paper by Matoušek [38].

1.3 Pitfalls

While the general proof ideas are fairly straightforward, our arguments in Section 2 may at first glance appear a bit tedious. In the following, we highlight one of the appearing challenges. It is easy to see that each special edge lies inside its tunnel in any geometric embedding. It follows that the projection of the special edge lies also inside the projection of the tunnel on the sphere centered at the apex. Furthermore, we know that the roofs of the tunnels are seen by the apex. One may be tempted to (directly) conclude that the projection of the special edge is thus also contained in the projection of the roof; the underlying thought being that the projection of the tunnel bottom lies below the tunnel roof in the geometric representation and thus the projection of the tunnel bottom is contained in the projection of the tunnel roof. Yet, the latter is not true in general, as can be seen in Figure 3. In the

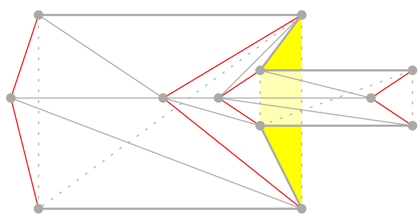

Figure 3 From the perspective of u, the tunnel bottom is not always hidden below the tunnel roof: From the three sections displayed, the bottom (yellow) of the middle one is partially visible.

figure, the tunnel bottom is not covered by the roof. We (implicitly) show that the projection of the special edge lies inside the projection of the roof by establishing some even stronger topological and geometric properties.

2 The Proof

In this section, we prove Theorem 1. Our proof consists of the following three parts.

a) Establishing $\exists\mathbb{R}$-membership (Section 2.1: Lemma 3).
b) Showing $\exists\mathbb{R}$-hardness in $\mathbb{R}^3$, i.e., of $\mathrm{GEM}_{2\to3}$ and $\mathrm{GEM}_{3\to3}$ (Section 2.2: Theorem 4 and Corollary 9).
c) Reducing $\mathrm{GEM}_{k\to d}$ to $\mathrm{GEM}_{k+1\to d+1}$ (Section 2.3: Lemma 10).

Together Lemmas 3 and 10, Theorem 4 and Corollary 9 prove Theorem 1.

2.1 Membership

In this subsection, we show $\exists\mathbb{R}$-membership of $\mathrm{GEM}_{k\to d}$. Note that this is essentially folklore [14]. We present a proof for the sake of completeness.

▶ **Lemma 3.** *For all $k, d \in \mathbb{N}$, the decision problem $\mathrm{GEM}_{k\to d}$ is contained in $\exists\mathbb{R}$.*

Proof. In order to show membership in $\exists\mathbb{R}$, we use the following characterization by Erickson, Hoog and Miltzow [23]: A problem P lies in $\exists\mathbb{R}$ if and only if there exists a verification algorithm A for P that runs in polynomial time on the real RAM, which we refer to as a *real verification algorithm*. In particular, for every yes-instance I of P there exists a polynomial sized witness w such that $A(I, w)$ returns yes, and for every no-instance I of P and any witness w, $A(I, w)$ returns no. In contrast to the definition of the complexity class NP, we also allow witnesses that consist of real numbers. Consequently, we execute A on the real RAM as well.

It remains to present a real verification algorithm for $\mathrm{GEM}_{k\to d}$. While the witness describes the coordinates of the vertices, the algorithm checks for intersections between any two simplices. Note that each simplex is a convex set and the intersection of convex sets is a convex set as well. For any simplex S with n vertices, we can efficiently determine n linear inequalities and at most one linear equality that together describe S: the inequalities may describe the n facets and the equality describes the subspace in case S is not d-dimensional. Then checking for intersections can be reduced to a linear program, which is polynomial time solvable in any fixed dimension. This finishes the description of the real verification algorithm. ◀

We note that one does not need to resort to the characterization of $\exists\mathbb{R}$ with verifiers as in [23]. It is possible to directly construct a polynomial system of polynomial size (in fixed dimension) in the coordinates of the vertices of the given complex in order to encode its geometric realizability. It may appear to be overly complicated to use the tools from [23], if you do not know this tool. However, if you know this tool it appears strange not to use it.

2.2 Hardness in three dimensions

This section is dedicated to proving Theorem 1 for $d = 3$ and $k \in \{2, 3\}$. The crucial part lies in the case $k = 2$.

▶ **Theorem 4.** *The decision problem $GEM_{2\to 3}$ is ∃ℝ-hard.*

Proof. We reduce from the ∃ℝ-hard problem Stretchability, as described in Section 1.2. In particular, for each pseudoline arrangement L, we construct a 2-dimensional complex C in time polynomial in L such that C geometrically embeds in $\mathbb{R}^3$ if and only if L is stretchable.

Let L be an arrangement of n pseudolines in the plane. Every pseudoline arrangement has a representation as a wiring diagram in which each pseudoline is given by a monotone curve consisting of $2n - 1$ sections. For an illustration consider Figure 4; each section could be represented by a segment, however for a visual appealing display, the bend points are rounded. We add a pseudoline ℓ_0 that intersects all pseudolines in the beginning, see Figure 4, and call the resulting pseudoline arrangement L^*. Note that L^* is stretchable if and only if L is stretchable. For later reference, we endow a natural orientation upon each pseudoline from left to right. In the following, we construct a 2-complex $C = (V, E)$ that allows for a geometric realization if and only if L^* (and thus L) is stretchable. In order to define C, we add a helper triangle $\triangle$ (consisting of three segments!) to our arrangement that intersects the pseudolines of L^* as illustrated in Figure 4. In particular, the helper triangle contains all intersection points of L^*.

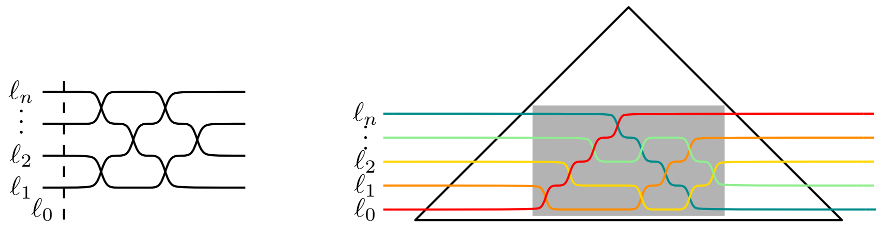

Figure 4 Adding an extra pseudoline ℓ_0 and the helper triangle $\triangle$ to the construction. (left) A pseudoline arrangement L^*. (right) The crossing diagram contains an additional helper triangle.

Construction of the 2-complex. In order to define C, we associate an almost geometric embedding of already defined parts along the way; where only a set of special edges is represented in a PL fashion, all other elements are already geometrically embedded. We will refer to the subsets in C as vertices, edges, and triangles depending on whether they contain one, two or three elements. The construction has five steps.

In the first step, we place the pseudolines and the helper triangle $\triangle$ in 3-space. Each pseudoline ℓ_i lies in the plane $z = i$ such that an observer high above (at infinity) sees the wiring diagram. Similarly, we place the segments of the helper triangle $\triangle$ in 3-space such that it lies in the plane $z = n + 1$. Note that no two pseudolines intersect. Therefore, we can surround each lifted pseudoline by a triangulated sphere which we call a *tunnel*; see also Figure 5. The tunnel T_i^+ of ℓ_i is formed by $2n + 3 + i$ sections; later, we will be particularly interested in a part of a tunnel, denoted by T_i, in which the first two and last two sections are removed. Each section consists of six triangles forming a triangulated triangular prism as illustrated in Figure 5. We close the tunnel with triangles at the ends and think of the *bottom* side of the prism to lie in the plane $z = i - 1/2$ (for now). The remaining part of the tunnel, i.e., the tunnel without its bottom, constitutes the *roof*, see Figure 5. The roof contains three disjoint paths on $2n + 4 + i$ vertices. The edges and vertices on the boundary of both the bottom and the roof form the *left* and *right roof path*; the edges of the closing triangles on either end do not belong to either path. The remaining vertices induce the *central roof path*. The three roof paths are thickened in Figure 5.

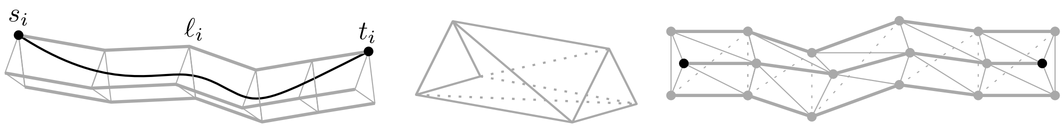

Figure 5 First step in the construction of the complex C – tunnel construction. (left) A tunnel viewed from side. (middle) A section of a tunnel. (right) A tunnel viewed from above.

Note that we do not add a tunnel for the helper triangle. We distribute the sections along T_i^+ to edges and crossings of the crossing diagram as follows: Generally, we associate one section per edge and one section per crossing of two pseudolines. Moreover, we associate one extra section of T_j^+ to a crossing of ℓ_i and ℓ_j whenever $i < j$. In order to represent the pseudoline ℓ_i, we insert a *special edge* e_i between the two top vertices on either end of the tunnel; for later reference, we denote the start vertex by s_i and the end vertex by t_i. In the associated almost geometric embedding, e_i is represented inside the tunnel by a concatenation of segments, one for each tunnel section. We aim for the fact that the special edge e_i lies inside the tunnel in every geometric embedding (if one exists).

In the second step, we identify parts of the tunnels. To this end, consider the tunnel sections assigned to a crossing of a pseudoline ℓ_i with ℓ_j, $i < j$. Recall that we assigned one section of T_i^+ and two sections of T_j^+ to the crossing. We identify the four triangles in the bottom of the two sections of T_j^+ with the four triangles in the roof of one section of T_i^+ as indicated in Figure 6. Note that we hereby identify six times two vertices, four of which belong to a left or right roof path of both tunnels, T_i^+ and T_j^+.

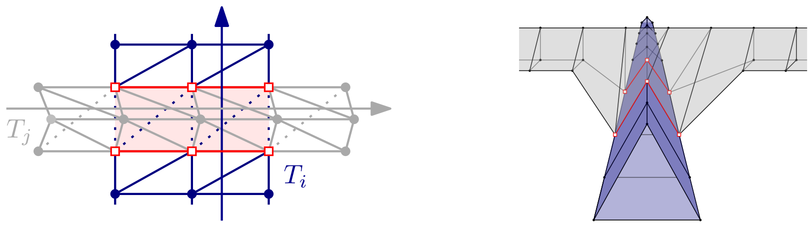

Figure 6 Second step in the construction of the complex C: (left) Gluing of tunnel parts viewed from above. (right) During the identification process, the vertices of the top tunnel are moved to the vertices of the bottom tunnel.

For the associated almost geometric embedding, we shortly explain here how to geometrically embed the tunnels. To this end, we may easily distribute the sections of the tunnels such that the six vertices of both tunnels (which will be pairwise identified) have the same x, y-coordinates. Then, during the identification process, we move the vertices of the top tunnel to the vertices of the lower tunnel.

In the third step, we add a new vertex to the construction that we call the *apex* and which we denote by u. We think of u as the observer high above (at infinity) and insert a triangle defined by u and the vertices of every edge that is *visible* from u. Clearly, every edge of the helper triangle $\triangle$ is visible. Moreover, note that every roof section that is neither glued in a crossing nor hidden by the helper triangle is visible. In contrast, no bottom of any tunnel is visible in the almost geometric embedding.

In the fourth step, we enhance the 1-complex induced by the neighborhood $N(u)$ of the apex u such that it corresponds to an essentially 3-connected planar graph G^+. We call a graph *essentially* 3-*connected* if it is a subdivision of a 3-connected graph. With the description so far, the 1-complex corresponds to the graph H depicted in black in Figure 7.

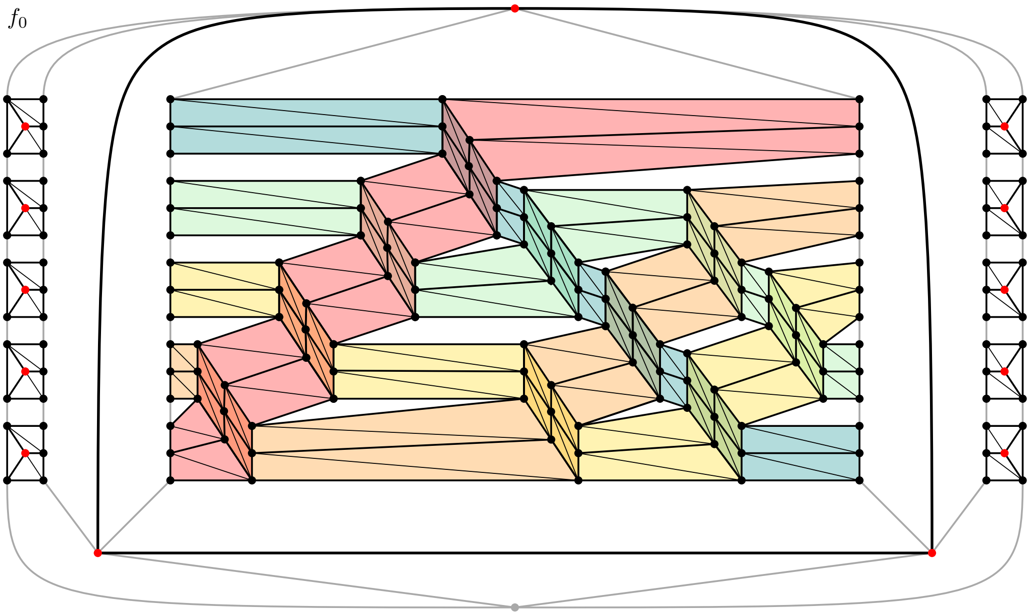

Figure 7 Third and fourth step in the construction of the complex C: neighborhood of the apex u. The graph Hafter the third step is depicted in black. Together with the gray edges, the graph is a candidate for the essentially 3-connected plane graph G^+ and its subgraph G inside $\triangle$. Each red vertex outside of $\triangle$ is a start vertex s_i or an end vertex t_i of some special edge e_i; its black component represents the first or last section of tunnel T_i^+, respectively.

To construct G^+, we make use of the following fact. We define the degree of a face in a potentially disconnected plane graph as the number of edges in the face boundary (counted with multiplicity), plus 1 for each but one component incident to the face. Note that the degree of a face is thus lower bounded by the number of incident vertices and upper bounded by twice the number of incident vertices.

▷ Claim 5. For every plane graph $G_1 = (V_1, E_1)$, there exists an essentially 3-connected plane graph $G_2 = (V_2, E_2)$ such that G_1 is a subgraph of G_2 and any straight-line drawing D_1 of G_1 in the plane can be extended to a straight-line drawing of G_2. Moreover, if the maximum face degree of G_1 is k, then the size of G_2 can be bounded by $|V_2| + |E_2| \leq O(k|V_1|)$.

Let $G^+ := G_2$ be an essentially 3-connected plane graph guaranteed by Claim 5 for the case that $G_1 = H$. Note that G_1 has $O(n^2)$ vertices and edges, and every face has degree $O(n)$. Hence, the size of G_2 is in $O(n^3)$. We denote the outer face of G_2 by f_0. The reader is invited to think about the far more sparse graph depicted in Figure 7, which also serves as a candidate for G^+. Indeed, the depicted graph also fulfills all properties necessary for our construction; however, not all properties of Claim 5. For example, the depicted graph is even 3-connected. The proof of this is straightforward, but a bit tedious. Thus, we leave it as an exercise to the interested reader to check that the graph remains connected even after the deletion of any two vertices or alternatively, that any pair of vertices is connected by three disjoint paths.

Later, the subgraph G of G^+ that is induced by all vertices of $\bigcup_i T_i$ will be of particular interest; in Figure 7, these vertices (and their convex hull) lie inside the helper triangle $\triangle$. Recall that T_i denotes the part of the tunnel T_i^+ obtained by deleting the first two and last two sections.

It is a well-known fact that all (straight-line or topological) planar drawings of a 3-connected planar graph on the sphere are equivalent [34]; for a definition of equivalent drawings consult Section 1.2. Consequently, the result extends to *essentially* 3-connected graphs as it also holds for topological drawings. For later reference, we note the following.

▷ Claim 6. The planar graph G^+ is essentially 3-connected. Therefore, all crossing-free drawings of G^+ on a sphere are equivalent. Furthermore, any straight-line drawing of H in the plane can be extended to a straight-line drawing of G^+.

We ensure that the neighborhood complex of u is the underlying planar graph of G^+, i.e., for each edge of G^+ not present in H, we insert a triangle formed by the vertices of this edge together with u and call the resulting complex $\overline{C}$.

In the fifth and last step, our final complex C consist of two copies of $\overline{C}$ in which the apex vertices are identified. We use these two copies in order to guarantee that in any geometric embedding the apex lies outside of all tunnels for one copy of $\overline{C}$. This finishes the construction of the abstract complex C.

It remains to show that our construction runs in polynomial time and fulfills the claimed properties.

Time Complexity. In order to verify that the construction shows $\exists\mathbb{R}$-hardness, we argue that it has a running time that is polynomial in the size of the input. To this end, note that a pseudoline arrangement with n pseudolines can be described by the sequence of crossings along each pseudoline, i.e., by the $O(n^2)$ crossings. Thus, the input size is $N = O(n^2)$. After adding the helper triangle and ℓ_0, the crossing diagram still has a size in $O(n^2)$. It is easy to see that our construction has a size proportional to $N^{3/2}$: For each segment and crossing of the diagram, we insert a constant number of objects. Moreover, we add a triangle for every (additional) edge in G^+; recall that G^+ has size $O(n^3)$. Consequently, the total construction has size $O(n^3) = O(N^{3/2})$. We remark, that a more careful choice of G^+, as in Figure 7, yields a construction that is linear in N.

It remains to show that the pseudoline arrangement L is stretchable if and only if C has a geometric embedding in $\mathbb{R}^3$.

Correctness. If L is stretchable, it is relatively straight-foward to construct a geometric embedding of C.

▷ Claim 7. If L is stretchable, then C has a geometric embedding.

The reverse direction is more involved and the interesting challenge.

▷ Claim 8. If C has a geometric embedding, then L is stretchable.

This finishes the proof of Theorem 4. ◀

Fattening the Complex. In the following, we present a simple modification for the proof of Theorem 4 to obtain hardness for pure 2- and 3-complexes.

▶ **Corollary 9.** *The decision problems $GEM_{2\to 3}$ and $GEM_{3\to 3}$ are ∃ℝ-hard, even when restricting to pure complexes.*

Proof. The constructed 2-complex C in the proof of Theorem 4 was not pure because the special edges are not contained in any triangle. We obtain a pure 2-complex $\hat{C}$ by adding one new vertex to each special edge such that it forms a special triangle. On the one hand, given a geometric embedding of C in $\mathbb{R}^3$, the new vertices can easily be added close enough to their defining set in C. On the other hand, any geometric embedding of $\hat{C}$ induces an embedding of C. Hence, C has a geometric embedding if and only if $\hat{C}$ has a geometric embedding in $\mathbb{R}^3$.

Analogously, we can add a private vertex to each triangle of $\hat{C}$ to form a pure 3-complex which has a geometric embedding if and only if C has a geometric embedding in $\mathbb{R}^3$. ◀

Alternatively for showing hardness of $\text{GEM}_{3\to 3}$, we remark that hardness of $\text{GEM}_{k\to d}$ for $k < d$ easily implies hardness of $\text{GEM}_{\ell\to d}$ for all $k \leq \ell \leq d$ by adding a disjoint ℓ-simplex to the construction which always has a geometric embedding in $\mathbb{R}^d$.

2.3 Dimension Reduction

In order to show hardness for all remaining cases of Theorem 1, we establish the following dimension reduction.

▶ **Lemma 10.** *The decision problem $GEM_{k\to d}$ reduces to $GEM_{k+1\to d+1}$.*

The idea is to add two apices to a k-complex C in order to obtain a $(k+1)$-complex C^+. We then argue that C has a geometric embedding in $\mathbb{R}^d$ if and only if C^+ has a geometric embedding in $\mathbb{R}^{d+1}$. More formally, for a complex $C = (V, E)$ and a disjoint vertex set U, $C * U$ denotes the *join* complex $(V \cup U, E')$ where $E' := \{e \cup u \mid e \in E, u \in U\}$. The following claim immediately implies Lemma 10.

▷ Claim 11. Let $C = (V, E)$ be a complex, $a, b \notin V$ two new vertices, and $C^+ := C * \{a, b\}$ their join complex. Then C has a geometric embedding in $\mathbb{R}^d$ if and only if C^+ has a geometric embedding in $\mathbb{R}^{d+1}$.

Proof. Let φ be a geometric embedding of C in $\mathbb{R}^d$. Then, we define for $v \in V \cup \{a, b\}$,

$$\varphi'(v) = \begin{cases} (\ \ \varphi(v)\ \ ,\ 0\ ) & \text{if } v \in V, \\ (0, \ldots, 0, +1) & \text{if } v = a, \\ (0, \ldots, 0, -1) & \text{if } v = b. \end{cases}$$

It is easy to check that φ' is a geometric embedding of C^+ in $\mathbb{R}^{d+1}$: By definition of the last coordinate, any two simplices where one possibly contains a and the other possibly contains b can only intersect in the subspace induced by the first d coordinates. Consequently, all (interesting) potential intersections happen in the d-dimensional subspace induced by the first d coordinates. Hence φ implies the correctness of the geometric embedding.

For the reverse direction, consider a geometric embedding φ of C^+ in $\mathbb{R}^{d+1}$. Let $\varphi_a := \varphi(a)$ and $\varphi_b := \varphi(b)$. Without loss of generality, we assume that $\varphi_a - \varphi_b$ is orthogonal to the first d coordinates, i.e., $\varphi_a - \varphi_b$ is parallel to the $(d+1)$-st coordinate axis. Let $\overline{\varphi}(C) := \bigcup_{e\in E} \overline{\varphi}(e)$

denote the induced geometric subrepresentation of C. We claim that the orthogonal projection $f : \overline{\varphi}(C) \to \mathbb{R}^d$ to the first d coordinates (i.e.,the effect of the function is the one of restricting to the first d coordinates, .) is injective. Thus $\varphi' := f \circ \varphi$ yields a representation of C in $\mathbb{R}^d$.

For the purpose of a contradiction, suppose that f is not injective. Then there exist two distinct points $p = (p_1, \ldots, p_{d+1})$ and $q = (q_1, \ldots, q_{d+1})$ with $p, q \in \overline{\varphi}(C)$ such that $(p_1, \ldots, p_d) = (q_1, \ldots, q_d)$ and $p_{d+1} \neq q_{d+1}$. Without loss of generality, we may assume that $p_{d+1} > q_{d+1}$. Consider the plane P spanned by φ_a, φ_b, p. Note that $q \in P$ because $\varphi_a - \varphi_b$ and $p - q$ are parallel (to the $(d+1)$st coordinate axis). For an illustration, see Figure 8.

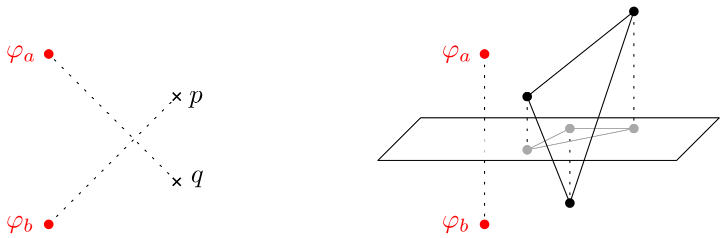

Figure 8 Illustration for the proof of Claim 11. The geometric embedding φ of C^+ gives a monotone embedding of C, otherwise we can find an intersection in C^+.

Let us denote with $e_p \in E$ and $e_q \in E$ any choice of hyperedges such that $p \in \overline{\varphi}(e_p)$ and $q \in \overline{\varphi}(e_q)$. Consider the two open segments $\text{seg}^\circ(\varphi_a, q) \in \overline{\varphi}(e_q \cup a)$ and $\text{seg}^\circ(\varphi_b, p) \in \overline{\varphi}(e_p \cup b)$. Clearly, these open segments intersect in a point x, as illustrated in Figure 8. Because φ is a geometric embedding, it holds that

$$x \in \overline{\varphi}(e_q \cup a) \cap \overline{\varphi}(e_p \cup b) = \overline{\varphi}(e_q \cap e_p) = \overline{\varphi}(e_q) \cap \overline{\varphi}(e_p).$$

In particular, this implies that $x \in \overline{\varphi}(e_q)$ and thus that $x \in \text{seg}^\circ(\varphi_a, q) \cap \overline{\varphi}(e_q)$. However, because $\overline{\varphi}(e_q \cup a)$ is a simplex, φ_a does not lie in $\text{span}(\overline{\varphi}(e_q))$ and thus $\text{seg}^\circ(\varphi_a, q) \cap \overline{\varphi}(e_q) = \emptyset$. A contradiction. $\lhd$

3 Conclusion

We established the computational complexity of $\text{GEM}_{k \to d}$ for all $d \geq 3$ and $k \in \{d-1, d\}$. In particular, we showed that for these values it is complete for $\exists\mathbb{R}$ to distinguish PL embeddable k-complexes in $\mathbb{R}^d$ from geometrically embeddable ones. Arguably, $\text{GEM}_{2 \to 3}$ is the most interesting case.

Investigating the computational complexity for the remaining open entries in Table 2 remains for future work. We strengthen the conjecture of Skopenkov [59] as follows.

▶ **Conjecture.** *The problem* $\text{GEM}_{k \to d}$ *is* $\exists\mathbb{R}$*-complete for all* k, d *such that* $\max\{3, k\} \leq d \leq 2k$.

References

1 Zachary Abel, Erik Demaine, Martin Demaine, Sarah Eisenstat, Jayson Lynch, and Tao Schardl. Who needs crossings? Hardness of plane graph rigidity. In *International Symposium on Computational Geometry (SoCG)*, pages 3:1–3:15, 2016. doi:10.4230/LIPIcs.SoCG.2016.3.

2 Mikkel Abrahamsen. Covering polygons is even harder. In *Foundations on Computer Science (FOCS)*, 2021. doi:10.1109/FOCS52979.2021.00045.

3 Mikkel Abrahamsen, Anna Adamaszek, and Tillmann Miltzow. The art gallery problem is ∃ℝ-complete. In *Symposium on Theory of Computing (STOC)*, pages 65–73, 2018. doi:10.1145/3188745.3188868.

4 Mikkel Abrahamsen, Linda Kleist, and Tillmann Miltzow. Training neural networks is ∃ℝ-complete. In *Conference on Neural Information Processing Systems (NeurIPS)*, volume 34, pages 18293–18306, 2021. URL: https://proceedings.neurips.cc/paper/2021/file/9813b270ed0288e7c0388f0fd4ec68f5-Paper.pdf.

5 Mikkel Abrahamsen, Tillmann Miltzow, and Nadja Seiferth. Framework for ER-completeness of two-dimensional packing problems. In *Foundations on Computer Science (FOCS)*, pages 1014–1021. IEEE, 2020. doi:10.1109/FOCS46700.2020.00098.

6 J. L. Ramírez Alfonsín. Knots and links in spatial graphs: a survey. *Discrete mathematics*, 302(1-3):225–242, 2005. doi:10.1016/j.disc.2004.07.035.

7 Vittorio Bilò and Marios Mavronicolas. A catalog of EXISTS-R-complete decision problems about Nash equilibria in multi-player games. In *Symposium on Theoretical Aspects of Computer Science (STACS)*, 2016. doi:10.4230/LIPIcs.STACS.2016.17.

8 R. H. Bing. An alternative proof that 3-manifolds can be triangulated. *Annals of Mathematics*, 69(1):37–65, 1959. doi:10.2307/1970092.

9 Jürgen Bokowski and A. Guedes de Oliveira. On the generation of oriented matroids. *Discrete & Computational Geometry (DCG)*, 24(2):197–208, 2000. doi:10.1007/s004540010027.

10 Ulrich Brehm. A nonpolyhedral triangulated Möbius strip. *Proceedings of the American Mathematical Society*, 89(3):519–522, 1983. doi:10.1090/S0002-9939-1983-0715878-1.

11 Ulrich Brehm and Karanbir S. Sarkaria. Linear vs. piecewise-linear embeddability of simplicial complexes. *Technical Report MPI Bonn*, pages 1–15, 1992. available at http://kssarkaria.org/docs/Linear%20vs.%20piecewise-linear%20embeddability%20of%20simplicial%20complexes.pdf.

12 J. L. Bryant. Approximating embeddings of polyhedra in codimension three. *Transactions of the American Mathematical Society*, 170:85–95, 1972. doi:10.1090/S0002-9947-1972-0307245-7.

13 Martin Čadek, Marek Krčál, Jiří Matoušek, Francis Sergeraert, Lukáš Vokřínek, and Uli Wagner. Computing all maps into a sphere. *Journal of the ACM (JACM)*, 61(3):1–44, 2014. doi:10.1145/2597629.

14 Martin Čadek, Marek Krčál, Jiří Matoušek, Lukáš Vokřínek, and Uli Wagner. Extendability of continuous maps is undecidable. *Discrete & Computational Geometry (DCG)*, 51(1):24–66, 2014. doi:10.1007/s00454-013-9551-8.

15 Martin Čadek, Marek Krčál, Jiří Matoušek, Lukáš Vokřínek, and Uli Wagner. Time computation of homotopy groups and Postnikov systems in fixed dimension. *SIAM Journal of Computing (SICOMP)*, 43(5):1728–1780, 2014. doi:10.1137/120899029.

16 Martin Čadek, Marek Krčál, and Lukáš Vokřínek. Algorithmic solvability of the lifting-extension problem. *Discrete & Computational Geometry (DCG)*, 57(4):915–965, 2017. doi:10.1007/s00454-016-9855-6.

17 John Canny. Some algebraic and geometric computations in PSPACE. In *Symposium on Theory of Computing (STOC)*, pages 460–467. ACM, 1988. doi:10.1145/62212.62257.

18 Jean Cardinal. Computational geometry column 62. *SIGACT News*, 46(4):69–78, 2015. doi:10.1145/2852040.2852053.

19 Jean Cardinal and Udo Hoffmann. Recognition and complexity of point visibility graphs. *Discrete & Computational Geometry (DCG)*, 57(1):164–178, 2017. doi:10.1007/s00454-016-9831-1.

20 Johannes Carmesin. Embedding simply connected 2-complexes in 3-space – I. A Kuratowski-type characterisation. arXiv preprint, 2019. arXiv:1709.04642.

21 Jean Dieudonné. *A history of algebraic and differential topology, 1900-1960*. Springer, 2009.

22 Michael Gene Dobbins, Andreas Holmsen, and Tillmann Miltzow. A universality theorem for nested polytopes. arXiv preprint, 2019. arXiv:1908.02213.

23 Jeff Erickson, Ivor van der Hoog, and Tillmann Miltzow. Smoothing the gap between NP and ER. In *Foundations on Computer Science (FOCS)*, pages 1022–1033. IEEE, 2020. `doi:10.1109/FOCS46700.2020.00099`.

24 Marek Filakovský, Uli Wagner, and Stephan Zhechev. Embeddability of simplicial complexes is undecidable. In *Symposium on Discrete Algorithms (SODA)*, pages 767–785, 2020. `doi:10.1137/1.9781611975994.47`.

25 Antonio Flores. Über n-dimensionale Komplexe, die im R_{2n+1} absolut selbstverschlungen sind. In *Ergeb. Math. Kolloq*, volume 34, pages 4–6, 1933.

26 Michael H. Freedman, Vyacheslav S. Krushkal, and Peter Teichner. Van Kampen's embedding obstruction is incomplete for 2-complexes in $\mathbb{R}^4$. *Mathematical Research Letters*, 1(2):167–176, 1994.

27 Florian Frick, Mirabel Hu, Nick Scheel, and Steven Simon. Embedding dimensions of simplicial complexes on few vertices. arXiv preprint, 2021. `arXiv:2109.04855`.

28 Jugal Garg, Ruta Mehta, Vijay V. Vazirani, and Sadra Yazdanbod. $\exists\mathbb{R}$-completeness for decision versions of multi-player (symmetric) Nash equilibria. *ACM Transactions on Economics and Computation*, 6(1):1:1–1:23, 2018. `doi:10.1145/3175494`.

29 Jonathan L. Gross and Ronald H. Rosen. A linear time planarity algorithm for 2-complexes. *Journal of the ACM (JACM)*, 26(4):611–617, 1979. `doi:10.1145/322154.322156`.

30 Branko Grünbaum. Imbeddings of simplicial complexes. *Commentarii Mathematici Helvetici*, 44(1):502–513, 1969. `doi:10.5169/seals-33795`.

31 Branko Grünbaum. Polytopes, graphs, and complexes. *Bulletin of the American Mathematical Society*, 76(6):1131–1201, 1970. `doi:10.1090/S0002-9904-1970-12601-5`.

32 Rudolf Halin and Heinz Jung. Charakterisierung der Komplexe der Ebene und der 2-Sphäre. *Archiv der Mathematik*, 15(1):466–469, 1964.

33 John Hopcroft and Robert Tarjan. Efficient planarity testing. *Journal of the ACM (JACM)*, 21(4):549–568, 1974. `doi:10.1145/321850.321852`.

34 Wilfried Imrich. On whitney's theorem on the unique embeddability of 3-connected planar graphs. In *Recent advances in graph theory: Proceedings of the Symposium held in Prague*, volume 1974, pages 303–306, 1975.

35 Fáry István. On straight-line representation of planar graphs. *Acta scientiarum mathematicarum*, 11(229-233):2, 1948.

36 Marek Krčál, Jiří Matoušek, and Francis Sergeraert. Polynomial-time homology for simplicial Eilenberg–MacLane spaces. *Foundations of Computational Mathematics (FoCM)*, 13(6):935–963, 2013. `doi:10.1007/s10208-013-9159-7`.

37 Anna Lubiw, Tillmann Miltzow, and Debajyoti Mondal. The complexity of drawing a graph in a polygonal region. In *International Symposium on Graph Drawing and Network Visualization (GD)*, pages 387–401. Springer, 2018. `doi:10.1007/978-3-030-04414-5_28`.

38 Jiří Matoušek. Intersection graphs of segments and $\exists\mathbb{R}$. arXiv preprint, 2014. `arXiv:1406.2636`.

39 Jiří Matoušek, Eric Sedgwick, Martin Tancer, and Uli Wagner. Embeddability in the 3-sphere is decidable. *Journal of the ACM (JACM)*, 65(1):1–49, 2018. `doi:10.1145/2582112.2582137`.

40 Jiří Matoušek, Martin Tancer, and Uli Wagner. Hardness of embedding simplicial complexes in $\mathbb{R}^d$. *Journal of the European Mathematical Society (JEMS)*, 13(2):259–295, 2011. `doi:10.4171/JEMS/252`.

41 Colin McDiarmid and Tobias Müller. Integer realizations of disk and segment graphs. *Journal of Combinatorial Theory, Series B*, 103(1):114–143, 2013. `doi:10.1016/j.jctb.2012.09.004`.

42 Karl Menger. *Dimensionstheorie*. Vieweg+Teubner Verlag, 1 edition, 1928. `doi:10.1007/978-3-663-16056-4`.

43 Arnaud de Mesmay, Yo'av Rieck, Eric Sedgwick, and Martin Tancer. Embeddability in $\mathbb{R}^3$ is NP-hard. *Journal of the ACM (JACM)*, 67(4):20:1–20:29, 2020. `doi:10.1145/3396593`.

44 Tillmann Miltzow and Reinier F. Schmiermann. On classifying continuous constraint satisfaction problems. In *Foundations of Computer Science (FOCS 2021)*, pages 781–791. IEEE, 2022. `doi:10.1109/FOCS52979.2021.00081`.

45 Nicolai Mnëv. The universality theorems on the classification problem of configuration varieties and convex polytopes varieties. In Oleg Y. Viro, editor, *Topology and geometry – Rohlin seminar*, pages 527–543. Springer, 1988. doi:10.1007/BFb0082792.
46 Isabella Novik. A note on geometric embeddings of simplicial complexes in a euclidean space. *Discrete & Computational Geometry (DCG)*, 23(2):293–302, 2000. doi:10.1007/s004549910019.
47 Patrice Ossona deMendez. Realization of posets. *Journal of Graph Algorithms and Applications (JGAA)*, 6(1):149–153, 2002. doi:10.7155/jgaa.00048.
48 Christos Papakyriakopoulos. A new proof of the invariance of the homology groups of a complex. *Bulletin of the Greek Mathematical Society*, 22:1–154, 1943.
49 S. Parsa and A. Skopenkov. On embeddability of joins and their 'factors'. arXiv preprint, 2020. arXiv:2003.12285.
50 Jürgen Richter-Gebert. *Realization spaces of polytopes*, volume 1643 of *LNM*. Springer, 1996. doi:10.1007/BFb0093761.
51 Jürgen Richter-Gebert and Günter M. Ziegler. Realization spaces of 4-polytopes are universal. *Bulletin of the American Mathematical Society*, 32(4):403–412, 1995. doi:10.1090/S0273-0979-1995-00604-X.
52 Marcus Schaefer. Complexity of some geometric and topological problems. In *International Symposium on Graph Drawing (GD)*, LNCS, pages 334–344. Springer, 2009. doi:10.1007/978-3-642-11805-0_32.
53 Egon Schulte and Ulrich Brehm. Polyhedral maps. In Csaba D. Toth, Jacob E. Goodman, and Joseph O'Rourke, editors, *Handbook of Discrete and Computational Geometry, Third Edition*, pages 533–548. Chapman and Hall/CRC, 2017. doi:10.1201/9781315119601.
54 Arnold Shapiro. Obstructions to the imbedding of a complex in a Euclidean space.: I. the first obstruction. *Annals of Mathematics*, pages 256–269, 1957. doi:10.2307/1969998.
55 Yaroslav Shitov. A universality theorem for nonnegative matrix factorizations. arXiv preprint, 2016. arXiv:1606.09068.
56 Peter Shor. Stretchability of pseudolines is NP-hard. In Peter Gritzmann and Bernd Sturmfels, editors, *Applied Geometry and Discrete Mathematics: The Victor Klee Festschrift*, DIMACS – Series in Discrete Mathematics and Theoretical Computer Science, pages 531–554. AMS, 1991.
57 Arkadiy Skopenkov. Realizability of hypergraphs and ramsey link theory. arXiv preprint, 2014. arXiv:1402.0658.
58 Arkadiy Skopenkov. Extendability of simplicial maps is undecidable. arXiv preprint, 2020. arXiv:2008.00492.
59 Arkadiy Skopenkov. Invariants of graph drawings in the plane. *Arnold Mathematical Journal*, 6:21–55, 2020. doi:10.1007/s40598-019-00128-5.
60 Arkadiy Skopenkov and Martin Tancer. Hardness of almost embedding simplicial complexes in $\mathbb{R}^d$. *Discrete & Computational Geometry (DCG)*, 61(2):452–463, 2019. doi:10.1007/s00454-018-0013-1.
61 Mikhail Skopenkov. Embedding products of graphs into euclidean spaces. arXiv preprint, 2016. arXiv:0808.1199.
62 Ernst Steinitz. Polyeder und Raumeinteilungen. In *Encyclopädie der mathematischen Wissenschaften*, volume 3-1-2 (Geometrie), chapter 12, pages 1–139. B. G. Teubner, Leipzig, 1922.
63 Dagmar Timmreck. Necessary conditions for geometric realizability of simplicial complexes. In A.I. Bobenko, P. Schröder, J.M. Sullivan, and G.M. Ziegler, editors, *Discrete Differential Geometry*, volume 38 of *Oberwolfach Seminars*, pages 215–233. Birkhäuser Basel, 2008. doi:10.1007/978-3-7643-8621-4_11.
64 Dagmar Ingrid Timmreck. *Realization Problems for Point Configurations and Polyhedral Surfaces*. PhD thesis, Freie Universität Berlin, 2015. doi:10.17169/refubium-14465.
65 Brian R. Ummel. The product of nonplanar complexes does not imbed in 4-space. *Transactions of the American Mathematical Society*, 242:319–328, 1978. doi:10.2307/1997741.

66 Egbert R. Van Kampen. Komplexe in euklidischen Räumen. In *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, volume 9, pages 72–78. Springer, 1933.

67 Wen-tsün Wu. *A theory of imbedding, immersion, and isotopy of polytopes in a Euclidean space*. Science Press, 1965.