A Structural Approach to Tree Decompositions of Knots and Spatial Graphs

Corentin Lunel

LIGM, CNRS, Univ. Gustave Eiffel, ESIEE Paris, F-77454 Marne-la-Vallée, France

Arnaud de Mesmay

LIGM, CNRS, Univ. Gustave Eiffel, ESIEE Paris, F-77454 Marne-la-Vallée, France

Abstract

Knots are commonly represented and manipulated via diagrams, which are decorated planar graphs. When such a knot diagram has low treewidth, parameterized graph algorithms can be leveraged to ensure the fast computation of many invariants and properties of the knot. It was recently proved that there exist knots which do not admit any diagram of low treewidth, and the proof relied on intricate low-dimensional topology techniques. In this work, we initiate a thorough investigation of tree decompositions of knot diagrams (or more generally, diagrams of spatial graphs) using ideas from structural graph theory. We define an obstruction on spatial embeddings that forbids low tree width diagrams, and we prove that it is optimal with respect to a related width invariant. We then show the existence of this obstruction for knots of high representativity, which include for example torus knots, providing a new and self-contained proof that those do not admit diagrams of low treewidth. This last step is inspired by a result of Pardon on knot distortion.

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1 Introduction

A (tame) **knot** is a polygonal embedding of the circle S^1 into $\mathbb{R}^3$, and two knots are considered equivalent if they are **isotopic**, i.e., if they can be continuously deformed one into the other without introducing self-intersections. The **trivial knot**, or **unknot**, is, up to equivalence, the embedding of S^1 as a triangle. The investigation of knots and their mathematical properties dates back to at least the nineteenth century [1] and has developed over the years into a very rich and mature mathematical theory. From a computational perspective, a fundamental question is to figure out the best algorithm testing whether a given knot is the unknot. Note that it is neither obvious from the definitions that a non-trivial knot exists, nor that the problem is decidable. This was famously posed as an open problem by Turing [50]. The current state of the art on this problem is that it lies in **NP** [21] and co-**NP** [29], a quasipolynomial time algorithm has been announced [30] but no polynomial-time algorithm is known. More generally, algorithmic questions surrounding knots typically display a wide gap between the best known algorithms (which are almost never polynomial-time, and sometimes the complexity is a tower of exponentials) and the best known complexity lower bounds. We refer to the survey of Lackenby for a panorama of algorithms in knot theory [28].

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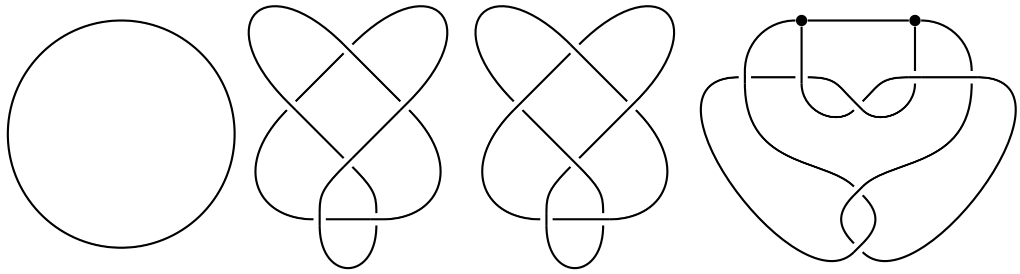

Figure 1 Diagrams of two trivial knots on the left, a bowline knot and a knotted spatial graph.

In recent years, many attempts have been made to attack such seemingly hard problems via the route of parameterized algorithms. In particular, the **treewidth** of a graph is a parameter quantifying how close a graph is to a tree, and thus algorithmic problems on graphs of low treewidth can often be solved very efficiently using dynamic programming techniques on the underlying tree structure of instance. The concept of **branchwidth**, which we also use below, is somewhat equivalent and always within a constant factor of treewidth [39]. One approach is to study a knot via one of its **diagrams** (see Figure 1), that is, a decorated graph obtained by a planar projection where it is indicated on each vertex which strand goes over or under. Then, if such a diagram has low treewidth, one can apply these standard dynamic programming techniques to solve seemingly hard problems very efficiently. While this approach has not yet been successful for unknot recognition beyond treewidth 2 [5], it has proved effective for the computation of many knot invariants, including: Jones and Kauffman polynomials [33] (which are known to be $\#P$-hard to compute in general [27]), HOMFLY-PT polynomials [6], and quantum invariants [34, 8]. Since any knot admits infinitely many diagrams, these algorithms naturally lead to the following question raised by Burton [7, p.2694], and Makowsky and Mariño [33, p.755]: do all knots admit diagrams of constant treewidth, or conversely does there exist a family of knots for which *all* the diagrams have treewidth going to infinity. This question was answered recently by de Mesmay, Purcell, Schleimer and Sedgwick [9] who proved that, among other examples, torus knots $T_{p,q}$ are such a family. The proof relies at its core on an intricate result of Hayashi and Shimokawa [23] on *thin position* of *multiple Heegaard splittings*.

Our results. The main purpose of this work is to provide new techniques to characterize which knots, or more generally which **spatial graphs** (polygonal embeddings of graphs into $\mathbb{R}^3$, considered up to isotopy, see for example Figure 1), do not admit diagrams of low treewidth. Our starting point is similar to the one in [9]: we first observe that if a knot or a spatial graph admits a diagram of low treewidth, then there is a way to sweep $\mathbb{R}^3$ using spheres arranged in a tree-like fashion which intersect the knot a small number of times (Proposition 5). This corresponds roughly to a map $f : \mathbb{R}^3 \to T$ where T is a trivalent tree, where the preimage of each point interior to an edge is a sphere with a small number of intersections with the knot (we refer to Section 2 for the precise technical definitions of all the concepts discussed in this introduction). We call this a **sphere decomposition**[1], and the resulting measure (maximal number of intersections) the **spherewidth** of the knot.

[1] Our sphere decompositions are different from the ones in [9] but functionally equivalent for knots.

Thus, in order to lower bound the treewidth of all the diagrams of a knot, it suffices to lower bound its spherewidth. We provide a systematic technique to do so using a perspective taken from structural graph theory. In the proof of the celebrated Graph Minor Theorem of Robertson and Seymour [41], handling families of graphs with bounded treewidth turns out not to be too hard [38], and in contrast, a large part is devoted to analyzing the structure shared by graphs of large treewidth. There, a fundamental contribution is the concept of **tangle**[2]. We refer to Diestel [13] or Grohe [18] for nice introductions to tangles and their applications. Informally, a tangle of order k in a graph G is a choice, for each separation of size at most k, of a "big side" of that separation, where the highly-connected part of the graph lies. In addition, there are some compatibility properties, in particular no three "small sides" should cover the whole graph. Such a tangle turns out to be exactly the structure dual to branchwidth, in the sense that, as is proved in [39], for any graph G, the maximal possible order of a tangle is exactly equal to its branchwidth. We develop a similar concept dual to sphere decompositions which we call a **bubble tangle**. Informally, a bubble tangle of order k for a knot or spatial graph K is a map that, for each sphere intersecting K at most k times, chooses a "big side" indicating where the complicated part of K lies. There are again some compatibility conditions which add topological information to the collection of "small sides". Then our first result is the following.

▶ **Theorem 1.** *For any knot or spatial graph K, the maximum order of a bubble tangle for K is equal to the spherewidth of K.*

This provides a convenient and systematic pathway to prove lower bounds on the spherewidth, and thus on the treewidth of all possible diagrams: it suffices to prove the existence of a bubble tangle of high order. However, making choices for the uncountable family of spheres with a small number of intersections with K, and then verifying the needed compatibility conditions is very unwieldy. Our second contribution is to provide a way to define such a bubble tangle in the case of knots (or spatial graphs) which are embedded on some surface Σ in $\mathbb{R}^3$. Given a surface Σ in $\mathbb{R}^3$, a **compression disk** is a disk properly embedded in $\mathbb{R}^3 \setminus \Sigma$ whose boundary is a non-contractible curve on Σ. The **compression-representativity of an embedding** of a knot or spatial graph K on a surface Σ in $\mathbb{R}^3$ is the smallest number of intersections between K and a cycle on Σ that bounds a compression disk. The **compression-representativity** of a knot or spatial graph is the supremum of that quantity over all embeddings on surfaces (this was originally defined by Ozawa [36]). Our second theorem is the following.

▶ **Theorem 2.** *For any knot or spatial graph K embedded on a surface Σ in $\mathbb{R}^3$, there exists a bubble tangle of order $2/3$ of the compression-representativity of the embedding. Therefore, for any knot or spatial graph K, there exists a bubble tangle of order $2/3$ of the compression-representativity of K.*

Combining together Theorems 1 and 2 with Proposition 5 provides a large class of knots of high spherewidth, and our tools are versatile enough to apply to spatial graphs, while previous ones did not. In particular, observing that torus knots $T_{p,q}$ have high compression-representativity, we obtain the following corollary, which improves the lower bound obtained by [9], without relying on deep knot-theoretical tools.

[2] It turns out that the word *tangle* holds a completely different meaning in knot theory, and, to avoid confusion, in this article we will always use it with the graph theory meaning.

▶ **Corollary 3.** *A torus knot $T(p,q)$ has spherewidth at least $2/3\min(p,q)$, and thus any diagram of $T(p,q)$ has treewidth at least $1/3\min(p,q)$.*

Related work and proof techniques. The results in this article and many of their proof techniques stem from two very distinct lineages in quite distant communities, the first one being knot theory or more generally low-dimensional topology, and the second one being structural graph theory. While there have been some recent works aiming at building bridges between combinatorial width parameters and topological quantities (for example the aforementioned [9], but also [25, 26, 35] for related problems in 3-manifold theory), the main contribution in this article is that we dive deeper in the structural graph theory perspective via the concept of a tangle. The latter has now proved to be a fundamental tool in graph theory and beyond (see for example Diestel [10, Preface to the 5th edition]).

The duality theorem of Robertson and Seymour between branchwidth and tangles in [39] has been generalized many times since its inception, for example in order to encompass other notions of decompositions and their obstructions [2, 32], to apply more generally to matroids [17] and to the wide-ranging setting of abstract separations systems [11, 12]. The key difference in our work, and why it does not fit into these generalizations, is that our notions of sphere decomposition and bubble tangles inherently feature the *topological* constraint of working with 2-spheres. This is a crucial constraint, as it would be easy to sweep any knot with width at most 2 if one were allowed to use arbitrary surfaces during the sweeping process. Furthermore, in planar graphs, it was shown by Seymour and Thomas [48] that the separations involved in an optimal branch decomposition can always be assumed to take the form of 1-spheres, i.e., Jordan curves. This property led to the ratcatcher algorithm to compute the branchwidth of planar graphs in polynomial time [48], and to sphere-cut decompositions and their algorithmic applications (see for example [14]). Our sphere decompositions are the generalization one dimension higher of these sphere-cut decompositions, and Theorem 1 identifies bubble tangles as a correct notion of dual obstruction for those. We believe that these notions could be of further interest beyond knots, in the study of graphs embedded in $\mathbb{R}^3$ with some topological constraints, e.g., linkless graphs [42].

The **representativity** (also called **facewidth**) of a graph embedded on a surface S is the smallest number of intersections of a non-contractible curve with that graph. Theorem 2 will not come as a surprise for readers accustomed to graph minor theory, as Robertson and Seymour proved a very similar-looking theorem in Graph Minors XI [40, Theorem 4.1], showing that that the branchwidth of a graph embedded on a surface is lower bounded by its representativity, which they prove by exhibiting a tangle. The key difference is that our notion of compression-representativity only takes into account the length of cycles bounding compression disks, instead of all the non-contractible cycles. Here again, this topological distinction is crucial to give a meaningful concept for knots, as for example the graph-theoretical representativity of a torus knot is zero. Due to this difference, the proof technique of Robertson and Seymour does not readily apply to prove Theorem 2; instead we have to rely on more topological arguments.

From the knot theory side, there is a long history in the study of the "best" way to sweep a knot while trying to minimize the number of intersections in this sweepout. One of the oldest knot invariants, the bridge number, can be seen through this lens (see for example [47]). A key concept in modern knot theory, introduced by Gabai in his proof of the Property R conjecture [16], is the notion of **thin position** which more precisely quantifies the best way to place a knot to minimize its width. It is at the core of many advances in modern knot theory (see for example Scharlemann [43]). Recent developments in thin position have highlighted that in order to obtain the best topological properties, it can be helpful to sweep the knot in a tree-like fashion compared to the classical linear one. This approach leads to definitions bearing close similarities to our sphere decompositions (this is one of the ideas behind generalized Heegaard splittings [45, 44], see also [23, 24, 49]). The concept of compression-representativity of a knot or a spatial graph finds its roots in the works of Ozawa [36], and Blair and Ozawa [4] who defined it under the simple name of representativity, taking inspiration from graph theory. They proved that it provides a lower bound on the bridge number and on more general linear width quantities. Our Theorem 2 strengthens their results by showing that it also lower bounds the width of tree-like decompositions. Furthermore, while specific tools have been developed to lower bound various notions of width of knots or 3-manifolds, we are not aware of duality theorems like our Theorem 1. It shows that our bubble tangles constitute an obstruction that is, in a precise sense, the optimal tool for the purpose of lower bounding spherewidth.

Finally, an important inspiration for our proof of Theorem 2 comes from a seemingly unrelated breakthrough of Pardon [37], who solved a famous open problem of Gromov [19] by proving the existence of knots with arbitrarily high **distortion**. The distortion for two points on an embedded curve in $\mathbb{R}^3$ is the ratio between the intrinsic and Euclidean distance between the points. The distortion of the entire curve is the supremum over all pairs of points. The distortion of a knot is the minimal distortion over all embeddings of the knot. While this metric quantity seems to have nothing to do with tree decompositions, it turns out that the technique developed by Pardon can be reinterpreted in our framework. With our terminology, his proofs amounts to first lower bounding the distortion by a constant factor of the spherewidth, and then defining a bubble tangle for knots of high representativity. The lower bound is nicely explained by Gromov and Guth [20, Lemma 4.2], where the simplicial map is similar to our sphere decompositions, up to a constant factor. Then our proof of Theorem 2 is inspired by the second part of Pardon's argument, with a quantitative strengthening to obtain the 2/3 factor, whereas his argument would only yield 1/2.

Organization of this paper. After providing background and defining our key concepts in Section 2, we prove Theorem 1 in Section 3, and Theorem 2 in Section 4. We provide examples in Section 5. Due to the line limitations, some proofs are not included in this extended abstract and are deferred to the full version [31].

2 Preliminaries

We include the most relevant definitions, but some familiarity with low-dimensional topology will help, see for example in the textbook of Schultens [46]. We refer to Diestel [10] for a nice introduction to graph theory and in particular its structural aspects. We denote by $V(G)$, $E(G)$, and $L(G)$ the vertices, edges and leaves (degree one vertices) of a graph G.

Low-dimensional topology. Following standard practice, instead of considering knots and spatial graphs within $\mathbb{R}^3$, we compactify it and work within $\mathbb{S}^3$. We denote by $C(A)$ the connected components of a subset A of $\mathbb{S}^3$, and thus by $|C(A)|$ its number of connected components. As is standard in low-dimensional topology, we work in the Piecewise-Linear (PL) category, which means that *all the objects that we use in this article are assumed to be piecewise-linear*, i.e., made of a finite number of linear pieces with respect to a fixed triangulation of $\mathbb{S}^3$. This allows us to avoid pathologies such as wild knots or the Alexander horned sphere. An **embedding** of a compact topological space X into another one Y is a

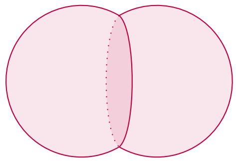

Figure 2 A double bubble: two spheres that intersect in a single disk.

continuous injective map, and it is **proper** if it maps the boundary ∂X within the boundary ∂Y. A 3-dimensional version of the Schoenflies theorem guarantees that for any PL 2-sphere S embedded in $\mathbb{S}^3$, both components of $\mathbb{S}^3 \setminus S$ are balls (see for example [3, Theorem XIV.1]). A **knot** is a PL embedding of S^1 into $\mathbb{S}^3$, a **link** is a disjoint union of knots, and a **spatial graph** is a PL embedding of a graph G into $\mathbb{S}^3$. All these objects are considered equivalent when they are ambient isotopic, i.e., when there exists a continuous deformation preserving the embeddedness. Knots and links are a special instance of spatial graphs, and henceforth we will mostly focus on spatial graphs, generally denoted by the letter G. For technical reasons, it is convenient to thicken our embedded graphs as follows. A **thickened embedding** φ of a graph G is an embedding of G in $\mathbb{S}^3$ where each vertex is thickened to a small ball, two balls are connected by a polygonal edge if and only if they are adjacent in the graph G, and pairs of edges are disjoint. We emphasize that we do not thicken edges, which might be considered nonstandard. We will also work with graphs embedded on surfaces which are themselves embedded in $\mathbb{S}^3$: such embeddings will also always be thickened, that is, vertices on the surface are thickened into small disks. *From now on, all the graph embeddings will be thickened, and thus to ease notations we will omit the word thickened.*

As mentioned in the introduction, for Σ a surface embedded in $\mathbb{S}^3$, a **compression disk** is a properly embedded disk D in $\mathbb{S}^3 \setminus \Sigma$ such that the boundary ∂D is a non-contractible curve on Σ. A **compressible curve** γ of Σ is the boundary of a compressing disk of Σ. For a spatial graph G embedded on a surface Σ in $\mathbb{S}^3$, the **compression representativity** of G on Σ, written c-rep(G, Σ) is $\min\{|C(\alpha \cap G|) \mid \alpha \text{ compressible curve of } \Sigma\}$ (we count connected components to correctly handle thickened vertices). The compression representativity c-rep(G) of G is the supremum of c-rep(G, Σ) over all nested embeddings $G \hookrightarrow \Sigma \hookrightarrow \mathbb{S}^3$.

In order to define spherewidth and bubble tangles, we require a precise control of the event when two spheres merge together to yield a third one, which is mainly encapsulated in the concept of double bubble. A **double bubble** is a triplet of closed disks (D_1, D_2, D_3) in $\mathbb{S}^3$, disjoint except on their boundaries, that they share: $D_1 \cap D_2 = D_1 \cap D_3 = D_2 \cap D_3 = D_1 \cap D_2 \cap D_3 = \partial D_1 = \partial D_2 = \partial D_3$, see Figure 2. Such a double bubble defines three spheres, which, by the PL Schoenflies theorem, bound three balls.

Two surfaces (resp. a knot and a surface) embedded in $\mathbb{S}^3$ are **transverse** if they intersect in a finite number of connected components, where the intersection is locally homeomorphic to the intersection of two orthogonal planes (resp. to the intersection of a plane and an orthogonal line). Likewise, we say that a surface is transverse to a ball if it is transverse to its boundary. A surface is transverse to a graph if it is transverse to all the thickened vertices and edges it intersects. A double bubble is transverse to a graph or a surface if each of its three spheres is and if the vertices of the graph do not intersect the spheres on their shared circle ∂D_i. Intersections are **tangent** when they are not transverse, and a sphere S is said **finitely tangent** to a graph G embedded in $\mathbb{S}^3$ if they do not intersect transversely but the number of intersections $|E(G) \cap S|$ is still finite.

Spherewidth. In this paragraph, we introduce sphere decompositions, which are the main way that we use to sweep knots and spatial graphs using spheres.

▶ **Definition 4** (Sphere decomposition). *Let G be a graph embedded in $\mathbb{S}^3$. A **sphere decomposition** of G is a continuous map $f : \mathbb{S}^3 \to T$ where T is a trivalent tree with at least one edge:*

- *For all $x \in L(T), f^{-1}(x)$ is a point disjoint from G.*
- *For all $x \in V(T) \setminus L(T), f^{-1}(x)$ is a PL double bubble transverse to G.*
- *For all x interior to an edge, $f^{-1}(x)$ is a sphere transverse or finitely tangent to G.*

The **weight** of a sphere S (with respect to G) is the number of connected components in its intersection with G. The **width** of a sphere decomposition f is the supremum of the weight of $f^{-1}(x)$ over all points x interior to edges of the tree T. The **spherewidth** of the graph G, denoted by $sw(G)$, is the infimum, over all sphere decompositions f, of the width of f: $sw(G) = \inf_{f:\mathbb{S}^3 \mapsto T} \sup_{x \in \mathring{e} \in E(T)} |C(f^{-1}(x) \cap G)|$. Therefore, a sphere decomposition is a way to continuously sweep $\mathbb{S}^3$ using spheres, which will occasionally merge or split in the form of double bubbles, and the spherewidth is a measure of how well we can sweep a graph G using sphere decompositions. This is similar to the level sets given by a Morse function, but note that our double-bubble singularities are not of Morse type, and those are key to the proof of Theorem 1.

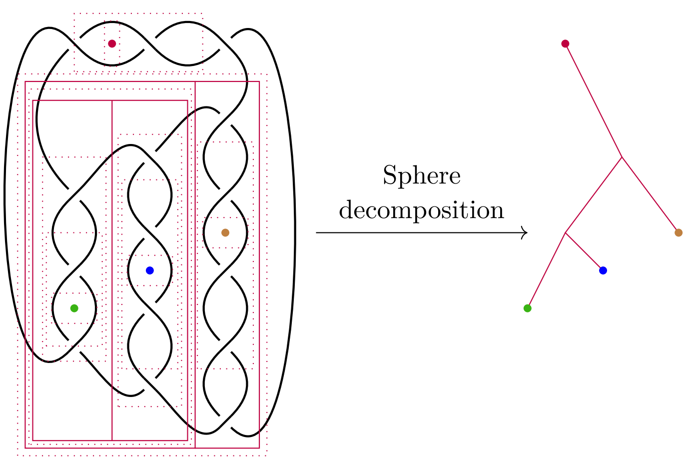

Figure 3 A width-4 sphere decomposition of a pretzel knot.

The point of using thickened embeddings instead of usual ones is that this allows disjoint spheres of a sphere decomposition to intersect a same vertex of a graph embedding. This is motivated by the following proposition, which provides a bridge between sphere decompositions and tree decompositions of diagrams of knots and spatial graphs.

▶ **Proposition 5.** *Let G be a knot or a graph embedded in $\mathbb{S}^3$, and D be a diagram of G. Then the spherewidth of G is at most twice the tree-width of D.*

The proof is very similar to that of Lemma 3.4 in [9] and is included to the full version [31].

Bubble tangle. Bubble tangles are our second main concept in this article. They will constitute an obstruction to spherewidth, by designating, for each sphere in $\mathbb{S}^3$ not intersecting the graph too many times, the side of the sphere that is easy to sweep. We first observe that some balls have to be easy to sweep: intuitively this will be the case of any unknotted segment or empty ball (see Figure 4). Let G be a graph embedded in $\mathbb{S}^3$. A closed ball B in $\mathbb{S}^3$ is said to be **G-trivial** if its boundary is transverse to G and one of the following holds (where $B(0,1)$ is the unit ball of $\mathbb{R}^3$):

- $B \cap G = \varnothing$.
- $B \setminus G$ is homeomorphic to $B(0,1) \setminus [-1,1] \times \{(0,0)\} \subset \mathbb{R}^3$.
- $B \setminus G$ is homeomorphic to $B(0,1) \setminus [-1,0] \times \{(0,0)\} \subset \mathbb{R}^3$.

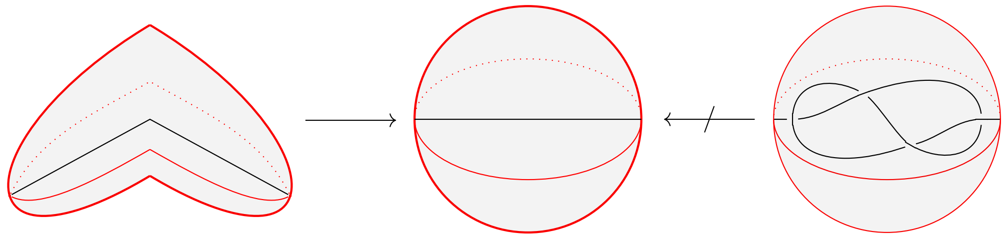

Figure 4 Representation of a G-trivial ball and a non G-trivial ball.

We can now introduce bubble tangles.

▶ **Definition 6.** *Let G be an embedding of a graph in $\mathbb{S}^3$ and $n \in \mathbb{N}$. A* ***bubble tangle $\mathcal{T}$ of order*** *$n \geq 2$, is a collection of closed balls in $\mathbb{S}^3$ such that:*

(T1) *For every closed ball B in $\mathcal{T}$, $|C(\partial B \cap G)| < n$.*

(T2) *For every sphere S in $\mathbb{S}^3$ transverse to G, if $|C(S \cap G)| < n$ then exactly one of the two closed balls $\bar{B}_1$ is in $\mathcal{T}$ or $\bar{B}_2$ is in $\mathcal{T}$, where $\mathbb{S}^3 \setminus S = \{B_1, B_2\}$.*

(T3) *For every triple of balls B_1, B_2 and B_3 induced by a double bubble transverse to G, $\{B_1, B_2, B_3\} \not\subset \mathcal{T}$.*

(T4) *For every closed ball B in $\mathbb{S}^3$, if B is G-trivial and $|C(\partial B \cap G)| < n$, then $B \in \mathcal{T}$.*

For every transverse sphere S such that $|C(S \cap G)| < n$, a bubble tangle chooses one of the two balls having S as the boundary. We think of the ball in $\mathcal{T}$ as being a "small side", since T4 stipulates that balls containing trivial parts of G are in $\mathcal{T}$, while the other one is the "big side". Then the key property T3 enforces that no three small sides forming a double bubble should cover the entire $\mathbb{S}^3$.

▶ Remark 7. Tangles in graph theory are often endowed with an additional axiom, specifying that small sides should be stable under inclusion (see e.g., [17, Axiom (T3A)]). Our bubble tangles are weaker in the sense that we do not enforce this axiom, but still strong enough to guarantee duality (Theorem 1) and the connection to compression-representativity (Theorem 2). Whether such an axiom can be additionally enforced in our definition of bubble tangle while preserving these properties is left as an open problem.

3 Obstruction and duality

In this section, we prove Theorem 1: given a graph G embedded in $\mathbb{S}^3$, the highest possible order of a bubble tangle is equal to the spherewidth of G. In the following, G is an embedding of a graph in $\mathbb{S}^3$ and the order of all bubble tangles that we consider is at least 3, the theorem being trivial otherwise. The proof is split into two inequalities: Proposition 8 and Proposition 11 which together immediately imply Theorem 1.

Bubble tangles as obstruction. We first show that a bubble tangle of order k and a sphere decomposition of width less than k cannot both exist at the same time.

▶ **Proposition 8.** *Let G be an embedding of a graph in $\mathbb{S}^3$. If G admits a bubble tangle $\mathcal{T}$ of order k then $sw(G) \geq k$.*

The proof of this proposition is similar to its graph-theoretical counterparts showing that tangles are an obstruction to branchwidth (see, e.g., [39]). The main difference with these proofs lies in the continuous aspects of our sphere decomposition, which we control using Lemmas 9 and 10.

Let S and S' be two disjoint spheres in $\mathbb{S}^3$. Then $\mathbb{S}^3 \setminus (S \cup S')$ has three connected components: two balls and a space I homeomorphic to $\mathbb{S}^2 \times [0, 1]$. The spheres S and S' are said to be **braid-equivalent** if $(I \cup S \cup S') \setminus G$ is homeomorphic to $S_k \times [0, 1]$ where S_k is the 2-sphere with k holes. The intuition behind this definition is that it means that G forms a braid between S and S'. The following lemma explains how braid-equivalent spheres interact with a bubble tangle.

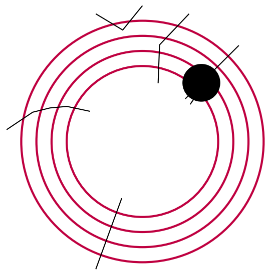

Figure 5 The three innermost spheres are braid-equivalent, not the fourth one.

▶ **Lemma 9.** *Let $\mathcal{T}$ be a bubble tangle and S, S' be two braid-equivalent spheres. Let us write $\mathbb{S}^3 \setminus S = \{B_1, B_2\}$ and $\mathbb{S}^3 \setminus S' = \{B'_1, B'_2\}$ such that $B_1 \subset B'_1$. If $B_1 \in \mathcal{T}$ then $B'_1 \in \mathcal{T}$.*

In the following, we will assume that there exists a bubble tangle $\mathcal{T}$ of order k and a sphere decomposition $f : \mathbb{S}^3 \to T$ of G of width less than k in order to reach a contradiction. Let $e = (u, v) \in E(T)$ be an edge and x be a point of e so that $f^{-1}(x)$ is transverse to G. Notice that x cuts T in two trees : $T_u(x)$ and $T_v(x)$ where $T_u(x)$ is the tree containing the endpoint u. By definition $f^{-1}(x) = S$ is a sphere in $\mathbb{S}^3$ such that $|C(G \cap S)| < k$. It follows by T2 that exactly one of $f^{-1}(T_u(x))$ or $f^{-1}(T_v(x))$ belongs to $\mathcal{T}$. We define an **orientation** $o : T \to V(T)$ induced by $\mathcal{T}$ as follows: if $f^{-1}(x)$ is transverse to G, $o(x) := v$ if $f^{-1}(T_u(x)) \in \mathcal{T}$, or $o(x) := u$ if $f^{-1}(T_v(x)) \in \mathcal{T}$. In other words, at a point x where $f^{-1}(x)$ is transverse to G the orientation o orients x outwards, toward the "big side". If $f^{-1}(x)$ has a tangency with G, note that for any close enough neighbor y of x, $f^{-1}(y)$ is transverse to G, and we define $o(x) := o(y)$, making an arbitrary choice if needed. As we consider edges of the tree T to be intervals, we will use interval notations: we write $[u, v]$ for the edge (u, v), and more generally $[x, y]$ to describe all the points on the edge between x and y. We say that an orientation o is **consistent** if for any x on some edge such that $f^{-1}(x)$ is transverse to G, o is constant on $[x, o(x)]$. The following lemma shows that the orientation o can be assumed to be consistent on all the edges of the tree T.

▶ **Lemma 10.** *Let us assume that there exists a bubble tangle $\mathcal{T}$ of order k and a sphere decomposition $f : \mathbb{S}^3 \to T$ of G of width less than k. Then there exists a sphere decomposition to the same tree such that o is consistent on T.*

Lemma 10 ensures that for any edge $e = (u, v)$ of T, there exists a point x_e so that all the points in (x_e, v) are oriented towards v, while all the points in (u, x_e) are oriented towards u. Hence, by subdividing each edge e of T at this x_e, we can think of o as assigning a direction to each edge. This directed tree is the main tool that we use in the proof of Proposition 8.

Proof of Proposition 8. Let us assume that there exists both a bubble tangle of order k and a sphere decomposition $f : \mathbb{S}^3 \to T$ of width less than k. By Lemma 10, there exists a sphere decomposition of width less than k so that the orientation o as defined above is consistent. Denoting by T' the tree T where each edge has been subdivided once, this orientation corresponds to a choice of direction for each edge of T'. Every directed acyclic graph, and thus in particular the tree T' contains at least one sink, see Figure 6.

This sink cannot be a leaf of the tree. Indeed, let $e = [\ell, u]$ be an edge of T incident to a leaf ℓ. By definition, $f^{-1}(\ell)$ is a point disjoint from G, and thus for any y in (ℓ, u) close enough to ℓ, $f^{-1}(y)$ is a sphere disjoint from G. Hence $f^{-1}(T_\ell(y))$ is a G-trivial ball and belongs to $\mathcal{T}$. It follows that all edges incident to leaves of T' are oriented inward. This sink cannot be a degree-two vertex either, as the tree T' was defined in such a way that the two edges adjacent to a degree-two vertex are always oriented outwards. Finally, this sink cannot be a degree-three vertex as this would mean that the three balls induced by a double bubble are in $\mathcal{T}$, which would violate T3. We have thus reached a contradiction. ◀

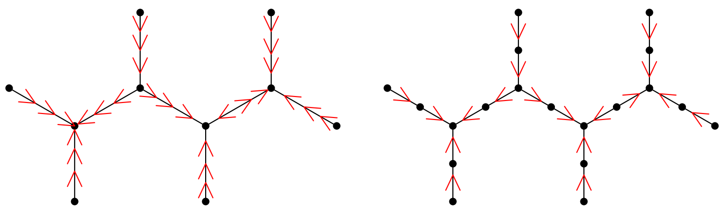

Figure 6 An example of T' from T leading to at least one sink.

Tightness of the obstruction. In this paragraph, we show that bubble tangles form a tight obstruction to sphere decompositions, in the sense that a bubble tangle of order k exists whenever a sphere decomposition of width less than k does not exist.

▶ **Proposition 11.** *Let G be an embedding of a graph in $\mathbb{S}^3$ and k be an integer at least three. If G does not admit a sphere decomposition of width less than k, then there exists a bubble tangle of order k.*

The idea of the proof is to show that, given a collection of closed balls satisfying the axioms T1 and T4 of bubble tangles, then either we can extend this collection to a bubble tangle, or there exists a **partial sphere decomposition** of width k which sweeps the space "between" the balls of the collection. We first introduce the relevant definition.

Let G a graph embedded in $\mathbb{S}^3$. A **partial sphere decomposition** of G is a continuous map $f : \mathbb{S}^3 \to T$ where T is a trivalent tree with at least one edge such that:

- For all $x \in L(T)$, $f^{-1}(x)$ is a point disjoint from G or a closed ball B.
- For all $x \in V(T) \setminus L(T)$, $f^{-1}(x)$ is a double bubble transverse to G.
- For all x interior to an edge, $f^{-1}(x)$ is a sphere transverse or finitely tangent to G.

The leaves of T having preimages by f which are not points are called **non-trivial leaves**. Let G be a graph embedding in $\mathbb{S}^3$ and $\mathcal{A}$ be a collection of closed balls in $\mathbb{S}^3$. A partial sphere decomposition f **conforms** to $\mathcal{A}$ if, for all $x \in L(T)$, $f^{-1}(x)$ is either a point disjoint from G, or a closed ball B such that there exists $A \in \mathcal{A}$ such that ∂B and ∂A are braid equivalent and $B \subset A$. In the latter case we say that x conforms to A. The width of a partial sphere decomposition is defined like the width of standard sphere decompositions: it is the supremal weight of spheres that are pre-images of points in the interiors of edges of T.

Now, the proof of Proposition 11 hinges on the following key lemma. Its proof is similar to branchwidth-tangle duality proofs [39] in that it builds a bubble tangle inductively, but the continuous nature of our objects makes us rely on transfinite induction in the form of Zorn's lemma.

▶ **Lemma 12.** *Let G be an embedding of a graph in $\mathbb{S}^3$, k be an integer at least 3 and $\mathcal{A}$ be a collection of closed balls in $\mathbb{S}^3$ satisfying T1 and T4. Then one of the following is true :*

- *$\mathcal{A}$ extends to a bubble tangle of order k.*
- *there is a partial sphere decomposition of width less than k that conforms to $\mathcal{A}$.*

Proof of Proposition 11. We denote by $\mathcal{A}$ the collection of G-trivial balls. By definition, $\mathcal{A}$ satisfies T4, and since G-trivial balls have weight at most two, it also satisfies T1 for k at least three. Therefore, by Lemma 12, either $\mathcal{A}$ extends to a bubble tangle of order k, or there exists a partial sphere decomposition of width less than k conforming to it. In the first case, we are done. In the second case, we are also done, since, given a partial sphere decomposition of width less than k conforming to G-trivial balls, it is straightforward to sweep within the G-trivial balls so as to obtain a sphere decomposition of width less than k. ◀

4 From compression representativity to bubble tangles

The goal of this section is to show Theorem 2: when a graph G is embedded on a compact, orientable, and non-zero genus surface Σ, there exists a bubble tangle naturally arising from the compression representativity of G on Σ. *In the following, we assume Σ is compact, orientable, and not a sphere.*

Under these hypotheses, the idea of the proof is to show that there exists a natural choice of small side for every sphere with fewer intersections with G than the compression representativity. Intuitively, such a sphere will only cut disks or "trivial parts" of Σ on one of its sides, which we will designate as the small one. That is justified by the following lemma.

▶ **Lemma 13.** *Let Σ be a surface embedded in $\mathbb{S}^3$ and S be a sphere in $\mathbb{S}^3$ that intersects Σ transversely such that there is at least one non-contractible curve in the intersection. Then one of the non-contractible curves is compressible.*

Proof. As Σ and S are transverse, the intersection of S and Σ consists of a disjoint union of simple closed curves. Each one of these curves bounds two disks on S. Let α be a curve of $S \cap \Sigma$ that is innermost in S, i.e. it bounds a disk D in S that does not contain any other curve of $S \cap \Sigma$. If α is non-contractible, then the disk D is a compression disk for α, and thus α is compressible. Otherwise, α bounds a disk D_Σ in Σ (see for example Epstein [15, Theorem 1.7]). We deform S continuously by "pushing" D through D_Σ while keeping S embedded (see Figure 7) until α disappears from $\Sigma \cap S$.

Repeating this process on a new innermost curve of S will eventually yield a non-contractible compressible curve. Indeed, the number of curves in the intersection is finite (recall that both surfaces are piecewise linear), decreases at each step, and one of the curves in $\Sigma \cap S$ is non-contractible. ◀

A direct consequence of this lemma is that if G is embedded on a surface Σ, a sphere S intersects Σ, and the intersection has weight less than c-rep(G, Σ), then all the simple closed curves in the intersection are contractible. Therefore, one of the two balls bounded

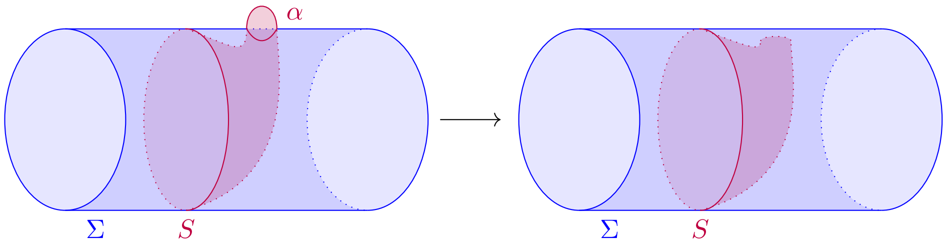

Figure 7 Removing a trivial curve from $S \cap \Sigma$.

by S contains the meaningful topology of Σ, while the other one only contains spheres with holes (see Figure 8). In order to formalize this, we will rely on fundamental groups (see for example Hatcher [22] for an introduction to this concept). The inclusion of a subsurface X on Σ induces a morphism $i_* : \pi_1(X) \to \pi_1(\Sigma)$. If this morphism is trivial, we say that X is π_1**-trivial** with respect to Σ.

▶ **Definition 14** (Compression bubble tangle on an embedded surface). *Let G be a graph embedded on Σ, a surface embedded in $\mathbb{S}^3$ such that c-rep$(G, \Sigma) \geq 3$ and set $k = \frac{2}{3}$c-rep(G, Σ). The* ***compression bubble tangle*** *c-$\mathcal{T}$, is the collection of balls in $\mathbb{S}^3$ defined as follows: for any sphere S in $\mathbb{S}^3$ transverse to G such that $|C(S \cap G)| < k$, by Lemma 13, there is exactly one connected component A of $\Sigma \setminus S$ that is π_1-trivial. Exactly one of the open balls B of $\mathbb{S}^3 \setminus S$ contains A, and we put the closed ball in c-$\mathcal{T}$: $\bar{B} \in$ c-$\mathcal{T}$.*

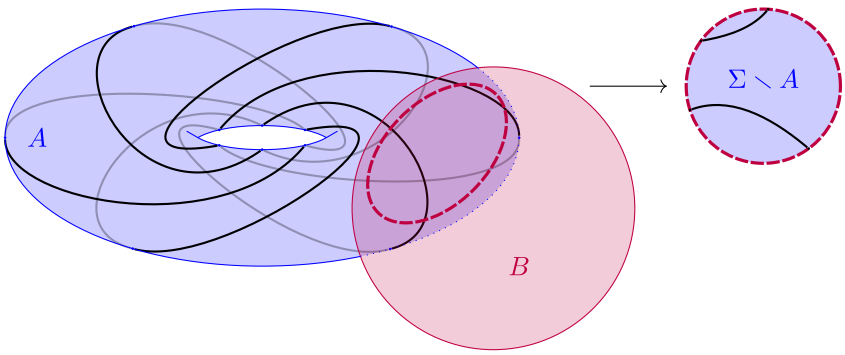

Figure 8 Intersection between a torus knot $T_{6,5}$ embedded on a torus and a sphere. Here the ball B containing the disk on the right is in the compression bubble tangle.

The main step in the proof of Theorem 2 is to prove that a compression bubble tangle on the torus is indeed a bubble tangle.

▶ **Proposition 15.** *A compression bubble tangle is a bubble tangle.*

Note that Propositions 15 and 8 directly imply Theorem 2 (the theorem is trivial if c-rep$(G, \Sigma) < 3$). Therefore, the rest of this section is devoted to proving Proposition 15.

By definition, a compression bubble tangle satisfies T1 and T2. We then notice that T4 is verified whenever the compression representativity of G on Σ is greater than 2.

▶ **Lemma 16.** *If c-rep$(G, \Sigma) \geq 3$ then for all G-trivial balls B, $B \cap \Sigma$ is π_1-trivial.*

The hard part of the proof is to show that T3 is satisfied. This is more delicate than it seems at first glance, since any surface can be obtained by gluing three disks, and these three disks can even come from a double bubble: we provide an example in the appendix of the full version [31].

Henceforth, we will proceed by contradiction and assume that we can cover $\mathbb{S}^3$ by three closed balls B_1, B_2, B_3 of c-$\mathcal{T}$ that induce a double bubble DB transverse to Σ and G. Thus Σ is covered by three surfaces with boundary: $\Sigma \cap B_1, \Sigma \cap B_2$ and $\Sigma \cap B_3$ which are π_1-trivial by definition of c-$\mathcal{T}$. In the following, we write $S_i = \partial B_i$. We first show that we can furthermore assume that these surfaces are a disjoint union of closed disks on Σ.

▶ **Lemma 17.** *Let G be a graph embedded on Σ, a surface embedded in $\mathbb{S}^3$. Let c-$\mathcal{T}$ be the compression bubble tangle associated to G and Σ. If there is a double bubble DB transverse to Σ, inducing three balls $B_1, B_2, B_3 \in$ c-$\mathcal{T}^3$ such that $B_1 \cup B_2 \cup B_3 = \mathbb{S}^3$, then we can isotope the double bubble so that we additionally have that $B_i \cap \Sigma$ is a union of closed disks.*

Then we define Γ **induced by the double bubble** DB to be the intersection of the double bubble with Σ: where vertices are the intersection of the common boundary of the three disks with Σ and edges are the intersections of Σ with the disks. By Lemma 17, we can assume that this graph is trivalent and cellularly embedded. It is naturally weighted by endowing each edge with its weight, i.e., the number of connected components in its intersection with G. Let us now state the lemma we will use for the sake of contradiction.

▶ **Lemma 18.** *The total weight of Γ is less than c-rep(G, Σ):*

$$\sum_{e \in E(\Gamma)} |C(e \cap G)| < \textit{c-rep}(G, \Sigma).$$

Proof. Since each edge of Γ bounds exactly two faces of Γ, i.e, disks of Σ; and $\Gamma = DB \cap \Sigma$ we get the following equality:

$$|C(S_1 \cap G)| + |C(S_3 \cap G)| + |C(S_3 \cap G)| = 2 \sum_{e \in E(\Gamma)} |C(e \cap G)| \tag{1}$$

By definition of c-$\mathcal{T}$, each ball B_i satisfies $|C(S_i \cap G)| < \frac{2}{3}$c-rep$(G, \Sigma)$ so that:

$$|C(S_1 \cap G)| + |C(S_3 \cap G)| + |C(S_3 \cap G)| < 3 \cdot \frac{2}{3}\text{c-rep}(G, \Sigma) = 2\text{c-rep}(G, \Sigma). \tag{2}$$

Combining (1) and (2) concludes the proof: $2 \sum_{e \in E(\Gamma)} |C(e \cap G)| < 2\text{c-rep}(G, \Sigma)$. ◀

Hence, if Γ contained a simple closed curve that is compressible, we would obtain the contradiction that we are looking for. The rest of the proof almost consists of finding such a compressible curve, leading to the following proposition.

▶ **Proposition 19.** *There exists a set of edges X on Γ such that:*

$$\sum_{e \in X} |C(e \cap G)| \geq \textit{c-rep}(G, \Sigma).$$

The proof of Proposition 19 is the technical crux of Theorem 2. It consists in defining a merging process, which gradually merges two balls of a double bubble, and proving that at some point in this merging process, one ball will intersect Σ in a non-trivial way, and thus yield a compressible curve via Lemma 13. An additional difficulty is that this curve might be non-simple in Γ; we circumvent this issue by finding a fractional version of such a curve instead, which will be strong enough to prove Proposition 19. This proposition directly implies Proposition 15, and thus Theorem 2:

Proof of Proposition 15. A compression bubble tangle immediately satisfies the bubble tangle axioms T1 and T2 by definition, and T4 by Lemma 16. For the axiom T3, assume by contradiction that there exist three closed balls $B_1, B_2, B_3 \in$ c-$\mathcal{T}$ covering $\mathbb{S}^3$ and inducing a double bubble transverse to Σ. By Lemma 17, we can assume the graph Γ induced by the intersection of the double bubble with Σ is cellularly embedded. Then by Proposition 19, the total weight of Γ is at least c-rep(G, Σ). This is a contradiction with Lemma 18. ◀

5 Examples

A torus knot $T_{p,q}$ is a knot embedded on an unknotted torus $\mathbb{T}$ in $\mathbb{S}^3$, for example a standard torus of revolution. It winds p times around the revolution axis, and q times around the core of the torus. We refer to Figure 8 for an illustration of $T_{6,5}$. The proof of Corollary 3 (see [31, Corollary 1.3]) follows by combining Proposition 5 and Theorems 1 and 2.

More generally, the same argument can be applied to lower bound the treewidth of the (p, q)-cabling [1, Section 5.2] of any nontrivial knot. We refer to Ozawa [36, Theorem 6] for examples of spatial embeddings of any graph with high compression representativity, and thus high spherewidth.

We conclude by observing that the proof of Theorem 2 offers more flexibility than what the theorem states and can also be applied in some settings where the compression-representativity is low. For example, a connected sum of two knots $K_1 \# K_2$ has compression-representativity two (see [36, Corollary 9]), but if one these two knots, say K_1, has high compression-representativity separately, then we can still define a bubble-tangle of high order by considering as big sides the balls containing the surface that K_1 is embedded on.

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