Distinguishing Classes of Intersection Graphs of Homothets or Similarities of Two Convex Disks

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Abstract

For smooth convex disks A, i.e., convex compact subsets of the plane with non-empty interior, we classify the classes $G^{\mathrm{hom}}(A)$ and $G^{\mathrm{sim}}(A)$ of intersection graphs that can be obtained from homothets and similarities of A, respectively. Namely, we prove that $G^{\mathrm{hom}}(A) = G^{\mathrm{hom}}(B)$ if and only if A and B are affine equivalent, and $G^{\mathrm{sim}}(A) = G^{\mathrm{sim}}(B)$ if and only if A and B are similar.

2012 ACM Subject Classification Theory of computation → Computational geometry

Keywords and phrases geometric intersection graph, convex disk, homothet, similarity

Digital Object Identifier 10.4230/LIPIcs.SoCG.2023.2

Related Version *Full Version*: https://arxiv.org/abs/2108.04588

Funding *Mikkel Abrahamsen*: The author is supported by Starting Grant 1054-00032B from the Independent Research Fund Denmark under the Sapere Aude research career programme. BARC is supported by the VILLUM Foundation grant 16582.
Bartosz Walczak: The author is partially supported by the National Science Center of Poland grant 2015/17/D/ST1/00585.

1 Introduction

Disk graphs have received much attention due to their ability to model graphs appearing in practice and their interesting structural properties. In a disk graph, each vertex corresponds to a (circular) disk, and there is an edge between two vertices if and only if the two corresponding disks intersect. Disk graphs appear naturally in problems related to radio and sensor networks. For instance, the region reached by the signal from each transmitter in a radio network can be modeled as a disk, and when two disks intersect, the interference of the signals may be an issue if the transmitters use the same frequency. The problem of avoiding interference while minimizing the number of used frequencies thus corresponds to finding the chromatic number of the disk graph. Applications like these are part of the motivation for various papers on algorithms or computational hardness for problems taking disk graphs in the input [2, 3, 7, 10, 13, 14, 16, 25] as well as papers studying disk graphs from a mathematical angle [21, 22].

Combinatorial analysis of problems such as chromatic number and minimum hitting set size has often been performed in greater generality, for intersection graphs of translated copies or homothetic (i.e., translated and scaled) copies of a fixed convex shape [11, 17, 18, 23], and recently also for translated, scaled, and rotated squares [6]. Algorithmic considerations have also been generalized in a similar way – Bonnet, Grelier, and Miltzow [4] studied the maximum clique problem and extended classic algorithms for disk graphs and unit disk graphs to intersection graphs of homothetic or translated copies of a fixed convex set.

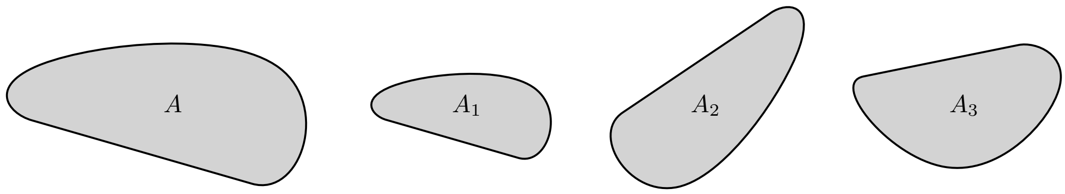

Figure 1 Here, A_1 is a homothet of A, A_2 is a similarity but not a homothet of A, and A_3 is affine equivalent to A, but not similar to A. By Theorem 1, A and A_3 induce the same intersection graphs of homothets, but Theorem 2 implies that the intersection graphs of similarities are different.

A well-established line of research in discrete and computational geometry has been aiming at understanding the relationships between classes of geometric intersection graphs such as whether two classes are equal or whether one class is a subclass of another [5, 8, 9, 15, 20]. In view of the above-mentioned research, it is natural to investigate the relationships between the classes of intersection graphs of translated copies, homothetic copies, and copies by similarity (translation, scaling, and rotation) of a fixed convex shape.

To be precise, consider an arbitrary *convex disk* A, that is, a convex and compact set in the plane with non-empty interior. A *translate* of A is a translated copy of A (with no scaling or rotation allowed). A *homothet* of A is a positively scaled and translated copy of A (with no rotation allowed). A *similarity* is a homothet rotated by an arbitrary angle. An *affine equivalent* of A is the image of A under an invertible affine transformation. See Figure 1. The *intersection graph* of a family $\mathcal{F}$ of sets in the plane is the graph with vertex set $\mathcal{F}$ and edge set $\{uv\colon u, v \in \mathcal{F},\ u \cap v \neq \emptyset\}$.

In a recent paper, Aamand, Abrahamsen, Knudsen, and Rasmussen [1] studied the question of when the translates of two convex disks induce the same intersection or contact graphs, where a *contact graph* is an intersection graph that can be realized by pairwise interior-disjoint disks. They proved for a large class of convex disks, including all strictly convex ones, that two disks A and B yield the same classes of contact and intersection graphs if and only if the central symmetrals of A and B are affine equivalent, where the *central symmetral* of a disk A is the centrally symmetric disk $\frac{1}{2}(A - A)$.

In this paper, we study the question of when the homothets or the similarities of two convex disks induce the same intersection graphs. We make the additional assumption that the convex disk A be *smooth*, that is, there is a unique tangent containing any point on the boundary of A. We let $\hom A$ and $\operatorname{sim} A$ denote the sets of homothets and similarities of A, respectively. We let $G^{\mathrm{hom}}(A)$ and $G^{\mathrm{sim}}(A)$ denote the classes of (finite) intersection graphs of homothets and similarities of A, respectively. For two smooth convex disks A and B, we are able to say exactly when $G^{\mathrm{hom}}(A) = G^{\mathrm{hom}}(B)$ and $G^{\mathrm{sim}}(A) = G^{\mathrm{sim}}(B)$, as follows.

▶ **Theorem 1.** *Let A and B be smooth convex disks. Then $G^{\mathrm{hom}}(A) = G^{\mathrm{hom}}(B)$ if and only if A and B are affine equivalent. Moreover, if A and B are not affine equivalent, then neither $G^{\mathrm{hom}}(A) \subseteq G^{\mathrm{hom}}(B)$ nor $G^{\mathrm{hom}}(B) \subseteq G^{\mathrm{hom}}(A)$.*

▶ **Theorem 2.** *Let A and B be smooth convex disks. Then $G^{\mathrm{sim}}(A) = G^{\mathrm{sim}}(B)$ if and only if B is similar to A or to the reflection $A^{\mathrm{r}} = \{(-x, y)\colon (x, y) \in A\}$.*

If A and B are affine equivalent, then $G^{\mathrm{hom}}(A) = G^{\mathrm{hom}}(B)$, because the affine transformation that maps A to B transforms every realization in $\hom A$ to a realization of the same graph in $\hom B$, and vice versa. Likewise, if B is similar to A or to A^{r}, then $G^{\mathrm{sim}}(A) = G^{\mathrm{sim}}(B)$, because the similarity transformation (possibly with reflection) that maps A to B transforms every realization in $\operatorname{sim} A$ to a realization of the same graph in $\operatorname{sim} B$, and vice versa. The difficult part is the necessity of these conditions.

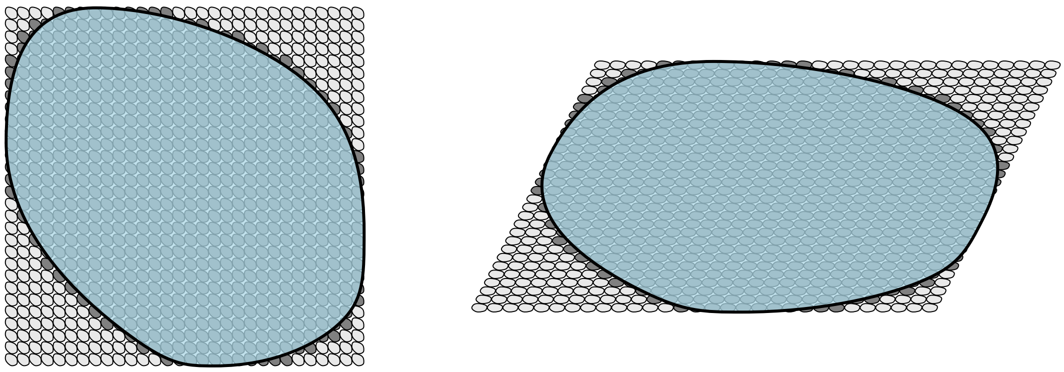

Figure 2 To the left is shown the grid of small copies of A and one large copy of A on top. The disks in the grid that are intersected (dark gray) define the shape of A to an arbitrarily high precision, if we make the grid sufficiently fine. To the right is shown the same graph realized by another disk B. As we will show, the arrangement must again form a grid of small disks with one large copy of B on top. An affine map that makes the two grids coincide then also maps B to A to within a small error, since the two disks intersect the same "pixels" in the grids.

When A and B are not affine equivalent, we point out graphs $G_A \in G^{\mathrm{hom}}(A)$ and $G_B \in G^{\mathrm{hom}}(B)$ such that $G_A \notin G^{\mathrm{hom}}(B)$ and $G_B \notin G^{\mathrm{hom}}(A)$, which yields the second part of Theorem 1. By contrast, when B is dissimilar to both A and A^{r}, then $G^{\mathrm{sim}}(A)$ and $G^{\mathrm{sim}}(B)$ may be properly nested. Indeed, if A is a circular disk and B is a non-circular filled ellipse, then $G^{\mathrm{sim}}(A) \subset G^{\mathrm{sim}}(B)$, because the affine stretch that maps A to B transforms every realization in $\hom A = \operatorname{sim} A$ to a realization of the same graph in $\hom B \subseteq \operatorname{sim} B$, while in the proof of Theorem 2, we construct a graph in $G^{\mathrm{sim}}(B)$ that is not in $G^{\mathrm{sim}}(A)$.

One may or may not allow scaling by negative numbers when defining the homothets of A, which corresponds to rotating A by 180°. We remark that Theorem 1 holds in either case (with the same proof). Likewise, one may or may not allow reflection along the y-axis when defining the similarities of A, and Theorem 2 holds in either case (with the same proof).

We note that although we establish results for more general families of graphs, our results are not generalizations of the ones in [1]. We also remark that the contact graphs of homothets or similarities of a smooth convex disk have already been characterized. The Koebe-Andreev-Thurston Circle Packing Theorem, first proved by Koebe in 1936 [19], asserts that every planar graph is the contact graph of some set of pairwise interior-disjoint circular disks. Since every contact graph is planar, the contact graphs are exactly the planar graphs. The Monster Packing Theorem by Schramm [24] generalizes the result in the following way. Suppose that a planar graph is given, together with a correspondence which assigns to each vertex of the graph a smooth convex disk. Then there exists a contact representation of the graph where each vertex is represented by a homothet of the associated disk. Hence the contact graphs of homothets or similarities of any smooth convex disk are the planar graphs.

Outline of the paper

In Section 2, we set our notation and define the central concepts. In Section 3, we introduce a notion of convergence of sequences of compact subsets of $\mathbb{R}^2$. The usual definition of convergence based on the Hausdorff distance between sets only allows us to talk about convergence towards a compact set, but in our case, we also need to be able to express, for instance, that a sequence of (growing) convex disks converges to a half-plane.

In Sections 4 and 5, we introduce the constructions that enable us to distinguish the graph classes. At an overall level, the idea behind our constructions is to define a graph G such that however G is realized as an intersection graph of homothets or similarities of a smooth convex disk A, then a subset of the disks in the realization will form a large and almost regular grid of small copies of A; see Figure 2. We use this grid in a somewhat similar manner as the grid of pixels in television: We put one large disk A on top of the grid. The disks in the grid that intersect A will then with high precision define the shape of A. If now another disk B is able to realize the same graph, then we can consider an affine transformation that makes the two grids "match", and it follows that A and B must be nearly identical under this transformation, since the same "pixels" in the two grids are intersected by the large disks on top. If B can realize the graph for an arbitrarily fine resolution of the grid, then we get in the limit a transformation $f^\star$ that maps A to B.

In the case of homothets (Section 6), the transformation $f^\star$ is an arbitrary affine transformation, which leads to Theorem 1. In the case of similarities (Section 7), we can further prove that the grid must be square-shaped. It then follows that the limit transformation $f^\star$ is angle preserving, so B must be similar to A or A^{r}.

The construction of this grid is rather delicate and relies on a careful analysis of various building blocks described in Section 4. Our first basic tool (Lemma 9) is that if the complete bipartite graph $K_{2,n}$ is realized as an intersection graph of similarities of a convex disk A, then the distance between the two disks U_1 and U_2 in the first vertex class can be made arbitrarily much smaller than the size of U_1 and U_2 by choosing n large enough. In other words, in the limit where $n \to \infty$, the two disks U_1 and U_2 behave as if they were in contact.

We are then able to define a larger graph L_n where a realization has two disks U_1, U_2 and n disks $V_1, \ldots, V_n$, such that by choosing n large enough, we know that all of the latter disks are arbitrarily small compared to both of U_1 and U_2 (Lemma 11), and they must furthermore be "squeezed in" between these disks. The disks in each row and each column of the aforementioned grid in the final construction will be a subset of the disks $V_1, \ldots, V_n$ in a realization of this graph L_n. Here, it is necessary to place chains of overlapping disks on top of each row and each column of the grid to ensure that when the grid becomes arbitrarily fine, it does not degenerate into a segment.

In the case of similarities, we introduce the concept of the *stretch* of a convex disk A, denoted ρ_A. We consider two parallel lines of distance 1 and a chain of n consecutively overlapping similarities of A, contained in the strip bounded by these lines. The stretch is the ratio between the (geometric) length of a longest such chain and n, as $n \to \infty$. Now if $\rho_B < \rho_A$, then it will be impossible for similarities of B to realize the graph that we construct for A, as there is no chain of similarities of B that can "reach far enough". If $\rho_B = \rho_A$, then for both A and B the graph can be realized only so that the grid is square-shaped, since otherwise some chains in the realizations will not be able to reach far enough.

We conclude the paper in Section 8 by mentioning some open questions.

2 Preliminaries

Let $\operatorname{int} X$ and ∂X denote the interior and the boundary of a set $X \subseteq \mathbb{R}^2$, respectively. A *convex disk* is a convex compact subset of $\mathbb{R}^2$ with non-empty interior. Every convex disk is the closure of its interior. Two non-empty subsets of $\mathbb{R}^2$ *touch* if they intersect but the interior of either one is disjoint from the other. A *tangent* to a convex disk A is a line that touches A (whence it follows that A lies in one of the two half-planes bounded by the line). For every convex disk A and every point $p \in \partial A$, there is at least one tangent to A containing p. A convex disk A is *smooth* if for every point $p \in \partial A$, there is exactly one tangent to A containing p. All convex disks that we consider are implicitly assumed to be smooth.

A *similarity* of a convex disk A is a rotated, scaled, and translated copy of A, that is, a set of the form

$$A' = \left\{ r \cdot \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} a + z \colon a \in A \right\},$$

where $r > 0$, $z \in \mathbb{R}^2$, and $\theta \in [0, 2\pi)$. We call r the *radius* of A' and denote it by $r_A(A')$. When A is clear from the context, we simplify the notation to $r(A')$. A similarity A' is a *homothet* of A if $\theta = 0$, that is, A' is a scaled and translated copy of A. We let $\operatorname{sim} A$ and $\operatorname{hom} A$ denote the set of similarities and the set of homothets of A, and we let $\operatorname{sim}^{\mathrm{r}} A = \operatorname{sim} A \cup \operatorname{sim} A^{\mathrm{r}}$, where A^{r} is the reflection of A about the y axis: $A^{\mathrm{r}} = \{(-x, y) \colon (x, y) \in A\}$.

A *realization* of a graph $G = (V, E)$ in a family $\mathcal{F}$ of subsets of $\mathbb{R}^2$ is a mapping $R \colon V \to \mathcal{F}$ such that $R(u) \cap R(v) \neq \emptyset$ if and only if $uv \in E$. We consider only finite graphs and their realizations with $\mathcal{F} = \operatorname{sim} A$ or $\mathcal{F} = \operatorname{hom} A$ for some convex disk A.

The Euclidean norm of a vector $a \in \mathbb{R}^2$ is denoted by $\|a\|$. The Euclidean distance between points $p, q \in \mathbb{R}^2$ is denoted by $\operatorname{dist}(p, q)$. This notation extends to the distance between a point $p \in \mathbb{R}^2$ and a set $X \subseteq \mathbb{R}^2$ or between two sets $X, Y \subseteq \mathbb{R}^2$:

$$\operatorname{dist}(p, X) = \inf_{x \in X} \operatorname{dist}(p, x), \qquad \operatorname{dist}(X, Y) = \inf_{x \in X} \inf_{y \in Y} \operatorname{dist}(x, y).$$

For a point $q \in \mathbb{R}^2$ and $\delta > 0$, let $\operatorname{ball}(q, \delta) = \{p \in \mathbb{R}^2 \colon \operatorname{dist}(p, q) \leqslant \delta\}$. For a compact set $X \subseteq \mathbb{R}^2$ and $\delta > 0$, let $\operatorname{ball}(X, \delta) = \{p \in \mathbb{R}^2 \colon \operatorname{dist}(p, X) \leqslant \delta\}$. The diameter of a set $X \subseteq \mathbb{R}^2$, which is $\sup_{x,y \in X} \operatorname{dist}(x, y)$, is denoted by $\operatorname{diam} X$. The *bounding box* of a compact set $X \subset \mathbb{R}^2$ is the unique minimal box of the form $[x_1, x_2] \times [y_1, y_2]$ containing X. Let $\mathbb{N} = \{1, 2, \ldots\}$ and $[n] = \{1, \ldots, n\}$ for $n \in \mathbb{N}$.

3 Convergence and limits

Recall the notion of Hausdorff distance between non-empty subsets X and Y of a metric space:

$$d_H(X, Y) = \max\left\{ \sup_{x \in X} \operatorname{dist}(x, Y),\ \sup_{y \in Y} \operatorname{dist}(y, X) \right\}.$$

It is well known that the family of non-empty compact subsets of a (compact) metric space equipped with this notion of distance forms a (compact) metric space. This leads to a notion of *convergence* of a sequence of non-empty compact subsets of $\mathbb{R}^2$ to a non-empty compact subset of $\mathbb{R}^2$ *in Hausdorff distance*. If a sequence of non-empty compact convex subsets of $\mathbb{R}^2$ converges in Hausdorff distance, then its limit is also convex. We need to extend the notion of convergence in Hausdorff distance by allowing the limit object to be an unbounded closed subset of $\mathbb{R}^2$ while assuming convexity of the members of the sequence.

A pair $(p, r) \in \mathbb{R}^2 \times \mathbb{R}_+$ is an *anchor* for a sequence $(X^n)_{n=1}^{\infty}$ of non-empty compact convex subsets of $\mathbb{R}^2$ if $\operatorname{dist}(p, X^n) \leqslant r$ for every $n \in \mathbb{N}$. A sequence of non-empty compact convex subsets of $\mathbb{R}^2$ is *anchored* if it has an anchor. We say that an anchored sequence $(X^n)_{n=1}^{\infty}$ of non-empty compact convex subsets of $\mathbb{R}^2$ *converges* to a set $X^\star \subseteq \mathbb{R}^2$ (and write $X^n \to X^\star$), and we call $X^\star$ the *limit* of $(X^n)_{n=1}^{\infty}$, if for every anchor (p, r) for it, the sequence $(X^n \cap \operatorname{ball}(p, r))_{n=1}^{\infty}$ converges to $X^\star \cap \operatorname{ball}(p, r)$ in Hausdorff distance. Since the latter limit is unique, so is the limit $X^\star = \bigcup_{(p,r)} (X^\star \cap \operatorname{ball}(p, r))$, where the union is taken over all anchors (p, r) for $(X^n)_{n=1}^{\infty}$. It is easy to see that the limit $X^\star$ is a closed convex set.

The following lemmas assert basic properties of this extended notion of convergence. See the full version of the paper for the proofs that are missing from the current version.

▶ **Lemma 3.** *If $(X^n)_{n=1}^{\infty}$ is a sequence of non-empty compact convex subsets of $\mathbb{R}^2$ with anchor (p, r) that converges to a set $X^\star \subseteq \mathbb{R}^2$ in Hausdorff distance, then the sequence $(X^n \cap \operatorname{ball}(p, r))_{n=1}^{\infty}$ converges to $X^\star \cap \operatorname{ball}(p, r)$ in Hausdorff distance.*

▶ **Lemma 4.** *Every anchored sequence of non-empty compact convex subsets of $\mathbb{R}^2$ has a convergent subsequence.*

▶ **Lemma 5.** *Let A be a convex disk and $\mathcal{F} = \operatorname{hom} A$ or $\mathcal{F} = \operatorname{sim} A$. Let $(X^n)_{n=1}^{\infty}$ be a sequence of members of $\mathcal{F}$ that converges to a set $X^\star \subseteq \mathbb{R}^2$. Then the sequence $(r(X^n))_{n=1}^{\infty}$ converges or diverges to ∞. Furthermore,*

- *if $r(X^n) \to r^\star \in \mathbb{R}$, where $r^\star > 0$, then $X^\star \in \mathcal{F}$,*
- *if $r(X^n) \to 0$, then $X^\star = \{z^\star\}$ for some point $z^\star \in \mathbb{R}^2$,*
- *if $r(X^n) \to \infty$, then $X^\star$ is a half-plane or $X^\star = \mathbb{R}^2$.*

▶ **Lemma 6.** *Let A be a convex disk and $\mathcal{F} = \operatorname{hom} A$ or $\mathcal{F} = \operatorname{sim} A$. For every set $X^\star$ that is a member of $\mathcal{F}$ or a half-plane, there is a sequence $(X^n)_{n=1}^{\infty}$ of members of $\mathcal{F}$ that converges to $X^\star$ and satisfies $X^n \subset \operatorname{int} X^\star$ for every $n \in \mathbb{N}$.*

An *interior-realization* of a graph $G = (V, E)$ in a family $\bar{\mathcal{F}}$ of subsets of $\mathbb{R}^2$ is a mapping $\bar{R}\colon V \to \bar{\mathcal{F}}$ such that $\operatorname{int} \bar{R}(u) \cap \operatorname{int} \bar{R}(v) \neq \emptyset$ if and only if $uv \in E$. Our main construction in Section 5 is easier to present in terms of interior-realizations rather than realizations, and the following lemma turns an interior-realization into a realization.

▶ **Lemma 7.** *Let A be a convex disk, $\mathcal{F} = \operatorname{hom} A$ or $\mathcal{F} = \operatorname{sim} A$, and $\mathcal{H}$ be the family of all half-planes. If a graph G has an interior-realization in $\mathcal{F} \cup \mathcal{H}$, then G has a realization in $\mathcal{F}$.*

Proof. Let $G = (V, E)$, and let $\bar{R}$ be an interior-realization of G in $\mathcal{F} \cup \mathcal{H}$. Let $p_{uv} \in \operatorname{int} \bar{R}(u) \cap \operatorname{int} \bar{R}(v)$ for every edge $uv \in E$. Let mappings $R^n\colon V \to \mathcal{F}$ for $n \in \mathbb{N}$ be such that the sequence $(R^n(v))_{n=1}^{\infty}$ converges to $\bar{R}(v)$ for every $v \in V$ and $R^n(v) \subset \operatorname{int} \bar{R}(v)$ for all $v \in V$ and $n \in \mathbb{N}$; they exist by Lemma 6. It follows that $R^n(u) \cap R^n(v) \neq \emptyset$ implies $\operatorname{int} \bar{R}(u) \cap \operatorname{int} \bar{R}(v) \neq \emptyset$ and thus $uv \in E$, for all $n \in \mathbb{N}$. If $n \in \mathbb{N}$ is sufficiently large that $p_{uv} \in R^n(u) \cap R^n(v)$ for every edge $uv \in E$, then R^n is a realization of G in $\mathcal{F}$. ◀

4 Basic configurations

Let $K_{m,n}$ denote the complete bipartite graph with vertices $u_1, \ldots, u_m$ on one side and $v_1, \ldots, v_n$ on the other side, so that $u_i v_j$ is an edge of $K_{m,n}$ for all $i \in [m]$ and $j \in [n]$. The following lemma is proved by a simple area argument.

▶ **Lemma 8.** *For every convex disk A and every $\varepsilon > 0$, if n is sufficiently large, then every realization R of $K_{1,n}$ in $\operatorname{sim} A$ satisfies $\min_{i \in [n]} r(R(v_i)) < \varepsilon r(R(u_1))$.*

▶ **Lemma 9.** *Let A be a convex disk and N be an infinite subset of $\mathbb{N}$. For every sequence $(R^n)_{n \in N}$ such that R^n is a realization of $K_{2,n}$ in $\operatorname{sim} A$ and $R^n(u_1)$ converges to a convex disk or singleton set $U_1^\star$, the sequence $(R^n(u_2))_{n \in N}$ is anchored and for all of its convergent subsequences, the limit touches $U_1^\star$.*

Proof. When $n \to \infty$, since $r(R^n(u_1)) \to r(U_1^\star)$, Lemma 8 yields $\operatorname{dist}(R^n(u_1), R^n(u_2)) \leqslant \min_{i \in [n]} \operatorname{diam} R^n(v_i) = \min_{i \in [n]} r(R^n(v_i)) \cdot \operatorname{diam} A \to 0$, which implies $\operatorname{dist}(U_1^\star, R^n(u_2)) \leqslant \operatorname{dist}(R^n(u_1), R^n(u_2)) + d_H(R^n(u_1), U_1^\star) \to 0$, and the lemma follows. ◀

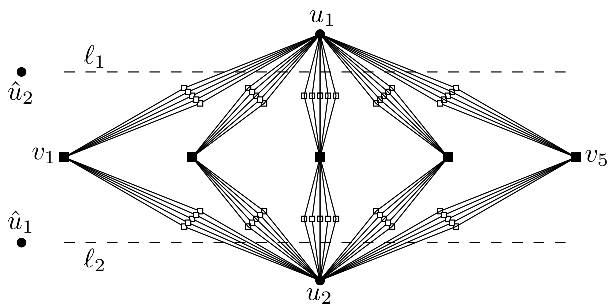

Figure 3 The graph L_5. Here, $\hat{u}_1$ has an edge to all vertices above the line ℓ_2, and $\hat{u}_2$ has an edge to all vertices below ℓ_1.

▶ **Construction 10** (the graph L_n)**.** The graph L_n has vertices $u_1, u_2, v_1, \ldots, v_n$, vertices w_{ijk} and edges $u_i w_{ijk}, w_{ijk} v_j$ for all $i \in [2]$ and $j, k \in [n]$ (so that $u_i, v_j, w_{ij1}, \ldots, w_{ijn}$ form a copy of $K_{2,n}$), and two additional vertices $\hat{u}_1, \hat{u}_2$ such that $\hat{u}_1$ has an edge to every vertex except u_2 and $\hat{u}_2$ has an edge to every vertex except u_1. See Figure 3.

When considering a specific realization R of L_n (possibly with a superscript), we write V_i, U_i, and $\hat{U}_i$ (with the same superscript) as shorthand for $R(v_i)$, $R(u_i)$, and $R(\hat{u}_i)$, respectively. The following lemma makes essential use of the assumption that A is smooth.

▶ **Lemma 11.** *For every convex disk A and every $\varepsilon > 0$, if n is sufficiently large, then every realization of L_n in $\operatorname{sim} A$ satisfies $\max_{j\in[n]} r(V_j) \leqslant \varepsilon \min\{r(U_1), r(U_2)\}$.*

Proof. Suppose for the sake of contradiction that there is $\varepsilon > 0$ such that for every n, there is a realization R^n of L_n in $\operatorname{sim} A$ such that $\max_{j\in[n]} r(V_j^n) > \varepsilon \min\{r(U_1^n), r(U_2^n)\}$. Assume without loss of generality that $r(U_1^n) \leqslant r(U_2^n)$ for all n. Furthermore, assume that U_1^n is constant (equal to U_1) while the other disks may change size and placement as a function of n.

Suppose there is $\rho > 0$ such that $\min_{i\in[n]} r(V_i^n) \geqslant \rho$ for every n. Let $k \in \mathbb{N}$. By Lemma 9, we can pass to a subsequence of $(R^n)_{n=k}^{\infty}$ in which $V_i^n \to V_i^\star$ and $V_i^\star$ touches U_1 for every $i \in [k]$. At least $k-2$ of these limits, say $V_1^\star, \ldots, V_{k-2}^\star$, are not half-planes. Along with U_1, they form a realization of $K_{1,k-2}$ in $\operatorname{sim} A$. When k is sufficiently large, Lemma 8 yields $\min_{i\in[k-2]} r(V_i^\star) < \rho$. This contradiction shows that $\min_{i\in[n]} r(V_i^n) \to 0$ as $n \to \infty$.

For each n, let $V_{\min}^n$ and $V_{\max}^n$ be disks among $V_1^n, \ldots, V_n^n$ with minimum and maximum radii, respectively, so that $r(V_{\max}^n) > \varepsilon r(U_1)$ and $r(V_{\min}^n) \to 0$ as $n \to \infty$. See Figure 4. Considering $n \to \infty$ and passing to a subsequence, by Lemmas 5 and 9, we can assume that

- $V_{\min}^n$ converges to a singleton set $\{p\}$, where $p \in \partial U_1$,
- U_2^n converges to a member of $\operatorname{sim} A$ or half-plane $U_2^\star$ that touches U_1 at p,
- $\hat{U}_1^n$ converges to a limit $\hat{U}_1^\star$ that touches $U_2^\star$ at p, as $p \in \hat{U}_1^\star$ and $\operatorname{int}(\hat{U}_1^\star \cap U_2^\star) = \emptyset$,
- $\hat{U}_2^n$ converges to a limit $\hat{U}_2^\star$ that touches U_1 at p, as $p \in \hat{U}_2^\star$ and $\operatorname{int}(U_1 \cap \hat{U}_2^\star) = \emptyset$,
- $V_{\max}^n$ converges to a member of $\operatorname{sim} A$ or half-plane $V_{\max}^\star$ that touches both U_1 and $U_2^\star$.

It follows that the unique line tangent to both U_1 and $U_2^\star$ at p splits the plane into two half-planes H_1 and H_2 such that $U_1, \hat{U}_1^\star \subseteq H_1$ and $U_2^\star, \hat{U}_2^\star \subseteq H_2$.

Suppose that at least one of $U_2^\star$, $V_{\max}^\star$ is a member of $\operatorname{sim} A$. By Lemma 8, there are disks W_1^n and W_2^n (members of $\operatorname{sim} A$) such that

- W_1^n intersects $V_{\max}^n$, U_1, and $\hat{U}_2^n$,
- W_2^n intersects $V_{\max}^n$, U_2^n, and $\hat{U}_1^n$,
- $r(W_1^n) \to 0$ and $r(W_2^n) \to 0$ as $n \to \infty$.

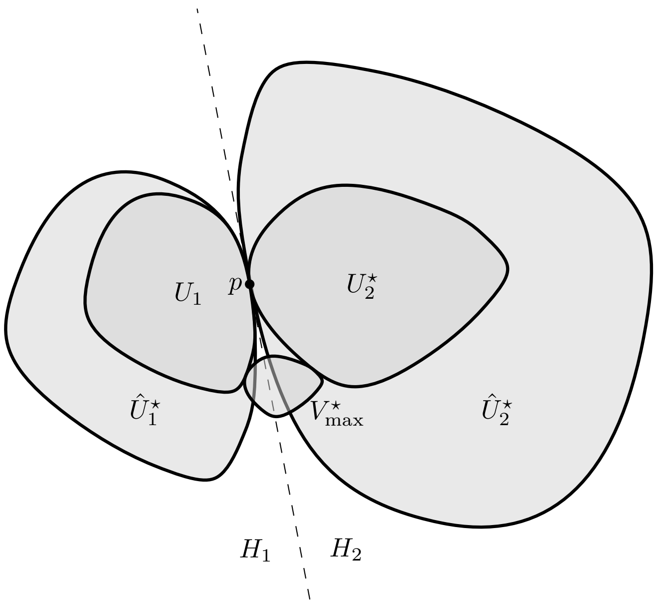

Figure 4 Situation from the proof of Lemma 11.

Considering $n \to \infty$ and passing to a subsequence, we can assume that $W_1^n \to \{q_1\}$ and $W_2^n \to \{q_2\}$, where $q_1 \in V_{\max}^\star \cap U_1 \cap \hat{U}_2^\star$ and $q_2 \in V_{\max}^\star \cap \hat{U}_1^\star \cap U_2^\star$. It follows that $V_{\max}^\star$ touches U_1 at q_1 and $U_2^\star$ at q_2, whereas both q_1 and q_2 lie on the boundary line between H_1 and H_2. This is possible only when $V_{\max}^\star = \{q_1\} = \{q_2\}$, which is a contradiction.

Now, suppose that both $U_2^\star$ and $V_{\max}^\star$ are half-planes (in particular $U_2^\star = H_2$). It follows that they are disjoint half-planes (as they must have disjoint interiors), while $\hat{U}_2^\star \subseteq H_2 = U_2^\star$, so $V_{\max}^\star$ and $\hat{U}_2^\star$ are disjoint, which is again a contradiction. ◀

▶ **Lemma 12.** *Let A be a convex disk and N be an infinite subset of $\mathbb{N}$. For each $n \in N$, let L'_n be a graph which contains, as induced subgraphs, L_n and a fixed connected graph H containing v_1 such that u_1 and u_2 have no edges to any vertex of H. Let $(R^n)_{n\in N}$ be a sequence such that R^n is a realization of L'_n in $\operatorname{sim} A$ for $n \in N$ and V_1^n converges to a convex disk $V_1^\star$. Then $(R^n)_{n\in N}$ has a subsequence in which*

- *U_1^n and U_2^n converge to disjoint half-planes $U_1^\star$ and $U_2^\star$,*
- *$\hat{U}_1^n$ and $\hat{U}_2^n$ converge to limits that touch $U_2^\star$ and $U_1^\star$, respectively,*
- *for every vertex w of H, $R^n(w)$ converges to a convex disk or singleton set.*

Proof sketch. By Lemma 9, the sequences $(U_1^n)_{n\in N}$ and $(U_2^n)_{n\in N}$ are anchored, and so are the sequences $(\hat{U}_1^n)_{n\in N}$ and $(\hat{U}_2^n)_{n\in N}$, so we can pass to a subsequence in which they converge to limits $U_1^\star$, $U_2^\star$, $\hat{U}_1^\star$, and $\hat{U}_2^\star$, respectively. Moreover, by Lemma 9, $U_1^\star$ touches $V_1^\star$ and $\hat{U}_2^\star$ at a common point, and $U_2^\star$ touches $V_1^\star$ and $\hat{U}_1^\star$ at a common point. By Lemma 11, $r(U_1^n) \to \infty$ and $r(U_2^n) \to \infty$, so $U_1^\star$ and $U_2^\star$ are disjoint half-planes. Simple induction shows that we can further pass to a subsequence in which $R^n(w)$ converges to a convex disk or singleton set for every vertex w of H. ◀

5 Main construction

An *n-chain* aligned to parallel lines ℓ_1, ℓ_2 is an n-tuple $A_1, \dots, A_n$ of convex disks all touching ℓ_1 and ℓ_2 and such that $A_i \cap A_{i+1} \neq \emptyset$ for all $i \in [n-1]$. The *length* of such an n-chain is the length of the orthogonal projection of $A_1 \cup \dots \cup A_n$ on ℓ_1 (or ℓ_2) divided by $\operatorname{dist}(\ell_1, \ell_2)$.

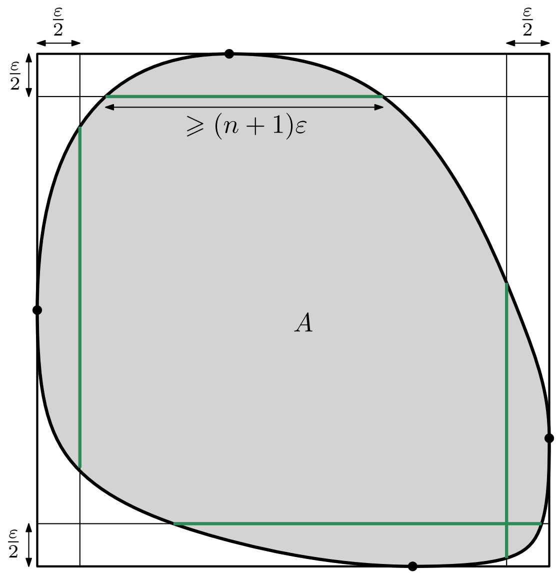

Figure 5 Lemma 13 asserts that for every $n \in \mathbb{N}$, if $\varepsilon > 0$ is sufficiently small, then the lengths of the four green segments are at least $(n+1)\varepsilon$.

Such an n-chain is *strict* if $\operatorname{int}(A_i \cap A_{i+1}) \neq \emptyset$ for all $i \in [n-1]$. A *horizontal* or *vertical n-chain* is an n-chain aligned to horizontal or vertical lines, respectively. Before using the n-chains to construct the key graph of our proof, we need the following lemma, which relies on the assumption that A is smooth; see Figure 5 for an illustration.

▶ **Lemma 13.** *For every convex disk A with bounding box $[0,1]^2$ and every $n \in \mathbb{N}$, there is $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, the lengths of the four segments $A \cap (\mathbb{R} \times \{\frac{\varepsilon}{2}\})$, $A \cap (\mathbb{R} \times \{1 - \frac{\varepsilon}{2}\})$, $A \cap (\{\frac{\varepsilon}{2}\} \times \mathbb{R})$, and $A \cap (\{1 - \frac{\varepsilon}{2}\} \times \mathbb{R})$ are at least $(n+1)\varepsilon$.*

For an illustration of the following construction, see Figure 6.

▶ **Construction 14** (the graph $G^{A,\mathcal{F}}_{mn}$)**.** Let A be a convex disk with bounding box $[0,1]^2$. Let $\mathcal{F} = \hom A$ or $\mathcal{F} = \operatorname{sim} A$. Let $m, n \in \mathbb{N}$ with $m \leqslant n$. Let $k \in \mathbb{N}$ be minimal such that there exist a strict horizontal k-chain and a strict vertical k-chain in $\mathcal{F}$ of length greater than m. Let $\varepsilon > 0$ be as in Lemma 13 for A and n. The graph $G^{A,\mathcal{F}}_{mn}$ has the following vertices and the following interior-realization $\bar{R}$ by members of $\mathcal{F}$ and half-planes:

- $\bar{R}(v_{ij}) = \frac{1}{m}A + (\frac{i-1}{m}, \frac{j-1}{m})$ for $(i,j) \in ([n] \times [m]) \cup ([m] \times [n])$,
- $\bar{R}(u_{1j}) = \mathbb{R} \times (-\infty, \frac{j-1}{m}]$ for $j = 1, \ldots, m+1$ and $\bar{R}(u_{2j}) = \mathbb{R} \times [\frac{j}{m}, +\infty)$ for $j = 0, \ldots, m$,
- $\bar{R}(\bar{u}_{i1}) = (-\infty, \frac{i-1}{m}] \times \mathbb{R}$ for $i = 1, \ldots, m+1$ and $\bar{R}(\bar{u}_{i2}) = [\frac{i}{m}, +\infty) \times \mathbb{R}$ for $i = 0, \ldots, m$,
- $\bar{R}(z_{1j}), \ldots, \bar{R}(z_{kj})$ that form a strict horizontal k-chain in $\mathcal{F}$ with bounding box $[-\delta, 1+\delta] \times [\frac{j-1}{m}, \frac{j}{m}]$ for $j = 1, \ldots, m$ and some sufficiently small $\delta > 0$,
- $\bar{R}(\bar{z}_{i1}), \ldots, \bar{R}(\bar{z}_{ik})$ that form a strict vertical k-chain in $\mathcal{F}$ with bounding box $[\frac{i-1}{m}, \frac{i}{m}] \times [-\delta, 1+\delta]$ for $i = 1, \ldots, m$ and some sufficiently small $\delta > 0$,
- $\bar{R}(w) = A$,
- $\bar{R}(x_{ij}) = \frac{\varepsilon}{m}A + (\frac{\varepsilon i}{m}, \frac{j}{m} - \frac{\varepsilon}{2m})$ for $i = 0, \ldots, \lceil\frac{n}{\varepsilon}\rceil - 1$ and $j = 0, \ldots, m$,
- $\bar{R}(\bar{x}_{ij}) = \frac{\varepsilon}{m}A + (\frac{i}{m} - \frac{\varepsilon}{2m}, \frac{\varepsilon j}{m})$ for $i = 0, \ldots, m$ and $j = 0, \ldots, \lceil\frac{n}{\varepsilon}\rceil - 1$.

By Lemma 7, $G^{A,\mathcal{F}}_{mn}$ has a realization in $\mathcal{F}$. When considering a specific realization R of $G^{A,\mathcal{F}}_{mn}$ (possibly with a superscript), we write V_{ij}, U_{ij}, $\bar{U}_{ij}$, Z_{ij}, $\bar{Z}_{ij}$, and W (with the same superscript) as shorthand for $R(v_{ij})$, $R(u_{ij})$, $R(\bar{u}_{ij})$, $R(z_{ij})$, $R(\bar{z}_{ij})$, and $R(w)$, respectively.

For $m \in \mathbb{N}$ and $i, j \in [m]$, let $S^m_{ij} = [\frac{i-1}{m}, \frac{i}{m}] \times [\frac{j-1}{m}, \frac{j}{m}]$. The following lemma asserts basic properties of Construction 14.

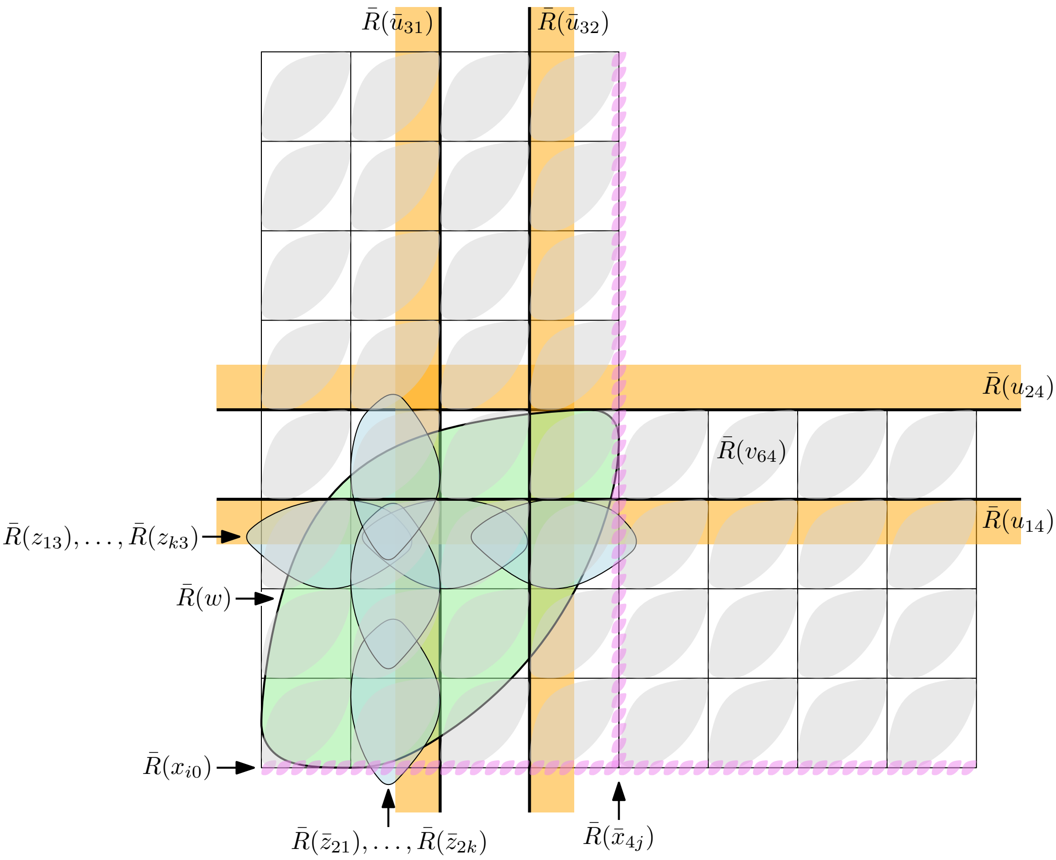

Figure 6 The interior-realization of the graph $G_{48}^{A,\mathrm{sim}}$. The figure is not to scale; in reality, the pink disks $\bar{R}(x_{ij})$ and $\bar{R}(\bar{x}_{ij})$ would be much smaller (and thus more numerous).

▶ **Lemma 15.** *Let $A, \mathcal{F}, m, n, k$ be as in Construction 14. The following hold for $G_{mn}^{A,\mathcal{F}}$:*

1. *For every $j \in [m]$, there is an induced subgraph isomorphic to L_n in which the vertices $u_{1j}, u_{2j}, u_{1(j+1)}, u_{2(j-1)}$, and $v_{1j}, \ldots, v_{nj}$ play the roles of $u_1, u_2, \hat{u}_1, \hat{u}_2$, and $v_1, \ldots, v_n$, respectively; for every $i \in [m]$, there is an induced subgraph isomorphic to L_n in which the vertices $\bar{u}_{i1}, \bar{u}_{i2}, \bar{u}_{(i+1)1}, \bar{u}_{(i-1)2}$, and $v_{i1}, \ldots, v_{in}$ play the roles of $u_1, u_2, \hat{u}_1, \hat{u}_2$, and $v_1, \ldots, v_n$, respectively.*
2. *For every $j \in [m]$, the subgraph induced on $v_{1j}, \ldots, v_{mj}, z_{1j}, \ldots, z_{kj}$ is connected and contains a path $z_{1j} \cdots z_{kj}$; for every $i \in [m]$, the subgraph induced on $v_{i1}, \ldots, v_{im}, \bar{z}_{i1}, \ldots, \bar{z}_{ik}$ is connected and contains a path $\bar{z}_{i1} \cdots \bar{z}_{ik}$.*
3. *The vertices $z_{11}, \ldots, z_{1m}$ are adjacent to $\bar{u}_{11}$, the vertices $z_{k1}, \ldots, z_{km}$ are adjacent to $\bar{u}_{m2}$, the vertices $\bar{z}_{11}, \ldots, \bar{z}_{m1}$ are adjacent to u_{11}, and the vertices $\bar{z}_{1k}, \ldots, \bar{z}_{mk}$ are adjacent to u_{2m}.*
4. *The vertex w is adjacent to at least one of $z_{1j}, \ldots, z_{kj}$ for every $j \in [m]$ and at least one of $\bar{z}_{i1}, \ldots, \bar{z}_{ik}$ for every $i \in [m]$; for every $u \in \{u_{11}, u_{2j}, \bar{u}_{11}, \bar{u}_{i2}\}$, there is an induced subgraph isomorphic to $K_{2,n}$ in which the vertices u and w form one of the parts of the bipartition.*
5. *For all $i, j \in [m]$, if $S_{ij}^m \subseteq A$, then $v_{ij}w$ is an edge, and if $v_{ij}w$ is an edge, then $S_{ij}^m \cap A \neq \emptyset$.*

An *m-grid* is a collection of two $(m+1)$-tuples of parallel lines $\ell_0, \ell_1, \ldots, \ell_m$ and $\bar{\ell}_0, \bar{\ell}_1, \ldots, \bar{\ell}_m$ that are images of horizontal lines at coordinates $0 = y_0 < y_1 < \cdots < y_m = 1$ and $m+1$ vertical lines at coordinates $0 = x_0 < x_1 < \cdots < x_m = 1$, respectively, under an

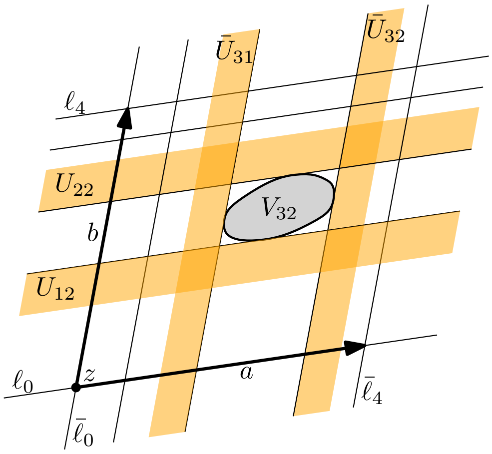

Figure 7 An example of a 4-grid with aligned disks and half-planes.

affine transformation $f\colon \mathbb{R}^2 \ni (x,y) \mapsto z + xa + yb \in \mathbb{R}^2$ for some point $z \in \mathbb{R}^2$ called the *origin* of the m-grid and some linearly independent vectors $a, b \in \mathbb{R}^2$ that form the *basis* of the m-grid; see Figure 7. The differences $x_1 - x_0, \dots, x_m - x_{m-1}$ and $y_1 - y_0, \dots, y_m - y_{m-1}$ are the *horizontal* and *vertical distances* of the m-grid, respectively. A configuration of convex disks V_{ij} with $i, j \in [m]$ and half-planes $U_{11}, U_{21} \dots, U_{1m}, U_{2m}, \bar{U}_{11}, \bar{U}_{12}, \dots, \bar{U}_{m1}, \bar{U}_{m2}$ is *aligned* to such an m-grid if the following holds:

- $U_{1j} = f(\mathbb{R} \times (-\infty, y_{j-1}])$ and $U_{2j} = f(\mathbb{R} \times [y_j, +\infty))$ for $j \in [m]$,
- $\bar{U}_{i1} = f((-\infty, x_{i-1}] \times \mathbb{R})$ and $\bar{U}_{i2} = f([x_i, +\infty) \times \mathbb{R})$ for $i \in [m]$,
- V_{ij} touches the four half-planes $U_{1j}, U_{2j}, \bar{U}_{i1}, \bar{U}_{i2}$ for $i, j \in [m]$.

The following lemma is at the heart of our argument. Among other things, it asserts that in realizations of the graph $G^{A,\mathcal{F}}_{mn}$, the disks V^n_{ij}, for $i, j \in [m]$, are indeed forced to form an aligned m-grid as $n \to \infty$. This will be the foundation for the proofs of Theorems 1 and 2.

▶ **Lemma 16.** *Let A and B be convex disks such that A has bounding box $[0,1]^2$. Let $\mathcal{F} = \operatorname{hom} A$ or $\mathcal{F} = \operatorname{sim} A$. Let $m \in \mathbb{N}$. Let $k \in \mathbb{N}$ be minimal such that there exist a strict horizontal k-chain and a strict vertical k-chain in $\mathcal{F}$ of length greater than m. Every sequence $(R^n)^\infty_{n=m}$ such that R^n is a realization of $G^{A,\mathcal{F}}_{mn}$ in $\operatorname{sim} B$ and V^n_{11} is constant has a subsequence in which the disks V^n_{ij} with $i, j \in [m]$, U^n_{1j}, U^n_{2j} with $j \in [m]$, and $\bar{U}^n_{i1}, \bar{U}^n_{i2}$ with $i \in [m]$ converge to convex disks $V^\star_{ij}$ and half-planes $U^\star_{1j}, U^\star_{2j}$ and $\bar{U}^\star_{i1}, \bar{U}^\star_{i2}$, respectively, that are aligned to an m-grid, and the disks $Z^n_{1j}, \dots, Z^n_{kj}$ with $j \in [m]$, $\bar{Z}^n_{i1}, \dots, \bar{Z}^n_{ik}$ with $i \in [m]$, and W^n converge to convex disks $Z^\star_{1j}, \dots, Z^\star_{kj}$, $\bar{Z}^\star_{i1}, \dots, \bar{Z}^\star_{ik}$, and $W^\star$, respectively, where $W^\star$ touches $U^\star_{11}, U^\star_{2m}, \bar{U}^\star_{11}, \bar{U}^\star_{m2}$.*

Proof. Let $(R^n)^\infty_{n=m}$ be a sequence of realizations R^n of $G^{A,\mathcal{F}}_{mn}$ in $\operatorname{sim} B$ such that V^n_{11} is constant. By Lemma 15 (1 and 2), we can apply Lemma 12 repeatedly as follows, in order:

- with vertices $u_{11}, u_{21}, u_{12}, u_{20}$, and $v_{11}, \dots, v_{n1}$ playing the roles of $u_1, u_2, \hat{u}_1, \hat{u}_2$, and $v_1, \dots, v_n$ (respectively) in L_n, and with the graph H formed by $v_{11}, \dots, v_{m1}, z_{11}, \dots, z_{k1}$,
- for each $i \in [m]$, with vertices $\bar{u}_{i1}, \bar{u}_{i2}, \bar{u}_{(i+1)1}, \bar{u}_{(i-1)2}$, and $v_{i1}, \dots, v_{in}$ playing the roles of $u_1, u_2, \hat{u}_1, \hat{u}_2$, and $v_1, \dots, v_n$ (respectively) in L_n, and with the graph H formed by $v_{i1}, \dots, v_{im}, \bar{z}_{i1}, \dots, \bar{z}_{ik}$,
- for each $j \in [m] \smallsetminus \{1\}$, with vertices $u_{1j}, u_{2j}, u_{1(j+1)}, u_{2(j-1)}$, and $v_{1j}, \dots, v_{nj}$ playing the roles of $u_1, u_2, \hat{u}_1, \hat{u}_2$, and $v_1, \dots, v_n$ (respectively) in L_n, and with the graph H formed by $v_{1j}, \dots, v_{mj}, z_{1j}, \dots, z_{kj}$.

This yields a subsequence in which the disks V_{ij}^n with $i,j \in [m]$, $U_{1j}^n, U_{2j}^n, Z_{1j}^n, \dots, Z_{kj}^n$ with $j \in [m]$, and $\bar{U}_{i1}^n, \bar{U}_{i2}^n, \bar{Z}_{i1}^n, \dots, \bar{Z}_{ik}^n$ with $i \in [m]$ converge to limits $V_{ij}^\star$, $U_{1j}^\star, U_{2j}^\star, Z_{1j}^\star, \dots, Z_{kj}^\star$, and $\bar{U}_{i1}^\star, \bar{U}_{i2}^\star, \bar{Z}_{i1}^\star, \dots, \bar{Z}_{ik}^\star$, respectively, where

- $V_{ij}^\star$ is a convex disk for $i,j \in [m]$,
- $U_{1j}^\star$ and $U_{2j}^\star$ are disjoint half-planes for $j \in [m]$,
- $U_{1(j+1)}^\star$ and $U_{2j}^\star$ touch and therefore share the boundary line, for $j \in [m-1]$,
- $\bar{U}_{i1}^\star$ and $\bar{U}_{i2}^\star$ are disjoint half-planes for $i \in [m]$,
- $\bar{U}_{(i+1)1}^\star$ and $\bar{U}_{i2}^\star$ touch and therefore share the boundary line, for $i \in [m-1]$.

Let

- $\ell_0 = \partial U_{11}^\star$, $\ell_j = \partial U_{1(j+1)}^\star = \partial U_{2j}^\star$ for $j \in [m-1]$, and $\ell_m = \partial U_{2m}^\star$,
- $\bar{\ell}_0 = \partial \bar{U}_{11}^\star$, $\bar{\ell}_i = \partial \bar{U}_{(i+1)1}^\star = \partial \bar{U}_{i2}^\star$ for $i \in [m-1]$, and $\bar{\ell}_m = \partial \bar{U}_{m2}^\star$.

It follows that the lines $\ell_0, \dots, \ell_m$ are parallel and occur in this order, and so do the lines $\bar{\ell}_0, \dots, \bar{\ell}_m$. Consequently, they form an m-grid, the origin of which is the intersection point of ℓ_0 and $\bar{\ell}_0$, and the basis vectors of which are the vectors from the origin to the intersection point of ℓ_0 and $\bar{\ell}_m$ and from the origin to the intersection point of ℓ_m and $\bar{\ell}_0$. Furthermore, Lemma 9 implies that $V_{ij}^\star$ touches $U_{1j}^\star, U_{2j}^\star, \bar{U}_{i1}^\star, \bar{U}_{i2}^\star$ for $i,j \in [m]$. This shows that the disks $V_{ij}^\star$ with $i,j \in [m]$, $U_{1j}^\star, U_{2j}^\star$ with $j \in [m]$, and $\bar{U}_{i1}^\star, \bar{U}_{i2}^\star$ with $i \in [m]$ are aligned to the m-grid.

By Lemma 15 (4), for every n, the vertex w has an edge to at least one of the vertices z_{ij} in $G_{mn}^{A,\mathcal{F}}$ and therefore $W^n \cap Z_{ij}^n \neq \emptyset$. It follows that the sequence $(W^n)_{n \in N}$ (where N comprises the indices of the considered subsequence) is anchored and therefore, passing yet to a subsequence, W^n converges to a limit $W^\star$. Moreover, by Lemma 15 (4) and Lemma 9, $W^\star$ touches $U_{11}^\star, U_{2m}^\star, \bar{U}_{11}^\star, \bar{U}_{m2}^\star$; in particular, it is a convex disk. ◀

6 Classifying intersection graphs of homothets

The proof of Theorem 1 is based on the following lemma.

▶ **Lemma 17.** *Let A and B be convex disks such that A has bounding box $[0,1]^2$. If for all $m,n \in \mathbb{N}$ with $m \leqslant n$, there is a realization of $G_{mn}^{A,\hom A}$ in $\hom B$, then there is an affine transformation that maps A to B.*

Before proving Lemma 17, let us see how Theorem 1 follows.

Proof of Theorem 1. Let A and B be convex disks. As we already observed, if A and B are affine equivalent, then $G^{\hom}(A) = G^{\hom}(B)$, because the affine transformation that maps A to B transforms every realization in $\hom A$ to a realization of the same graph in $\hom B$, and vice versa. Now, suppose $G^{\hom}(A) = G^{\hom}(B)$. We can assume without loss of generality that the bounding box of A is $[0,1]^2$, otherwise we can apply an affine transformation to A to obtain a convex disk with that bounding box; as observed before, such a transformation does not change the intersection graphs realized in $\hom A$. Now, since $G_{mn}^{A,\hom A} \in G^{\hom}(B)$ for all $m,n \in \mathbb{N}$ with $m \leqslant n$, the lemma asserts that A and B are affine equivalent.

The last statement of the theorem asserts that when A and B are not affine equivalent, then the classes of intersection graphs are not nested. Under this assumption, the lemma yields $G_{mn}^{A,\hom A} \notin G^{\hom}(B)$ for some m and n. Using the lemma with A and B interchanged, we also have $G_{mn}^{B,\hom B} \notin G^{\hom}(A)$ for m and n. Therefore, the graph classes are not nested. ◀

Proof of Lemma 17. For all $m,n \in \mathbb{N}$ with $m \leqslant n$, let R^{mn} be a realization of $G_{mn}^{A,\hom A}$ in $\hom B$. We first fix m and consider the sequence of realizations $(R^{mn})_{n=m}^\infty$. Without loss of generality, V_{11}^{mn} is constant in this sequence. By Lemma 16, we can pass to a subsequence such that the disks V_{ij}^{mn} with $i,j \in [m]$, U_{1j}^{mn}, U_{2j}^{mn} with $j \in [m]$, and $\bar{U}_{i1}^{mn}, \bar{U}_{i2}^{mn}$ with $i \in [m]$

converge to disks $V_{ij}^{m\star} \in \hom B$ and half-planes $U_{1j}^{m\star}, U_{21}^{m\star}$ and $\bar{U}_{i1}^{m\star}, \bar{U}_{i2}^{m\star}$, respectively, that are aligned to an m-grid, and the disks W^{mn} converge to a disk $W^{m\star} \in \hom B$. It follows that all $V_{ij}^{m\star}$ with $i, j \in [m]$ have the same radius, so the horizontal and vertical distances of the m-grid are all equal to $\frac{1}{m}$. Without loss of generality, the origin of the m-grid is $(0,0)$ and $r(W^{m\star}) = 1$. Let $a^m, b^m \in \mathbb{R}^2$ be the basis vectors of the m-grid, and let $f^m\colon \mathbb{R}^2 \ni (x,y) \mapsto xa^m + yb^m \in \mathbb{R}^2$. It follows that $V_{ij}^{m\star} \subseteq f^m(S_{ij}^m)$ for $i, j \in [m]$ and $W^{m\star} \subseteq f^m([0,1]^2)$.

Recall that in Construction 14, the edges between w and the vertices v_{ij} with $i, j \in [m]$ are meant to "encode" the shape of A. The following two claims are implied by the existence and non-existence of these edges in the realizations R^{mn}.

▷ Claim 17.1. There is a constant $\eta > 0$ such that $\|a^m\| + \|b^m\| \leqslant \eta$ for all m.

▷ Claim 17.2. For every $\varepsilon > 0$, if m is sufficiently large, then $d_H(W^{m\star}, f^m(A)) \leqslant \varepsilon$, where d_H denotes the Hausdorff distance.

Since $\|a^m\| + \|b^m\| \leqslant \eta$ (by Claim 17.1), we can find an infinite set of indices m such that a^m and b^m converge to vectors $a^\star, b^\star \in \mathbb{R}^2$, respectively, as $m \to \infty$ over that set of indices. Let $f^\star\colon \mathbb{R}^2 \ni (x,y) \mapsto xa^\star + yb^\star \in \mathbb{R}^2$. We show that $W^{m\star} \to f^\star(A)$ in Hausdorff distance. To this end, let $\varepsilon > 0$, and let m be sufficiently large that $d_H(W^{m\star}, f^m(A)) \leqslant \frac{\varepsilon}{2}$ (by Claim 17.2) and $\|a^m - a^\star\| + \|b^m - b^\star\| \leqslant \frac{\varepsilon}{2}$. Since $A \subseteq [0,1]^2$, we have $\operatorname{dist}(f^m((x,y)), f^\star((x,y))) = \|(a^m - a^\star)x + (b^m - b^\star)y\| \leqslant \|a^m - a^\star\| + \|b^m - b^\star\| \leqslant \frac{\varepsilon}{2}$ for every point $(x,y) \in A$, whence it follows that $d_H(f^m(A), f^\star(A)) \leqslant \frac{\varepsilon}{2}$. This yields $d_H(W^{m\star}, f^\star(A)) \leqslant d_H(W^{m\star}, f^m(A)) + d_H(f^m(A), f^\star(A)) \leqslant \varepsilon$. Since $W^{m\star} \to f^\star(A)$, Lemma 5 yields $f^\star(A) \in \hom B$, that is, there is a homothetic transformation $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ that maps B to $f^\star(A)$. We conclude that $h^{-1} \circ f^\star$ is an affine transformation that maps A to B. ◀

7 Classifying intersection graphs of similarities

For a convex disk A and $n \in \mathbb{N}$, we define $\sigma_A(n)$ as the maximum length of an n-chain in $\operatorname{sim} A$. The sequence $(\sigma_A(n))_{n=1}^{\infty}$ is subadditive, that is, $\sigma_A(n_1 + n_2) \leqslant \sigma_A(n_1) + \sigma_A(n_2)$ for all $n_1, n_2 \in \mathbb{N}$. Indeed, in an (n_1+n_2)-chain realizing the value $\sigma_A(n_1+n_2)$, the first n_1 disks form an n_1-chain of length $x_1 \leqslant \sigma_A(n_1)$, and the last n_2 disks form an n_2-chain of length $x_2 \leqslant \sigma_A(n_2)$, whence it follows that $\sigma_A(n_1 + n_2) \leqslant x_1 + x_2 \leqslant \sigma_A(n_1) + \sigma_A(n_2)$. By Fekete's Subadditive Lemma [12], the limit $\lim_{n\to\infty} \sigma_A(n)/n$ exists and is equal to $\inf_{n\in\mathbb{N}} \sigma_A(n)/n$. We call this limit the *stretch* of A and denote it by ρ_A.

▶ **Lemma 18.** *For every $k \in \mathbb{N}$, $\rho_A k \leqslant \sigma_A(k) \leqslant \rho_A k + \sigma_A(1)$.*

The proof of Theorem 2 is based on the following lemma.

▶ **Lemma 19.** *Let A and B be convex disks such that A has bounding box $[0,1]^2$ and $\rho_A \geqslant \rho_B$. If for all $m, n \in \mathbb{N}$ with $m \leqslant n$, there is a realization of $G_{mn}^{A,\operatorname{sim} A}$ in $\operatorname{sim} B$, then $B \in \operatorname{sim}^{\mathrm{r}} A$.*

Before proving the lemma, let us see how Theorem 2 follows.

Proof of Theorem 2. Let A and B be convex disks. As we have already observed, if B is similar to A or to A^{r}, then $G^{\mathrm{sim}}(A) = G^{\mathrm{sim}}(B)$, because the similarity transformation (possibly with reflection) that maps A to B transforms every realization in $\operatorname{sim} A$ to a realization of the same graph in $\operatorname{sim} B$, and vice versa. Now, suppose $G^{\mathrm{sim}}(A) = G^{\mathrm{sim}}(B)$. We can assume without loss of generality that $\rho_A \geqslant \rho_B$. We can further assume that the bounding box of A is $[0,1]^2$, otherwise we can rotate, scale, and translate A to obtain a disk

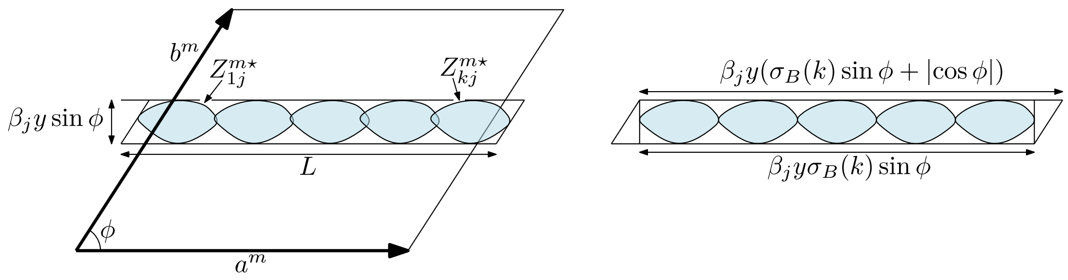

Figure 8 To the left is shown the definition of the length L. To the right is shown a maximum k-chain between two lines of distance $\beta_j y \sin\phi$. It holds that $x = \|a^m\| \leqslant L \leqslant \beta_j y(\sigma_B(k)\sin\phi+|\cos\phi|)$.

with this bounding box, and that transformation does not change the intersection graphs realized in $\operatorname{sim} A$. Since $G_{mn}^{A,\operatorname{sim} A} \in G^{\operatorname{sim}}(B)$ for all $m, n \in \mathbb{N}$ with $m \leqslant n$, we get from the lemma that $B \in \operatorname{sim}^{\mathrm{r}} A$, as claimed. ◀

Proof of Lemma 19. For all $m, n \in \mathbb{N}$ with $m \leqslant n$, let R^{mn} be a realization of $G_{mn}^{A,\operatorname{sim} A}$ in $\operatorname{sim} B$. We first fix m and consider the sequence of realizations $(R^{mn})_{n=m}^{\infty}$. Without loss of generality, V_{11}^{mn} is constant in this sequence. By Lemma 16, we can pass to a subsequence such that the disks V_{ij}^{mn} with $i, j \in [m]$, U_{1j}^{mn}, U_{2j}^{mn} with $j \in [m]$, and $\bar{U}_{i1}^{mn}, \bar{U}_{i2}^{mn}$ with $i \in [m]$ converge to disks $V_{ij}^{m\star} \in \hom B$ and half-planes $U_{1j}^{m\star}, U_{21}^{m\star}$ and $\bar{U}_{i1}^{m\star}, \bar{U}_{i2}^{m\star}$, respectively, that are aligned to an m-grid, the disks Z_{ij}^{mn} and $\bar{Z}_{ij}^{mn}$ converge to disks $Z_{ij}^{m\star} \in \operatorname{sim} B$ and $\bar{Z}_{ij}^{m\star} \in \operatorname{sim} B$, respectively, and the disks W^{mn} converge to a disk $W^{m\star} \in \operatorname{sim} B$ that touches $U_{11}^{\star}, U_{2m}^{\star}, \bar{U}_{11}^{\star}, \bar{U}_{m2}^{\star}$. Without loss of generality, the origin of the m-grid is $(0,0)$ and $r(W^{m\star}) = 1$. Let $a^m, b^m \in \mathbb{R}^2$ be the basis vectors of the m-grid, and let $f^m \colon \mathbb{R}^2 \ni (x, y) \mapsto xa^m + yb^m \in \mathbb{R}^2$. Let $\alpha_1^m, \ldots, \alpha_m^m$ and $\beta_1^m, \ldots, \beta_m^m$ be the horizontal and vertical distances of the m-grid, respectively, where $\sum_{i=1}^m \alpha_i = \sum_{j=1}^m \beta_j = 1$.

▷ Claim 19.1. There is a constant $c > 0$ (which depends only on B) such that for every m, if $x = \|a^m\|$, $y = \|b^m\|$, and $\phi \in (0, \pi)$ is the angle between a^m and b^m, then

$$\frac{x}{y} \leqslant 1 + \frac{c}{m}, \qquad \frac{y}{x} \leqslant 1 + \frac{c}{m}, \qquad \sin\phi \leqslant 1 - \frac{c}{m},$$
$$\frac{i}{m} - \frac{2c}{m} < \alpha_1 + \cdots + \alpha_i < \frac{i}{m} + \frac{2c}{m} \quad \text{for every } i \in [m-1],$$
$$\frac{j}{m} - \frac{2c}{m} < \beta_1 + \cdots + \beta_j < \frac{j}{m} + \frac{2c}{m} \quad \text{for every } j \in [m-1].$$

The following claims are analogous to Claims 17.1 and 17.2.

▷ Claim 19.2. There is a constant $\eta > 0$ such that $\|a^m\| + \|b^m\| \leqslant \eta$ for all m.

▷ Claim 19.3. For every $\varepsilon > 0$, if m is sufficiently large, then $d_H(W^{m\star}, f^m(A)) \leqslant \varepsilon$.

Since $\|a^m\| + \|b^m\| \leqslant \eta$ (by Claim 19.2), we can find an infinite set of indices m such that a^m and b^m converge to vectors $a^\star, b^\star \in \mathbb{R}^2$, respectively, as $m \to \infty$ over that set of indices. Let $f^\star \colon \mathbb{R}^2 \ni (x, y) \mapsto xa^\star + yb^\star \in \mathbb{R}^2$. It follows from Claim 19.1 that $\|a^\star\| = \|b^\star\|$ and the vectors $a^\star$ and $b^\star$ are orthogonal, so $f^\star$ is a similarity transformation or similarity transformation with reflection. The same argument as in the proof of Lemma 17, using Claim 19.3, shows that $W^{m\star} \to f^\star(A)$ in Hausdorff distance. Since $W^{m\star} \to f^\star(A)$, Lemma 5 yields $f^\star(A) \in \operatorname{sim} B$, and we have $f^\star(A) \in \operatorname{sim}^{\mathrm{r}} A$, so $B \in \operatorname{sim}^{\mathrm{r}} A$. ◀

8 Open problems

For our row construction to work, we need the disks to be smooth. In particular, Lemmas 5, 11, and 13 do not hold if A is not smooth. Distinguishing the classes of intersection graphs for non-smooth convex disks remains an interesting question.

One may also consider the even larger class $G^{\text{aff}}(A)$ of intersection graphs of disks that are affine equivalent to a convex disk A and ask when $G^{\text{aff}}(A) = G^{\text{aff}}(B)$ for two convex disks A and B. Other classes that have so far not been investigated are the contact and intersection graphs that can be obtained from rotated translations of a disk A, i.e., with no scaling allowed.

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