Convolution Products on Double Categories and Categorification of Rule Algebras

Nicolas Behr
CNRS, Université Paris Cité, IRIF, France

Paul-André Melliès
CNRS, Université Paris Cité, INRIA, France

Noam Zeilberger
École Polytechnique, LIX, Palaiseau, France

Abstract

Motivated by compositional categorical rewriting theory, we introduce a convolution product over presheaves of double categories which generalizes the usual Day tensor product of presheaves of monoidal categories. One interesting aspect of the construction is that this convolution product is in general only oplax associative. For that reason, we identify several classes of double categories for which the convolution product is not just oplax associative, but fully associative. This includes in particular framed bicategories on the one hand, and double categories of compositional rewriting theories on the other. For the latter, we establish a formula which justifies the view that the convolution product categorifies the rule algebra product.

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1 Introduction

Our main motivation in this work is to categorify notions coming from *compositional rewriting theory* in the sense of [1–5, 8, 9] and more specifically the concepts of *rule algebra* [1, 3, 8, 9] and of *tracelet* [2, 6]. There, a *rewriting theory* is specified by a base category $\mathbf{C}$ together with a specific categorical description of *direct derivations*, defined as rewriting steps $s : X \rightharpoonup Y$ obtained by applying a rewriting rule $r : A \rightharpoonup B$ to a given object $X \in \mathbf{C}$ of the base category. Typical descriptions include *double-pushout (DPO)* [13] and *sesqui-pushout (SqPO)* [10] formalisms. A rewriting theory defined in this way is called *compositional* when it satisfies a technical property of two- and three-step derivation traces, ensuring that the two theorems below are satisfied:

- the *concurrency theorem* [1, 4, 5, 7, 13] states that every two-step derivation trace may be (essentially uniquely) characterized by a one-step trace (i.e., a direct derivation) along a *composite rule* capturing the causal interactions between the two rules,
- the *associativity theorem* [1, 4, 5, 7, 9] states that whenever the concurrency theorem is applied twice in order to convert a three-step trace into a one-step trace along a composite rule, either possible nesting order of two-step rule composition operations yields essentially the same one-step trace (i.e., up to universal isomorphisms).

One important benefit of compositionality is that every compositional rewriting theory gives rise to a *rule algebra* defined as a vector space $\mathcal{R}$ (over a suitable field $\mathbf{k}$ such as $\mathbf{k} = \mathbb{R}$) with a basis indexed by (isomorphism classes of) rules, and equipped with a bilinear product that maps a pair of basis elements to a sum over basis elements indexed by composite rules. More

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explicitly, letting $\delta(r)$ denote the basis vector of $\mathcal{R}$ indexed by (the isomorphism class of) a rule r, writing $\cdot \star \cdot$ for the aforementioned binary product, $\mathcal{M}_r(s)$ for admissible matches of rule r into rule s, we have

$$\delta(r) \star \delta(s) = \sum_{\mu \in \mathcal{M}_r(s)} \delta(r_\mu s) \tag{1}$$

where $r_\mu s$ denotes one possible way to obtain a composite rule from r and s. Another natural idea when reasoning about compositional rewriting systems is to study *sets of direct derivations* as follows: we introduce a vector space $\mathcal{S}$ together with a notation $|X\rangle$ for a basis vector of $\mathcal{S}$ indexed by an (isomorphism class of an) object X of the underlying category. We then define the algebra morphism (or representation) $\rho : \mathcal{R} \to \mathit{Endo}_{\mathbf{k}}(\mathcal{S})$ as follows:

$$\rho(\delta(r))\,|X\rangle := \sum_{m \in M_r(X)} |r_m(X)\rangle \,, \tag{2}$$

where the right-hand side of the equation ranges over possible matches $m \in M_r(X)$ of the rule r into the object X, and where $|r_m(X)\rangle$ is the vector indexed by the isomorphism class of the outcome of applying r to X via m. The crucial fact that ρ satisfies the equation

$$\rho(\delta(r))\rho(\delta(s)) = \rho(\delta(r) \star \delta(s)) \tag{3}$$

and thus defines a *representation in $\mathcal{S}$ of the rule algebra* $(\mathcal{R}, \star)$ is far from trivial, and comes from a subtle interplay between the concurrency and the associativity theorems [1, 4, 5, 7, 9].

In the present paper, our primary purpose is to begin to *categorify* the rule algebra formalism, starting from the observation that the traditional frameworks for categorical rewriting (including the double-pushout and sesqui-pushout formalisms) can be neatly expressed using double categories. The idea is to associate to any such categorical rewriting framework a specific double category $\mathbb{D}$ whose objects are the objects of the original base category $\mathbf{C}$ and whose horizontal 1-cells $X \rightharpoonup Y$ are transformations typically defined as spans $X \leftarrow S \to Y$ in $\mathbf{C}$, defined in such a way that they include both the rewriting rules $r : A \rightharpoonup B$ as well as the derivation traces $s : X \rightharpoonup Y$ of the underlying rewriting theory. The double category $\mathbb{D}$ is then carefully designed in such a way that a direct derivation θ applying the rewriting rule $r : A \rightharpoonup B$ to define a rewriting step $s : X \rightharpoonup Y$ is the same thing as a double cell $\theta : r \to s$ of the form below, in the double category $\mathbb{D}$.

$$\begin{array}{ccc} B & \xleftharpoondown{\;r\;} & A \\ {\scriptstyle g}\big\downarrow & \Downarrow{\scriptstyle\theta} & \big\downarrow{\scriptstyle f} \\ Y & \xleftharpoondown[\;s\;]{} & X \end{array} \tag{4}$$

Here, the vertical maps f and g of the double category $\mathbb{D}$ indicate how the source A and the target B of the rewriting rule $r : A \rightharpoonup B$ are "embedded" in the objects X and Y, respectively, in order to define the direct derivation $\theta : r \to s$ exhibiting $s : X \rightharpoonup Y$ as an instance of the rewriting rule r. Given a rewriting rule $r : A \rightharpoonup B$ and a horizontal 1-cell $s : X \rightharpoonup Y$, it makes sense to look at the set $\hat{\Delta}_r(s)$ of double cells $\theta : r \to s$ of the form (4), which describes all the possible embeddings $f : A \rightarrowtail X$ and $g : Y \rightarrowtail B$ and all the possible ways $\theta : r \to s$ the horizontal 1-cell $s : X \rightharpoonup Y$ can be seen as an instance of the rewriting rule r. An important observation is that $\hat{\Delta}_r$ defines a covariant presheaf $\hat{\Delta}_r : \mathbb{D}_1 \to \mathsf{Set}$ over the vertical cell category $\mathbb{D}_1$ whose objects are horizontal 1-cells, and whose morphisms $\varphi : s \to s'$ are double cells of the form

$$\begin{array}{ccc} Y & \xleftharpoondown{\;s\;} & X \\ {\scriptstyle h_Y}\big\downarrow & \Downarrow{\scriptstyle\varphi} & \big\downarrow{\scriptstyle h_X} \\ Y' & \xleftharpoondown[\;s'\;]{} & X'' \end{array} \tag{5}$$

Note that the covariant presheaf $\hat{\Delta}_r : \mathbb{D}_1 \to \mathsf{Set}$ is the representable presheaf $\hat{\Delta}_r = \mathbb{D}_1(r, -)$ which associates to every 1-cell $s : X \rightharpoondown Y$ the set $\hat{\Delta}_r(s) = \mathbb{D}_1(r,-)(s)$ of all morphisms (4) from r to s in the category $\mathbb{D}_1$. One main intuition guiding us in the process of categorification is that the representable presheaf $\hat{\Delta}_r : \mathbb{D}_1 \to \mathsf{Set}$ should play the role of the basis vector $\delta(r)$ of rule algebra $(\mathcal{R}, \star)$.

This fundamental intuition brought us to develop a larger picture, and to associate to any double category $\mathbb{D}$ the category $\widehat{\mathbb{D}}$ of its vertical presheaves, simply defined as the category $\widehat{\mathbb{D}} = [\mathbb{D}_1, \mathsf{Set}]$ of covariant presheaves $G, F : \mathbb{D}_1 \to \mathsf{Set}$ over the vertical cell category $\mathbb{D}_1$ with natural transformations $G \Rightarrow F$ between them. One main contribution of the paper is the discovery of a convolution product $* : \widehat{\mathbb{D}} \times \widehat{\mathbb{D}} \to \widehat{\mathbb{D}}$ which generalizes the usual Day convolution product of presheaves over a monoidal category, and is of interest in its own right. In particular, we explain in §3 that, somewhat unexpectedly, the convolution product is only *oplax associative* in general. We then examine in the paper a number of additional *fibrational properties* of the double category $\mathbb{D}$ in order to recover strong associativity. We establish in §4 that strong associativity is guaranteed for framed bicategories and then study in §5 how the story unfolds for the case of double categories coming from rewriting frameworks. Finally, we establish in §6 that the convolution product $\hat{\Delta}_{r_2} * \hat{\Delta}_{r_1}$ of two representable presheaves $\hat{\Delta}_{r_1}$ and $\hat{\Delta}_{r_2}$ can in certain situations be decomposed as a finite sum of representables, categorifying equation (1).

2 Double categories

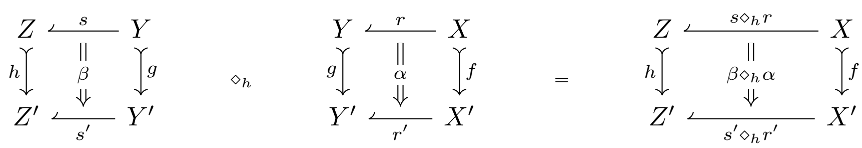
Throughout this paper, we consider double categories as weakly internal categories in CAT [14, Ch. 12.3]. This means that a (weak) *double category* $\mathbb{D}$ consists of a pair of categories $\mathbb{D}_0$ and $\mathbb{D}_1$ and a collection of functors

$$S, T : \mathbb{D}_1 \longrightarrow \mathbb{D}_0\,, \quad U : \mathbb{D}_0 \longrightarrow \mathbb{D}_1\,, \quad \diamond_h : \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \longrightarrow \mathbb{D}_1\,, \tag{6}$$

(where $\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1$ denotes the pullback of S and T) making the diagrams

commute, together with natural isomorphisms $(r \diamond_h s) \diamond_h t \xrightarrow{\sim} r \diamond_h (s \diamond_h t)$ and $U \diamond_h r \xrightarrow{\sim} r \xrightarrow{\sim} r \diamond_h U$ expressing associativity and neutrality of the structure up to isomorphism, and satisfying a number of coherence axioms.

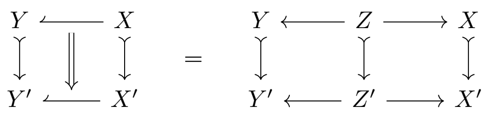
We refer to the objects of $\mathbb{D}_0$ as 0*-cells*, to the morphisms of $\mathbb{D}_0$ as *vertical 1-cells*, to the objects of $\mathbb{D}_1$ as *horizontal 1-cells*, and to the morphisms of $\mathbb{D}_1$ as *double cells*. We employ a slightly non-standard convention in writing horizontal 1-cells from right to left, using $\leftharpoondown$ arrows, and we reserve the arrow type $\rightarrowtail$ for vertical 1-cells. With these conventions, horizontal composition, denoted $\diamond_h$, reads as follows:



We emphasize that horizontal composition is only associative up to isomorphism. On the other hand, *vertical composition*, denoted $\diamond_v$, is a strictly associative operation, corresponding to composition of morphisms in the category $\mathbb{D}_1$ and of their images along the functors S and T in the category $\mathbb{D}_0$.

▶ **Example 2.1.** A prototypical example of a double category is the *double category of spans* $\mathbb{S}\mathsf{pan}(\mathbf{C})$ in some category $\mathbf{C}$ with chosen pullbacks, where:

- 0-cells and vertical 1-cells of $\mathbb{S}\mathsf{pan}(\mathbf{C})$ are given by objects and morphisms of $\mathbf{C}$;
- horizontal 1-cells $Y \leftharpoondown X$ are given by spans $Y \leftarrow Z \rightarrow X$ in $\mathbf{C}$;
- double cells are given by morphisms of spans in the sense of a pair of commuting squares

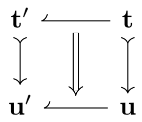


- horizontal composition of spans is defined by pullback, with unit $U_X = X \overset{id_X}{\leftarrow} X \overset{id_X}{\rightarrow} X$.

Observe that horizontal composition in $\mathbb{S}\mathsf{pan}(\mathbf{C})$ is indeed only associative up to isomorphism. For that reason, the notion of double category we have just introduced is sometimes called weak double category.

▶ **Example 2.2.** In order to describe term rewriting in the language of double categories, we consider the *double category* $\mathbb{T}\mathsf{RS}[\Sigma]$ associated to a fixed signature Σ of operations, defined as follows:

- 0-cells are lists of terms $\mathbf{t} = t_1, \ldots, t_n$ over Σ with set of variables denoted $\mathsf{Var}(\mathbf{t})$;
- vertical 1-cells $\mathbf{t} \rightarrowtail \mathbf{u}$ represent subterm matchings, given by a pair $(C \mid \sigma)$ of a multi-hole context C and a substitution σ such that $\mathbf{u} = C[\mathbf{t}\sigma]$;
- there is a unique horizontal 1-cell $\mathbf{t} \rightharpoonup \mathbf{t}'$ for every pair of lists of terms of the same length $|\mathbf{t}| = |\mathbf{t}'|$ such that $\mathsf{Var}(\mathbf{t}') \subseteq \mathsf{Var}(\mathbf{t})$;
- double cells



  are given by a pair $(C \mid \sigma)$ of a multi-hole context C and a substitution σ such that $\mathbf{u} = C[\mathbf{t}\sigma]$ and $\mathbf{u}' = C[\mathbf{t}'\sigma]$.

The idea is that the horizontal 1-cells of $\mathbb{T}\mathsf{RS}[\Sigma]$ describe all possible shapes of (potentially parallel) rewriting rules, and double cells close those rules under context extension and substitution.

Any bicategory may be seen as a double category in which all vertical 1-cells are identities, i.e., such that $\mathbb{D}_0$ is a discrete category. (As a special case, any monoidal category may be seen as a double category with $\mathbb{D}_0 = 1$ the terminal category.) Conversely, every double category $\mathbb{D}$ has an underlying *horizontal bicategory* $\mathbb{D}_\bullet$, defined as the double category with the same 0-cells and horizontal 1-cells as $\mathbb{D}$, but restricted to double cells whose vertical components are identities (i.e., morphisms α in $\mathbb{D}_1$ such that $S(\alpha)$ and $T(\alpha)$ are identity morphisms), also said to be *globular*.

It will be convenient to also consider an "unbiased" (in the sense of [16, §3.1]) definition of double category, starting from a pair of categories $\mathbb{D}_0$ and $\mathbb{D}_1$ and a family of functors $(h_n : \mathbb{D}_n \to \mathbb{D}_1)_{n \geqslant 0}$ where $\mathbb{D}_n := \underbrace{\mathbb{D}_1 \times_{\mathbb{D}_0} \ldots \times_{\mathbb{D}_0} \mathbb{D}_1}_{n \text{ times}}$ is the limit in Set of the "zig-zag" diagram of functors:

where the category $\mathbb{D}_1$ appears n times and $\mathbb{D}_0$ appears $n+1$ times. The objects (respectively morphisms) of $\mathbb{D}_n$ may be seen as sequences $(s_n, \ldots, s_1)$ of n composable horizontal 1-cells (resp. double cells), with the functor

$$h_n \; = \; (s_n, \ldots, s_1) \;\mapsto\; h_n(s_n, \ldots, s_1) \in \mathsf{obj}(\mathbb{D}_1)$$

performing the horizontal composition "all at once". In this *presentation*, both associativity and neutrality are represented by a single family of natural isomorphisms

$$h_n \circ (h_{i_1}, \ldots, h_{i_n}) \quad \cong \quad h_{i_1+\cdots+i_n}$$

satisfying a number of coherence axioms.

We will go back and forth between the biased and unbiased definitions of a double category, which are equivalent. In particular, given a double category with unit U and (binary) horizontal composition $\diamond_h$, we can obtain a family of n-ary composition functors h_n by taking $h_0 = U$, $h_1 = id$, and h_n to be any bracketing of $n-1$ $\diamond_h$'s for $n \geqslant 2$.

3 A convolution product of presheaves over double categories

3.1 Presheaves over double categories and the convolution product

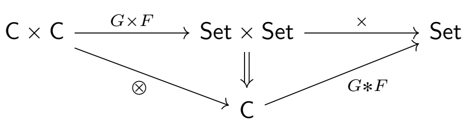
One starting point for our work was the observation that the Day convolution product on presheaves over monoidal categories [11] may be extended to a convolution product for *vertical presheaves* over double categories. As explained in the introduction, a *vertical presheaf over a double category* $\mathbb{D}$ is simply defined as a covariant Set-valued presheaf over the category $\mathbb{D}_1$ whose objects are the horizontal 1-cells and whose morphisms are the double cells of $\mathbb{D}$. We write $\widehat{\mathbb{D}} = [\mathbb{D}_1, \mathsf{Set}]$ for the category of vertical presheaves over $\mathbb{D}$ and natural transformations between them. A vertical presheaf F over a double category $\mathbb{D}$ thus assigns a set $F(r)$ to every horizontal 1-cell $r : Y \leftharpoonup X$, and a function $F(\alpha) : F(r) \to F(r')$ to every double cell $\alpha : r \to r'$. As also explained in the introduction, an important example is provided by *representable presheaves*, which we notate $\hat{\Delta}_r := \mathbb{D}_1(r, -)$.

▶ **Example 3.1.** Term rewriting systems may be modeled as vertical presheaves over the double category $\mathbb{T}\mathsf{RS}[\Sigma]$. For example, suppose Σ contains a binary operation m and a constant e, and consider the rewriting rule $r : m(e, x) \rightharpoonup x$. The presheaf $\hat{\Delta}_r$ in a sense encapsulates all ways of applying r once to a subterm. For instance, $\hat{\Delta}_r(r' : m(e, m(e, x)) \rightharpoonup m(e, x))$ contains exactly two elements, corresponding to the double cells $\alpha_1 : r \to r'$ and $\alpha_2 : r \to r'$ defined by the context/substitution pairs $(C_1 = - \mid \sigma_1 = m(e, x)/x)$ and $(C_2 = m(e, -) \mid \sigma_2 = x/x)$ respectively.

At this stage, we explain how we extend the Day convolution product to double categories. We find it instructive to begin by recalling the usual definition of the Day convolution product on presheaves over a category C equipped with a monoidal product $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$. Given a pair of presheaves G and F over C, the convolution product $G * F$ is defined as the left Kan extension of the presheaf over the product category $\mathsf{C} \times \mathsf{C}$

$$\mathsf{C} \times \mathsf{C} \xrightarrow{G \times F} \mathsf{Set} \times \mathsf{Set} \xrightarrow{\times} \mathsf{Set}$$

along the monoidal product functor $\otimes$:



Equivalently, $G * F$ may be defined by the following well-known coend formula:

$$G * F = a \mapsto \int^{(c,b)\in \mathbf{C}\times\mathbf{C}} \mathbf{C}(c \otimes b, a) \times G(c) \times F(b) \tag{7}$$

Recall that in general, the coend $\int^{x\in \mathsf{C}} M(x,x)$ of a functor $M : \mathsf{C}^{op} \times \mathsf{C} \to \mathsf{Set}$ may be explicitly computed as a quotient of the coproduct $\coprod_{x\in\mathsf{C}} M(x,x)$ modulo an equivalence relation induced by the co- and contravariant actions of M [15, IX.6].

This definition of convolution product based on a left Kan extension and a coend formula can be adapted to double categories in the following way. Given two presheaves G and F over the category $\mathbb{D}_1$, the *convolution product* $G * F$ is the presheaf over the category $\mathbb{D}_1$ defined as the left Kan extension of the presheaf

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{proj} \mathbb{D}_1 \times \mathbb{D}_1 \xrightarrow{G\times F} \mathsf{Set}\times\mathsf{Set} \xrightarrow{\times} \mathsf{Set}$$

along the horizontal composition functor $\diamond_h$:

$$\begin{array}{ccccccc} \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 & \xrightarrow{proj} & \mathbb{D}_1 \times \mathbb{D}_1 & \xrightarrow{G\times F} & \mathsf{Set}\times\mathsf{Set} & \xrightarrow{\times} & \mathsf{Set} \\ & \searrow^{\diamond_h} & & \Downarrow & & \nearrow_{G*F} & \\ & & & \mathbb{D}_1 & & & \end{array}$$

As in the case of the convolution product for monoidal categories, the left Kan extension can be also neatly expressed as a coend formula:

$$G * F = r \mapsto \int^{(s_2,s_1)\in \mathbb{D}_2} \mathbb{D}_1(s_2 \diamond_h s_1, r) \times G(s_2) \times F(s_1) \tag{8}$$

As in the case of the Day convolution product, it follows from the definition that

▶ **Proposition 3.2.** *The convolution product* $* : \widehat{\mathbb{D}} \times \widehat{\mathbb{D}} \to \widehat{\mathbb{D}}$ *preserves colimits component-wise.*

Before proceeding further, let us consider an example from term rewriting that illustrates the motivation for the definition of the convolution product.

▶ **Example 3.3.** Let $r : m(e,x) \rightharpoonup x$ as in Example 3.1, and consider the convolution product $\hat{\Delta}_r * \hat{\Delta}_r$ of $\hat{\Delta}_r$ with itself. Intuitively, this presheaf encapsulates all ways of applying the rewriting rule r twice, possibly in parallel. One can verify that $\hat{\Delta}_r * \hat{\Delta}_r$ decomposes as a sum of representables:

$$\hat{\Delta}_r * \hat{\Delta}_r \cong \hat{\Delta}_{r_1} + \hat{\Delta}_{r_1} + \hat{\Delta}_{r_2} + \hat{\Delta}_{r_3} \tag{9}$$

where

$$r_1 : m(e, m(e,x)) \rightharpoonup x\,, \quad r_2 : m(m(e,e),x) \rightharpoonup x\,, \quad r_3 : m(e,x), m(e,y) \rightharpoonup x, y\,.$$

The convolution product therefore allows us to express neatly in algebraic form by the formula (9) the fact that there are four canonical ways of applying the rule $r : m(e,x) \rightharpoonup x$ twice, corresponding to the two ways of deriving the r_1 rule and the unique derivation of r_2, as well as the r_3 rule corresponding to two parallel applications of r. In Section 6 we will give a more general analysis of this phenomenon.

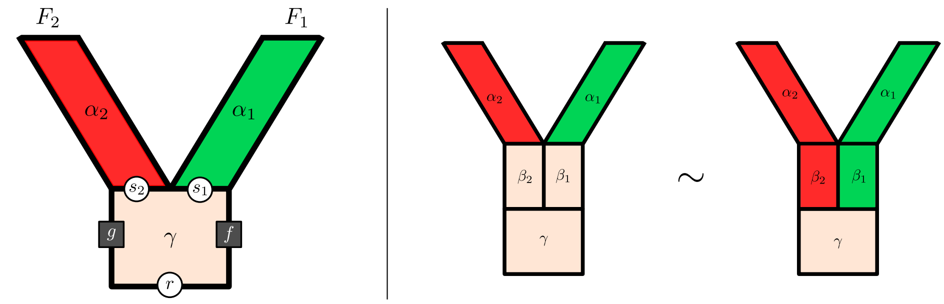


Figure 1 Left: diagram illustrating an element of the convolution product $F_2 * F_1$ evaluated at a horizontal cell r, where $\alpha_1 \in F_1(s_1)$ and $\alpha_2 \in F_2(s_2)$, and $\gamma : s_2 \diamond_h s_1 \to r$. Right: equivalence relation on diagrams induced by the coend formula.

3.2 Oplax associativity of the convolution product

The binary convolution product naturally generalizes to an n-ary convolution product of presheaves, defined in terms of the functors $h_n : \mathbb{D}_n \to \mathbb{D}_1$ discussed in Section 2.

▶ **Definition 3.4.** *Let $F_1, \ldots, F_n : \mathbb{D}_1 \to \mathsf{Set}$ be an n-tuple of covariant presheaves over $\mathbb{D}_1$. We define their* convolution product *by the coend formula*

$$F_n * \ldots * F_1 = r \mapsto \int^{(s_n,\ldots,s_1)\in\mathbb{D}_n} \mathbb{D}_1(h_n(s_n,\ldots,s_1), r) \times F_n(s_n) \times \cdots \times F_1(s_n) \qquad (10)$$

with the understanding that the formula specializes to $r \mapsto \int^{X\in\mathbb{D}_0} \mathbb{D}_1(U_X, r)$ in the nullary case $n = 0$. Equivalently, $F_n * \ldots * F_1$ is defined by the following left Kan extension diagram:

$$\begin{array}{ccccccc} \mathbb{D}_n & \xrightarrow{proj} & \mathbb{D}_1^n & \xrightarrow{F_n\times\cdots\times F_1} & \mathsf{Set}^n & \xrightarrow{\times} & \mathsf{Set} \\ & \searrow^{h_n} & & \Downarrow & & \nearrow_{F_n*\ldots*F_1} & \\ & & & \mathbb{D}_1 & & & \end{array} \qquad (11)$$

For convenience, we write $\circledast_n : \widehat{\mathbb{D}}^n \to \widehat{\mathbb{D}}$ for the resulting n-ary convolution product, and we also sometimes write $\bar{U}$ for the nullary case $\bar{U} = \circledast_0$.

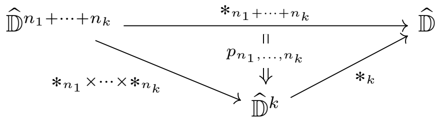
We find it evocative to visualize the elements of the convolution product $F_n * \ldots * F_1$ evaluated at a horizontal 1-cell r by a kind of "rabbit diagram", as illustrated on the left side of Figure 1 for the case $n = 2$. In the diagram, α_1 and α_2 represent elements of $F_1(s_1)$ and $F_2(s_2)$ respectively, where s_1 and s_2 are arbitrary horizontal 1-cells, and γ represents a double cell $s_2 \diamond_h s_1 \to r$ with $S(\gamma) = f$ and $T(\gamma) = g$. (We will sometimes omit the labels of the various cells in diagrams when they are unimportant.) On the right side of the figure, we also depict the equivalence relation on tuples

$$(\gamma \diamond_v (\beta_1 \diamond_h \beta_2), \alpha_2, \alpha_1) \sim (\gamma, F_2(\beta_2)(\alpha_1), F_1(\beta_1)(\alpha_1)) \qquad (12)$$

that is forced by the coend formula (8).

Perhaps surprisingly, it turns out that in general convolution only defines an *oplax* monoidal product on the presheaf category $\widehat{\mathbb{D}}$.

▶ **Theorem 3.5.** *The convolution product on the category $\widehat{\mathbb{D}}$ of vertical presheaves is oplax associative and oplax unital in the sense that there exists a family of natural transformations*

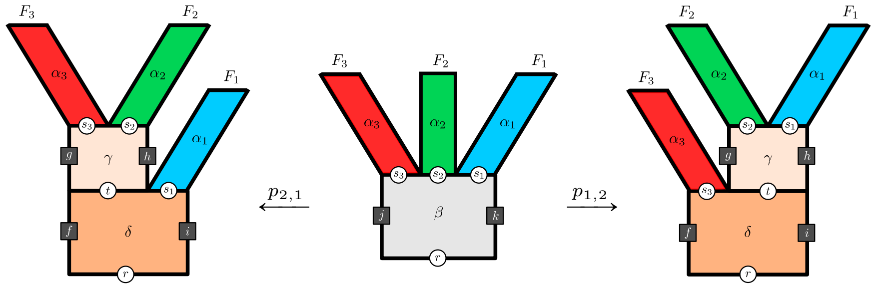


satisfying the coherence laws of an oplax monoidal product. In other words, $\widehat{\mathbb{D}}$ is an oplax monoidal category under the convolution product functors $\circledast_n : \widehat{\mathbb{D}}^n \to \widehat{\mathbb{D}}$.

An illustrative example is given by the natural transformations

$$(F_3 * F_2) * F_1 \xleftarrow{p_{2,1}} F_3 * F_2 * F_1 \xrightarrow{p_{1,2}} F_3 * (F_2 * F_1)$$

for any triple (F_3, F_2, F_1) of presheaves of $\widehat{\mathbb{D}}$. To understand these natural transformations, let us consider the following diagrams, which depict generic elements of $(F_3 * F_2) * F_1$, $F_3 * F_2 * F_1$, and $F_3 * (F_2 * F_1)$ evaluated at a horizontal 1-cell r:



(13)

Here s_1, s_2, s_3, t are horizontal 1-cells, β, γ, and δ are double cells, and each α_i is an element of $F_i(s_i)$, while $f, \ldots, k$ are vertical 1-cells corresponding to the projections of the respective double cells $f = T(\delta), g = T(\gamma), h = S(\gamma), i = S(\delta), j = T(\beta), k = S(\beta)$. The diagram in the middle of (13) may be seen as a degenerate case of the ones on the outside, in the sense that it corresponds to taking γ to be the identity double cell (on $s_3 \diamond_h s_2$ and $s_2 \diamond_h s_1$ respectively) and taking $\delta = \beta$. This simple observation yields the natural transformations $p_{2,1}$ and $p_{1,2}$. On the other hand, these natural transformations need not be invertible in general for an arbitrary double category, since the diagrams on the outside of (13) cannot necessarily be transformed into the one in the middle, in particular if the vertical 1-cells g and h are non-trivial.

One special case where the oplaxity maps are easily seen to be invertible is when the underlying vertical category $\mathbb{D}_0$ is discrete, i.e., when $\mathbb{D}$ is a bicategory.

▶ **Definition 3.6.** *We say the convolution product on $\widehat{\mathbb{D}}$ is* strongly associative *if the natural transformations $p_{n_1,\ldots,n_k}$ of Theorem 3.5 are invertible for all $n_1, \ldots, n_k \geqslant 1$. We say that it is strongly associative and unital if the $p_{n1,\ldots,n_k}$ are invertible for all $n_1, \ldots, n_k \geqslant 0$.*

▶ **Proposition 3.7.** *If $\mathbb{D}_0$ is a discrete category then the convolution product on $\widehat{\mathbb{D}}$ is strongly associative and unital.*

Proposition 3.7 covers in particular the case of the usual Day convolution product on presheaves over a monoidal category, seen as a double category over the terminal category $\mathbb{D}_0 = 1$. In Section 5 we will establish sufficient conditions under which the convolution product is strongly associative and unital. First, though, let us take a bit of time to consider a well-known class of double categories for which it turns out that the convolution product is strongly associative, but not strongly unital.

4 Non-unital associativity in a framed bicategory

A special situation in which the convolution product becomes strongly associative is when the horizontal 1-cells of the underlying double category may be "pushed" along vertical 1-cells independently with respect to their source and target, while leaving the other end fixed. Such situations are captured precisely by the notion of framed bicategory [18].

▶ **Definition 4.1** (Shulman [18, Definition 4.2]). *A double category $\mathbb{D}$ is said to be a* framed bicategory *if the pairing of the source and target functors $(T, S) : \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ is an opfibration (or if it satisfies any of the equivalent conditions of [18, Theorem 4.1]).*

The double category $\mathbb{S}\mathsf{pan} = \mathbb{S}\mathsf{pan}(\mathsf{Set})$ is an example of a framed bicategory. Indeed, the pushforward of a span of sets $Y \overset{a}{\leftarrow} Z \overset{b}{\rightarrow} X$ along a pair of functions $(g, h) : (Y, X) \to (Y', X')$ is given by composing the legs of the span with the two functions $Y' \overset{g}{\leftarrow} Y \overset{a}{\leftarrow} Z \overset{b}{\rightarrow} X \overset{h}{\rightarrow} X'$.

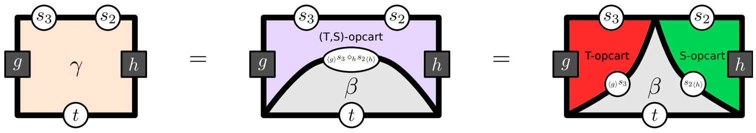
One important property of framed bicategories is that pushing forward along a pair of vertical cells may be decomposed into a pair of pushforward operations with respect to the source and the target. Given a horizontal 1-cell $r : Y \leftharpoondown X$ and a pair of vertical 1-cells $g : Y \to Y'$ and $h : X \to X'$, we thus write ${}_{\langle g \rangle}r_{\langle h \rangle} : Y' \leftharpoondown X'$ for the pushforward of r along (g, h), which may be equivalently read as $({}_{\langle g \rangle}r)_{\langle h \rangle}$ (push r along g relative to T and then along h relative to S) or as ${}_{\langle g \rangle}(r_{\langle h \rangle})$ (push r along h relative to S and then along g relative to T). In particular, the pushforward operations are compatible with horizontal composition, in the following sense.

▶ **Proposition 4.2** ([18, Corollary 4.3]). *In a framed bicategory, ${}_{\langle g \rangle}(t \diamond_h s)_{\langle h \rangle} \cong ({}_{\langle g \rangle}t) \diamond_h (s_{\langle h \rangle})$.*

This property is crucial in the proof of associativity of the convolution product for presheaves on framed bicategories.

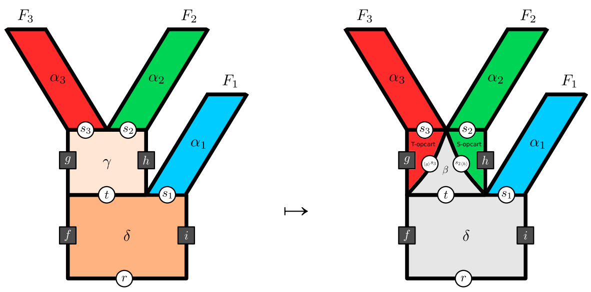
▶ **Theorem 4.3.** *If $\mathbb{D}$ is a framed bicategory then the convolution product on $\widehat{\mathbb{D}}$ is strongly associative.*

Proof sketch. The idea is summarized in the following series of diagrams:



(14)

Here, γ is the double cell appearing on the left side of (13), in the depiction of a generic element of $(F_3 * F_2) * F_1$. In the middle, by pushing $s_3 \diamond_h s_2$ forward along (g, h), we have factored γ as an op-Cartesian double cell followed by a globular cell β. On the right side of (14), by applying Proposition 4.2, we have factored this op-Cartesian double cell as the horizontal composition of a pair of op-Cartesian "triangles" (i.e., double cells with one side being an identity vertical cell). Finally, using this factorization, we can turn a generic element of $(F_3 * F_2) * F_1$ into an element of $F_3 * F_2 * F_1$:



(15)

Note that here we rely on the covariant action of the presheaves F_3 and F_2 to extend the elements $\alpha_3 \in F(s_3)$ and $\alpha_2 \in F(s_2)$ by the respective op-Cartesian triangles to obtain elements of $F_3({}_{\langle g \rangle}s_3)$ and $F_2(s_{2\langle h \rangle})$. It is routine to verify that the transformation (15) defines an inverse to the natural transformation $p_{2,1} : F_3 * F_2 * F_1 \to (F_3 * F_2) * F_1$: the equation $q_{2,1} \circ p_{2,1} = id$ is trivial, while the equation $p_{2,1} \circ q_{2,1} = id$ holds because the two sides of (15) are equivalent modulo equation (12) (cf. right side of Figure 1). This argument generalizes easily to inverting $p_{n_1,\ldots,n_k}$ for any $n_1, \ldots, n_k \geqslant 1$. ◀

However, the proof of the invertibility of the natural transformations $p_{n_1,\ldots,n_k}$ does *not* extend to arbitrary sequences of non-negative integers $n_1, \ldots, n_k \geqslant 0$, and indeed in general the convolution product on presheaves over framed bicategories is not strongly unital. Before demonstrating this, we first state an easy observation about representability of the nullary convolution product in the presence of an initial object.

▶ **Proposition 4.4.** *If $\mathbb{D}_0$ has an initial object 0, then $\bar{U} \cong \hat{\Delta}_{U_0}$.*

▶ **Example 4.5** (Counterexample to unitality of convolution over framed bicategories)**.** Consider the framed bicategory $\mathbb{S}\mathsf{pan}$, and let us write $\mathsf{Span} = \mathbb{S}\mathsf{pan}_1$ for its underlying category of spans and double cells between them. Since Set has an initial object given by the empty set $\varnothing$, the nullary convolution is representable (Prop. 4.4) by $U_\varnothing = \varnothing \leftarrow \varnothing \to \varnothing$. Note that the functor $U : \mathsf{Set} \to \mathsf{Span}$ has a right adjoint given by the functor sending a span $Y \leftarrow Z \to X$ to its underlying carrier Z, which implies (and indeed it is easy to verify) that $\varnothing \leftarrow \varnothing \to \varnothing$ is itself an initial object in Span. This entails that $\bar{U}$ is isomorphic to the terminal presheaf on Span.

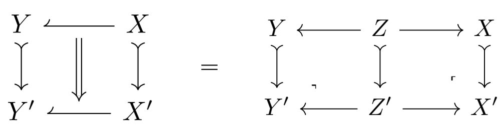
Now, let $F = \hat{\Delta}_{U_1}$ be the presheaf represented by the identity span over the one-element set. By the aforementioned adjunction, for an arbitrary span $r = Y \leftarrow Z \to X$, elements of $F(r) \cong \mathsf{Span}(U_1, r)$ are in bijection with elements of Z, that is $F(r) \cong Z$. In particular, $F(r)$ is empty if Z is empty. On the other hand, since $U_Y \diamond_h r \xrightarrow{\sim} r$ and since $\bar{U}(r) \cong \{*\}$, every element of $F(U_Y) \cong Y$ induces an element of $(F * \bar{U})(r)$. So $(F * \bar{U})(r)$ is non-empty if Y is non-empty. But this implies that there is no natural transformation $q_{1,0} : F * \bar{U} \to F$, and hence the oplax unitor $p_{1,0} : F \to F * \bar{U}$ is not invertible.

5 Associativity from the positive cylindrical decomposition property

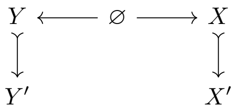
5.1 Cylindrical Decomposition Property

At this point, it must be stressed that our motivating examples of double categories coming from rewriting theory are *not* framed bicategories. For many of these examples, each of the source and target functors $S, T : \mathbb{D}_1 \to \mathbb{D}_0$ separately has some kind of opfibrational structure (or multi-opfibrational structure, see [5]), yet the pairing $(T, S) : \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ is typically not an opfibration, because pushing one end of a horizontal 1-cell along a vertical 1-cell will not leave the other end fixed. Nevertheless, we will see that such double categories admit a strongly associative convolution product.

As an illustration, consider the *double category* $\mathbb{D}\mathsf{PO} = \mathbb{D}\mathsf{PO}(\mathsf{Set})$ defined as a sub-double category of $\mathbb{S}\mathsf{pan} = \mathbb{S}\mathsf{pan}(\mathsf{Set})$ with the same 0-cells and horizontal 1-cells, but restricting vertical 1-cells to injections, and restricting double cells to pairs of pushout squares:



($\mathbb{D}$PO and similar double categories play a role in DPO-rewriting, see [5].) Now suppose that we want to push a span of the form $Y \leftarrow \varnothing \rightarrow X$ along a pair of injective functions $Y \rightarrow Y'$ and $X \rightarrow X'$:



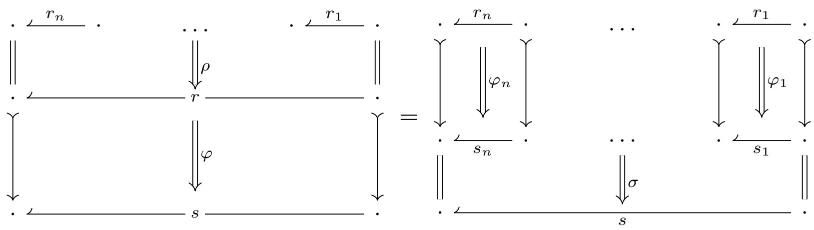
Such a horn may be completed to a double cell in $\mathbb{D}$PO just in case there exists a set Z' such that $Y' \cong Y + Z'$ and $X' \cong X + Z'$. But it is easy to construct examples for which no such set exists, for instance taking $|X| = |Y| = 1$, $|Y'| = 2$, and $|X'| = 3$. So the functor $(T, S) : \mathsf{DPO} \rightarrow \mathsf{Inj} \times \mathsf{Inj}$ is not an opfibration.

However, despite $\mathbb{D}$PO not being a framed bicategory, let us observe that it enjoys a similar factorization property to the one used in the proof of Theorem 4.3, and which is sufficient for proving associativity.

▶ **Definition 5.1.** *We say that a double category $\mathbb{D}$ has the n-*cylindrical decomposition property *if for every globular cell $\rho : h_n(r_n, \ldots, r_1) \rightarrow r$ and for every double cell $\varphi : r \rightarrow s$ there exists a family of n double cells $(\varphi_n, \ldots, \varphi_1) : (r_n, \ldots, r_1) \rightarrow (s_n, \ldots, s_1)$ in $\mathbb{D}_n$ and a globular cell $\sigma : h_n(s_n, \ldots, s_1) \rightarrow s$ such that*

$$\varphi \diamond_v \rho \quad = \quad \sigma \diamond_v h_n(\varphi_n, \ldots, \varphi_1) \tag{16}$$

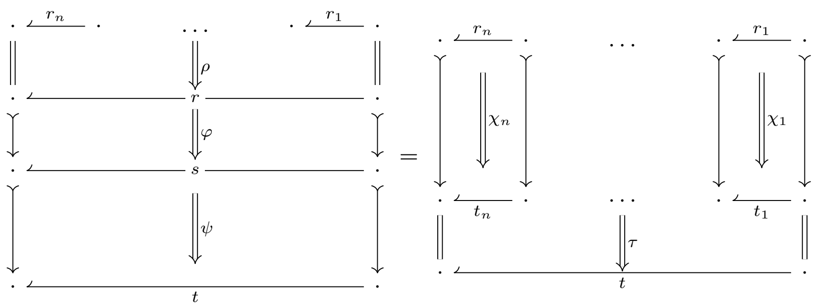
which means that the double cell $\varphi \diamond_v \rho$ factors as the vertical composition $\sigma \diamond_v h_n(\varphi_n, \ldots, \varphi_1)$ of the globular cell σ after the horizontal composition $h_n(\varphi_n, \ldots, \varphi_1)$, as depicted below:



Moreover, the family of n cells $(\varphi_n, \ldots, \varphi_1)$ and the globular cell σ are universal, in the sense that for every family of n double cells $(\chi_n, \ldots, \chi_1) : (r_n, \ldots, r_1) \rightarrow (t_n, \ldots, t_1)$ in $\mathbb{D}_n$, for every globular cell $\tau : h_n(t_n, \ldots, t_1) \rightarrow t$ and for every double cell $\psi : s \rightarrow t$ such that the equation

$$\psi \diamond_v \varphi \diamond_v \rho \quad = \quad \tau \diamond_v h_n(\chi_n, \ldots, \chi_1) \tag{17}$$

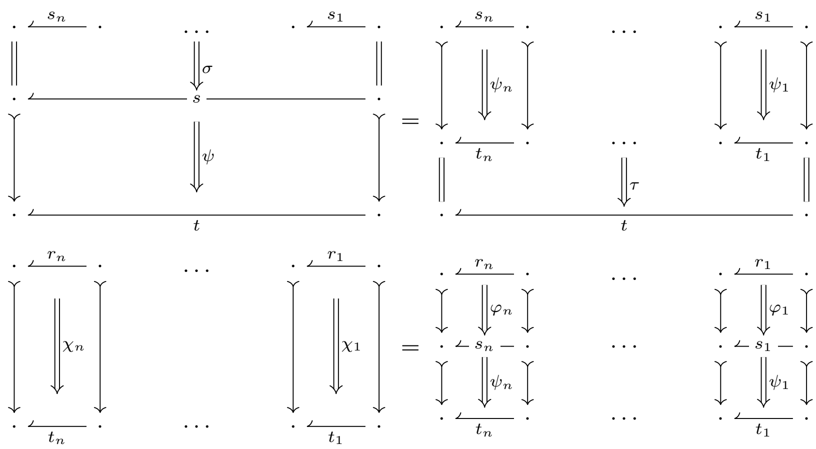
holds, as depicted below



there exists a unique family of n double cells $(\psi_n, \ldots, \psi_1) : (s_n, \ldots, s_1) \to (t_n, \ldots, t_1)$ in $\mathbb{D}_n$ such that the two equations

$$\begin{aligned} \psi \diamond_v \sigma &= \tau \diamond_v h_n(\psi_n, \ldots, \psi_1) \\ (\chi_n, \ldots, \chi_1) &= (\psi_n, \ldots, \psi_1) \diamond_v (\varphi_n, \ldots, \varphi_1) \end{aligned} \tag{18}$$

are satisfied, as depicted below:



(19)

We say that $\mathbb{D}$ has the cylindrical decomposition property *(or CDP) if it has the n-CDP for all $n \geqslant 0$, and the* positive CDP *if it has the n-CDP for all $n \geqslant 1$.*

▶ **Example 5.2.** Any framed bicategory has the positive CDP.

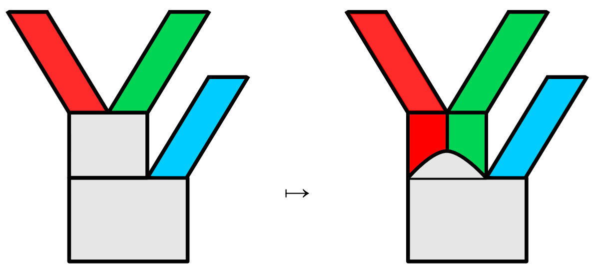
▶ **Example 5.3** ([5], Prop. 6.5)**.** $\mathbb{D}$PO has the positive CDP.

▶ **Example 5.4.** $\mathbb{T}$RS$[\Sigma]$ has the n-CDP for all $n \geqslant 0$, in the even stronger sense that any double cell $\varphi : h_n(r_n, \ldots, r_1) \to s$ factors uniquely as a horizontal composition $\varphi = h_n(\alpha_n, \ldots, \alpha_1)$.

One can establish that

▶ **Theorem 5.5.** *If $\mathbb{D}$ has the positive CDP then the convolution product on $\widehat{\mathbb{D}}$ is strongly associative. If it also has the 0-CDP then convolution is strongly associative and unital.*

Momentarily putting aside the full motivation for the universality condition in Definition 5.1 (which will become clearer in Section 5.2 below), we can already give an intuitive explanation for why the positive CDP entails strong associativity. Indeed, the factorization (16) applied to the trivial globular cell $\gamma : s_3 \circ s_2 \to s_3 \circ s_2$ generalizes equation (14), allowing us to reuse essentially the same procedure to invert the natural transformations $p_{n_1,\ldots,n_k}$ for any positive $n_1, \ldots, n_k \geqslant 1$. For example, we can define an inverse $q_{2,1} : (F_3 * F_2) * F_1 \to F_3 * F_2 * F_1$ to $p_{2,1}$ in an analogous way to the map (15) we used in proving strong associativity of the convolution product over framed bicategories:


(20)

Once again, the reason $q_{2,1}$ defines an inverse to $p_{2,1}$ boils down to the fact that the two sides of (20) are equated by the coend formula defining the convolution product (recall Figure 1). A similar argument can also be used to establish strong unitality when the underlying double category $\mathbb{D}$ also has the 0-CDP.

Note however that neither example of $\mathbb{S}\mathsf{pan}$ nor $\mathbb{D}\mathsf{PO}$ has the 0-CDP. It turns out nevertheless that convolution over the latter is strongly unital, as we will briefly address in Section 5.4.

5.2 Relative opfibrations

Here we explain how the cylindrical decomposition property can be reformulated in fibrational terms, for a natural common generalization of the notions of Grothendieck opfibration and of Street opfibration.

Let $F : \mathsf{E} \to \mathsf{B}$ be a functor, and let $\mathsf{G} \xrightarrow{\iota} \mathsf{B}$ be a wide subcategory of B (i.e., ι is bijective on objects). We define a new category $F[\mathsf{G}]$ as the category whose objects are given by triples (e, b, γ) of an object $e \in \mathsf{E}$, an object $b \in \mathsf{B}$, and an arrow $\gamma : Fe \twoheadrightarrow b$ in G, and whose morphisms

$$(e, b, \gamma) \xrightarrow{(\varepsilon,\beta)} (e', b', \gamma')$$

are given by pairs of morphisms $\varepsilon : e \to e'$ and $\beta : b \to b'$ such that $\beta \circ \gamma = \gamma' \circ F\varepsilon$. This category comes equipped with evident forgetful functors $\pi_{\mathsf{E}} : F[\mathsf{G}] \to \mathsf{E}$ and $\pi_{\mathsf{B}} : F[\mathsf{G}] \to \mathsf{B}$. Moreover, there is a functor $in_{\mathsf{E}} : \mathsf{E} \to F[\mathsf{G}]$ defined on objects by $in_{\mathsf{E}}(e) = (e, Fe, id_{Fe})$, in such a way that $F = \pi_{\mathsf{B}} \circ in_{\mathsf{E}}$. We remark that the functor in_{E} is a left adjoint to the functor π_{E}, since

$$\begin{aligned} Hom_{F[\mathsf{G}]}(in_{\mathsf{E}}(e), (e', b', \gamma')) &= Hom_{F[\mathsf{G}]}((e, Fe, id_{Fe}), (e', b', \gamma')) \\ &= \{(\varepsilon, \beta) \mid \varepsilon : e \to e', \beta : Fe \to b', \beta \circ id_{Fe} = \gamma' \circ F\varepsilon\} \\ &\cong Hom_{\mathsf{E}}(e, e') = Hom_{\mathsf{E}}(e, \pi_{\mathsf{E}}((e', b', \gamma'))) \,. \end{aligned} \tag{21}$$

The construction generalizes to every wide subcategory $\mathsf{G} \xrightarrow{\iota} \mathsf{B}$ the construction of the free opfibration $\pi_{\mathsf{B}} : F[\mathsf{B}] \to \mathsf{B}$ associated to a functor $F : \mathsf{E} \to \mathsf{B}$, which one recovers when G is the category B itself. This leads us to the following definition:

▶ **Definition 5.6.** *Let $F : \mathsf{E} \to \mathsf{B}$ be a functor, and let G be a wide subcategory of B. We say that F is a* G-relative opfibration *if $\pi_{\mathsf{B}} : F[\mathsf{G}] \to \mathsf{B}$ is a Grothendieck opfibration.*

▶ **Example 5.7.** The definition above subsumes three standard notions of functor $F : \mathsf{E} \to \mathsf{B}$:

- When $\mathsf{G} = |\mathsf{B}|$ is the discrete wide subcategory of B, a G-relative opfibration is the same thing as a Grothendieck opfibration $F : \mathsf{E} \to \mathsf{B}$,
- When $\mathsf{G} = \mathsf{core}(\mathsf{B})$ is the wide subcategory of reversible maps in B, a G-relative opfibration is the same thing as a Street opfibration $F : \mathsf{E} \to \mathsf{B}$,
- When $\mathsf{G} = \mathsf{B}$ is the category B itself, a G-relative opfibration $F : \mathsf{E} \to \mathsf{B}$ is the same thing a general functor $F : \mathsf{E} \to \mathsf{B}$.

Moreover, one can readily verify that the cylindrical decomposition property we introduced above (Def. 5.1) corresponds to the following particular instance of G-relative opfibration.

▶ **Proposition 5.8.** *A double category $\mathbb{D}$ satisfies the n-CDP precisely when the functor $h_n : \mathbb{D}_n \to \mathbb{D}_1$ is a* globular opfibration*, that is, a G-relative opfibration where $\mathsf{G} = \mathbb{D}_1^{\bullet}$ is the wide subcategory of $\mathbb{D}_1$ of globular double cells.*

The reason is that the category $h_n[\mathsf{G}]$ can be neatly described as the category of *cylindric maps* whose objects the tuples $(s_1, \ldots, s_n, s, \sigma)$ where $\sigma : h_n(s_n, \ldots, s_1) \to s$ is a globular double cell, and whose morphisms

$$(\psi_n, \ldots, \psi_1, \psi) \quad : \quad (s_n, \ldots, s_1, s, \sigma) \longrightarrow (t_n, \ldots, t_1, t, \tau)$$

consist of a family of n double cells $(\psi_n, \ldots, \psi_1) : (s_n, \ldots, s_1) \to (t_n, \ldots, t_1)$ in $\mathbb{D}_n$ and of a double cell $\psi : s \to t$ in $\mathbb{D}_1$ satisfying the equation

$$\psi \diamond_v \sigma \quad = \quad \tau \diamond_v h_n(\psi_n, \ldots, \psi_1).$$

depicted in (19).

5.3 Kan extensions along relative opfibrations

We now state a basic result on left Kan extensions along G-relative opfibrations, analogous to a standard result about Kan extension along Grothendieck opfibrations that is extremely useful in practice:

▶ **Lemma 5.9** (cf. e.g. [17], Cor. 5.8)**.** *Let $F : \mathsf{E} \to \mathsf{B}$ be a Grothendieck opfibration between small categories, and let C be cocomplete and locally small. Then the (point-wise) left Kan extension of a functor $G : \mathsf{E} \to \mathsf{C}$ along F at $b \in \mathsf{obj}(\mathsf{B})$ can be computed as a colimit over the fiber $F^{-1}(b)$,*

$$\mathsf{Lan}_F G(b) \cong \operatorname*{colim}_{e \in F^{-1}(b)} G(e)\,. \tag{22}$$

To develop a variant and generalization of this result for G-relative opfibrations, let us first recall a few standard constructions and facts from category theory.

▶ **Definition 5.10** ("Global" definition of left Kan extensions)**.** *Let $p : \mathsf{C} \to \mathsf{C}'$ be a functor, and D a category. If $p^* := - \circ p : [\mathsf{C}', \mathsf{D}] \to [\mathsf{C}, \mathsf{D}]$ has a left adjoint $p_! : [\mathsf{C}, \mathsf{D}] \to [\mathsf{C}', \mathsf{D}]$, i.e., if $p_! \dashv p^*$, then for all functors $F : \mathsf{C} \to \mathsf{D}$ the left Kan extension $\mathsf{Lan}_p F$ exists and is given by $\mathsf{Lan}_p F = p_! F$.*

▶ **Lemma 5.11.** *Let $L \dashv R$ be a pair of adjoint functors, with $L : \mathsf{C} \to \mathsf{C}'$ and $R : \mathsf{C}' \to \mathsf{C}$. Then for every category D, there is an induced pair of adjoint functors $- \circ R \dashv - \circ L : [\mathsf{C}', \mathsf{D}] \to [\mathsf{C}, \mathsf{D}]$. Therefore, for any functor $F : \mathsf{C} \to \mathsf{D}$, the left Kan extension $\mathsf{Lan}_L F$ of F along L is given by $\mathsf{Lan}_L F = F \circ R$.*

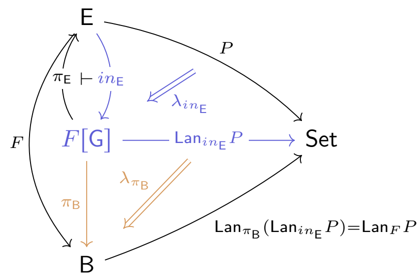
▶ **Theorem 5.12.** *Let $F : \mathsf{E} \to \mathsf{B}$ be a G-relative opfibration for some wide subcategory G of B, and let $P : \mathsf{E} \to \mathsf{Set}$ be a covariant presheaf (for small categories E and B). Then for every object b of B, we find that*

$$\mathsf{Lan}_F P(b) \cong \mathsf{Lan}_{\pi_{\mathsf{B}}} P \circ \pi_{\mathsf{E}}(b) \cong \operatorname*{colim}_{(e,b,\gamma) \in \pi_{\mathsf{B}}^{-1}(b)} P(e) \cong \left(\coprod_{e \in \mathsf{E}} \mathsf{G}(\bar{F}(e), \bar{b}) \times P(e) \right) \big/ {\sim_{\mathsf{G}}}\,. \tag{23}$$

Here, we used convenient shorthand notations $\bar{F}(e) := \iota^{-1} \circ F(e)$ and $\bar{b} := \iota^{-1}(b)$, and the equivalence relation $\sim_{\mathsf{G}}$ is defined as

$$\begin{aligned} (e, (\gamma, p)) \sim_{\mathsf{G}} (e', (\gamma', p')) :\Leftrightarrow\ & \exists e - \varepsilon \to e' \in \mathsf{E},\ (\delta, q) \in \mathsf{G}(\bar{F}(e'), \bar{b}) \times P(e) : \\ & (\gamma, p) = (\delta \circ \iota^{-1} \circ F(\varepsilon) \circ \iota, q) \ \wedge\ (\gamma', p') = (\delta, P(\varepsilon) q)\,. \end{aligned} \tag{24}$$

Proof. Recall that F factorizes uniquely as $F = \pi_{\mathsf{B}} \circ in_{\mathsf{E}}$, and that $in_{\mathsf{E}} \dashv \pi_{\mathsf{E}}$. Let us then compute $\mathsf{Lan}_F P$ step-wise via $\mathsf{Lan}_F P = \mathsf{Lan}_{\pi_{\mathsf{B}} \circ in_{\mathsf{E}}} P = \mathsf{Lan}_{\pi_{\mathsf{B}}} \mathsf{Lan}_{in_{\mathsf{E}}} P$:



(25)

By Lemma 5.11, since $in_{\mathsf{E}} \dashv \pi_{\mathsf{E}}$, $\mathsf{Lan}_{in_{\mathsf{E}}} P = P \circ \pi_{\mathsf{E}}$. Since F by assumption is a G-relative opfibration, π_{B} is in particular a Grothendieck opfirbation, hence according to Lemma 5.9,

$$\mathsf{Lan}_{\pi_{\mathsf{B}}} P \circ \pi_{\mathsf{E}}(b) \cong \operatorname*{colim}_{(e,b,\gamma) \in \pi_{\mathsf{B}}^{-1}(b)} P(e) \cong \left(\coprod_{(e,b,\gamma) \in \pi_{\mathsf{B}}^{-1}(b)} P(e) \right) \big/ \sim_{F[\mathsf{G}]} .$$

Here, the equivalence relation $\sim_{F[\mathsf{G}]}$ is the least equivalence relation such that

$$((e,b,\gamma),p) \sim_{F[\mathsf{G}]} ((e',b,\gamma'),p') :\Leftrightarrow \exists (e,b,\gamma) \xrightarrow{(\varepsilon, id_b)} (e',b,\gamma') \in \pi_{\mathsf{B}}^{-1}(b) : p' = P(\varepsilon)p \,.$$

Finally, according to the definition of morphisms in $F[\mathsf{G}]$, for a morphism (ε, id_b) in the above equation exists only if $\gamma = \gamma' \circ F(\varepsilon)$, which explains the last isomorphism in (23). ◀

We will now demonstrate the utility of these results for evaluating convolution products. Invoking Theorem 5.12 yields the following results:

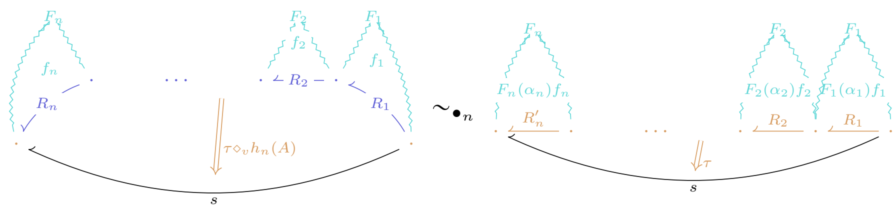
▶ **Lemma 5.13.** *Let $\mathbb{D}$ be a double category such that for all $n > 1$, the functors $h_n : \mathbb{D}_n \to \mathbb{D}_1$ are globular opfibrations (i.e., $\mathbb{D}$ has the positive CDP property). Denote by $\iota : \mathbb{D}_1^\bullet \to \mathbb{D}_1$ the inclusion functor from the wide subcategory of globular morphisms into $\mathbb{D}_1$, and define $\mathbb{D}_n^\bullet$ as the wide subcategory of $\mathbb{D}_n$ whose morphisms satisfy $h_n(\mathbb{D}_n) \in \mathbb{D}_1^\bullet$. Let $F_n, \ldots, F_1 : \mathbb{D}_1 \to \mathsf{Set}$ be covariant presheaves, and denote by $\mathbb{F}_n^\bullet : \mathbb{D}_n \to \mathsf{Set}$ the restriction of $F_n \times \ldots \times F_1$ to $\mathbb{D}_n^\bullet$. Then the convolution product formula simplifies as follows:*

$$(F_n * \ldots * F_1)(r) \cong \left(\coprod_{R \in \mathbb{D}_n^\bullet} \mathbb{D}_1^\bullet(h_n(R), r) \times \mathbb{F}_n^\bullet(R) \right) \big/ \sim_{\bullet_n} \tag{26}$$

where $\sim_{\bullet_n}$ is the least equivalence relation that satisfies

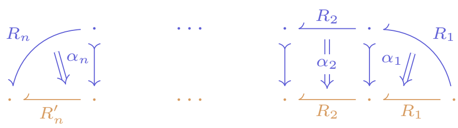
$$\begin{aligned} (R,(\sigma,f)) \sim_{\bullet_n} (R',(\sigma',f')) \Leftrightarrow \exists R - A \to R' \in \mathbb{D}_n^\bullet, (\gamma,g) \in \mathbb{D}_1^\bullet(h_n(R'),r) \times \mathbb{F}_n^\bullet(R) :\\ (\sigma,f) = (\gamma \circ h_n(A), g) \ \wedge \ (\sigma',f') = (\gamma, \mathbb{F}_n^\bullet(A)g) \,. \end{aligned} \tag{27}$$

In order to provide some intuition for the structure of convolution products within the refined framework, we provide below a graphical illustration of $\sim_{\bullet_n}$ (where $\sigma = \tau \diamond_v h_n(A)$):



(28)

$A =$



$\in \mathbb{D}_n^\bullet$

The preceding discussion allows us to give a fully rigorous proof of Theorem 5.5, which may be found in the long version of the paper (to appear).

5.4 A brief analysis of unitality

As already mentioned, neither the framed bicategory $\mathbb{S}\mathsf{pan}$ nor the double category $\mathbb{D}\mathsf{PO}$ modeling DPO-rewriting have the 0-CDP, in the sense that the functor $h_0 = U : \mathbb{D}_0 \to \mathbb{D}_1$ is not a globular opfibration. Nevertheless, the convolution product over $\mathbb{D}\mathsf{PO}$ is in fact strongly unital. One way to establish unitality is by showing that $\mathbb{D}\mathsf{PO}$ and similar double categories do satisfy a weakened version of the 0-CDP, equivalent to saying that h_0 is an opfibration relative to both the subcategory of S-vertical maps (i.e., double cells α such that $S(\alpha)$ is an identity in $\mathbb{D}_0$) and the subcategory of T-vertical maps. We leave a more detailed analysis of this phenomenon to future work.

6 Categorification of rule algebras

As a presheaf, the convolution product $\hat{\Delta}_s * \hat{\Delta}_r$ of two representable presheaves $\hat{\Delta}_r$ and $\hat{\Delta}_s$ is isomorphic to a *colimit of representables* by general considerations on categories of presheaves. Moreover, the fact that the convolution product $* : \widehat{\mathbb{D}} \times \widehat{\mathbb{D}} \to \widehat{\mathbb{D}}$ preserves colimits componentwise (Prop. 3.2) implies that it is entirely determined by its restriction $\mathbb{D}_1 \times \mathbb{D}_1 \to \widehat{\mathbb{D}}$ to representable presheaves. In the introduction, we recalled how the rule algebra product was typically defined as a *sum over admissible matchings* between two rules, see equation (1). We now categorify this formula by showing that in many situations the convolution product of representable presheaves is isomorphic to a *sum of representables* of the following form

$$\hat{\Delta}_{r_2} * \hat{\Delta}_{r_1} \quad \cong \quad \sum_{j \in J} \quad \hat{\Delta}_{s_j}$$

where the family of horizontal cells noted

$$r_2 \circledast r_1 \quad = \quad \big(s_j : A_j \rightharpoonup B_j\big)_{j \in J}$$

can be effectively computed from r_1 and r_2. The property means that, in a certain sense, the family $r_2 \circledast r_1 = (s_j)_{j \in J}$ *classifies* the convolution product of the presheaves $\hat{\Delta}_{r_2}$ with $\hat{\Delta}_{r_1}$, thus providing a categorified version of what is known as the concurrency theorem in rewriting theory (compare [1, 4, 5, 7, 13]). The intuition is that the composition of two rewrite rules can be classified into a number of different, disjoint cases, induced by all possible ways of matching the source of one rule with the target of the other.

In order to formalize this intuition in the language of double categories, we make from now on the assumption that our double category $\mathbb{D}$ satisfies the following property:

(i) the vertical category $\mathbb{D}_0$ has multi-sums.

We find useful to recall at this stage the notion of *multi-sum* due to Diers [12]. Suppose that A and B are objects in a category. A *multi-sum* (or multi-coproduct) of A and B is a family of cospans

$$\Big(A \xrightarrow{a_i} U_i \xleftarrow{b_i} B\Big)_{i \in I} \tag{29}$$

such that for any cospan $A \xrightarrow{f} X \xleftarrow{g} B$ there exists a unique $i \in I$ and a unique morphism $[f, g] : U_i \to X$ such that $f = [f, g] \circ a_i$ and $g = [f, g] \circ b_i$. The multi-sum generalizes the standard notion of coproduct $A \xrightarrow{a} A + B \xleftarrow{b} B$ to situations in which there may not necessarily be a single universal cospan through which all other cospans $A \xrightarrow{f} X \xleftarrow{g} B$ factor, but there is nonetheless a universal family (29) of such cospans. As with the ordinary coproduct of two objects, when it exists the multi-sum of A and B is unique up to unique isomorphism.

Notation: given two horizontal 1-cells $r_1 : A \rightharpoonup B$ and $r_2 : C \rightharpoonup D$ of the double category $\mathbb{D}$, we find sometimes convenient to write $\Sigma^*_{(r_2,r_1)}$ for the set of cospans $(m_2, m_1) = (c_i, b_i)$ appearing in the multi-sum of B and C. This notation is used in particular in §6.2.

6.1 A first easy version of categorification

We start by establishing a categorification of equation (1) under the general assumption that

(ii) the source and target functors $S, T : \mathbb{D}_1 \to \mathbb{D}_0$ are Grothendieck opfibrations.

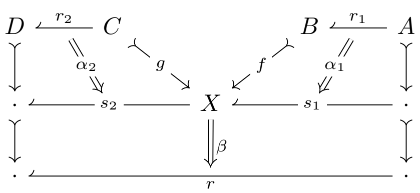
This assumption holds for framed bicategories since it is weaker than the assumption that the pairing (T, S) is an opfibration. It also holds for the double category $\mathbb{T}\mathsf{RS}[\Sigma]$ of term rewriting which is not a framed bicategory. On the other hand, this assumption is too strong for the double category $\mathbb{D}\mathsf{PO}$, and we will see further below how to weaken it to prove a more general formula that also applies in that case. We establish that

▶ **Theorem 6.1.** *Assume $\mathbb{D}$ is a small double category satisfying assumptions (i) and (ii) and suppose that $r_1 : A \rightharpoonup B$ and $r_2 : C \rightharpoonup D$ are horizontal 1-cells in $\mathbb{D}$. In that case, the convolution product of two representable presheaves is isomorphic to the sum of representables*

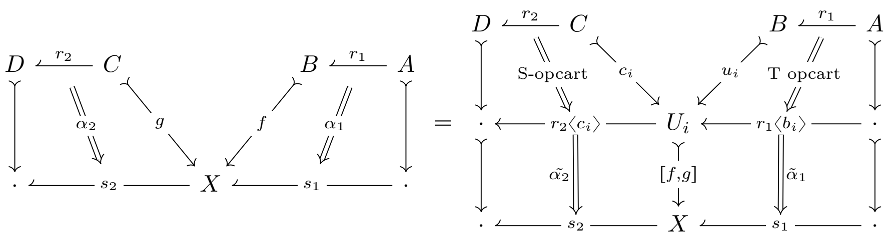
$$\hat{\Delta}_{r_2} * \hat{\Delta}_{r_1} \quad \cong \quad \sum_{i \in I} \quad \hat{\Delta}_{r_2 \langle c_i \rangle \diamond_h \langle b_i \rangle r_1} \tag{30}$$

where the multi-sum of B and C is given by a family of cospans $(B \xrightarrow{b_i} U_i \xleftarrow{c_i} C)_{i \in I}$, and where $r_2 \langle c_i \rangle$ denotes the S-pushforward of r_2 along c_i and $\langle b_i \rangle r_1$ denotes the T-pushforward of r_1 along b_i.

Proof. By definition, an element of $\hat{\Delta}_{r_2} * \hat{\Delta}_{r_1}$ evaluated at a generic horizontal 1-cell r consists of three double cells of the following shape:

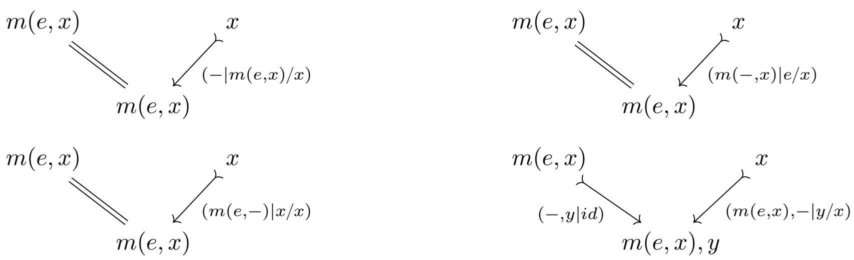


Since $(B \xrightarrow{b_i} U_i \xleftarrow{c_i} C)_{i \in I}$ is the multi-sum of B and C, there exists a unique $i \in I$ and a morphism $[f, g] : U_i \to X$ such that $f = [f, g] \circ b_i$ and $g = [f, g] \circ c_i$. By the assumption that S and T are opfibrations, the double cells α_1 and α_2 therefore factor as follows:



Observe that the double cell $(\tilde{\alpha}_2 \diamond_h \tilde{\alpha}_1) \diamond_v \beta$ is an element of the representable $\hat{\Delta}_{r_2 \langle c_i \rangle \diamond_h \langle b_i \rangle r_1}$ evaluated at r. This defines a natural transformation from $\hat{\Delta}_{r_2} * \hat{\Delta}_{r_1}$ to $\sum_{i \in I} \hat{\Delta}_{r_2 \langle c_i \rangle \diamond_h \langle b_i \rangle r_1}$, which is invertible by the universal properties of the multi-sum and the pushforward. ◀

▶ **Example 6.2.** In Example 3.3 we saw how the convolution product $\hat{\Delta}_r * \hat{\Delta}_r$ of the representable presheaf for the rewrite rule $r : m(e, x) \rightharpoonup x$ decomposes as the sum (9) of representables. This decomposition may be seen as a consequence of Theorem 6.1. First, note that the multi-sum of x and $m(e, x)$ exists in $\mathbb{TRS}[\Sigma]_0$, and is given by the minimal set of four unifying context/substitution pairs depicted below (with the last corresponding to the *disjoint matching* of the two terms, up to variable renaming).



Moreover, observe that the functors $S, T : \mathbb{TRS}[\Sigma]_1 \to \mathbb{TRS}[\Sigma]_0$ are Grothendieck opfibrations, indeed even discrete opfibrations: the S-pushforward of a rule $\mathbf{t} \rightharpoonup \mathbf{t}'$ along a vertical 1-cell $\mathbf{t} \rightarrowtail \mathbf{u} = C[\mathbf{t}\sigma]$ is the rule $C[\mathbf{t}\sigma] \rightharpoonup C[\mathbf{t}'\sigma]$, and similarly for the T-pushforward along a vertical 1-cell $\mathbf{t}' \rightarrowtail \mathbf{u}' = C[\mathbf{t}'\sigma]$. Instantiating (30), we recover (9).

6.2 A more advanced version of categorification

In this subsection, we refine the assumptions of the previous subsection in order to establish in a more general framework that the convolution product of two representable presheaves is a sum of representables. One main motivation is to include among our examples the double category $\mathbb{D}$PO and other double categories of interest in graph rewriting theory. From now on, we thus make the following two assumptions (iia) and (iib) on the double category $\mathbb{D}$, which generalize the assumption (ii) just made in the previous subsection:

(iia) the source functor $S : \mathbb{D}_1 \to \mathbb{D}_0$ is a multi-opfibration;
(iib) the target functor $T : \mathbb{D}_1 \to \mathbb{D}_0$ is a residual multi-opfibration.

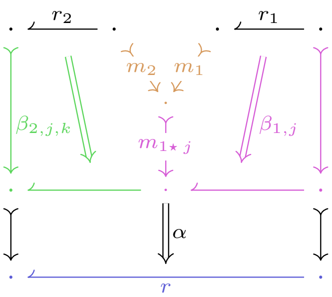
The three assumptions (i), (iia) and (iib) are part of the definition of *compositional rewriting double category (crDC)* formulated in [5] where the interested reader will find the notion of (residual) multi-opfibration. We establish that

▶ **Theorem 6.3.** *Assume $\mathbb{D}$ is a small double category satisfying assumptions (i), (iia) and (iib) and suppose that $r_1 : A \rightharpoonup B$ and $r_2 : C \rightharpoonup D$ are horizontal 1-cells in $\mathbb{D}$. In that case, the convolution product of two representable presheaves is isomorphic to the sum of representables*

$$\hat{\Delta}_{r_2} * \hat{\Delta}_{r_1} \cong \sum_{(m_2, m_1) \in \Sigma^*_{(r_2, r_1)}} \; \sum_{(m_{1\,\star j}, \beta_{1,j}) \in T^*(r_1; m_1)} \; \sum_{\beta_{2,j,k} \in S^*(r_2, m_{1\,\star j} \circ m_2)} \hat{\Delta}_{\beta_{2,j,k}(r_2) \diamond_h \beta_{1,j}(r_1)} \quad (31)$$

where $\Sigma^*_{(r_2, r_1)}$ denotes the set of cospans appearing in the multi-sum of B and C, and where a choice of cleavages S^*, T^* for S, T on $\mathbb{D}_0$, respectively.

For illustration, every element of the presheaf $\hat{\Delta}_{r_2} * \hat{\Delta}_{r_1}$ at instance the horizontal arrow r may be factored in the following way:



(32)

where the pair (m_2, m_1) of vertical arrows $m_1 : B \rightarrowtail U_i$ and $m_2 : C \rightarrowtail U_i$ is an element of the set $\Sigma^*_{(r_2,r_1)}$ of cospans appearing in the multi-sum of B and C.

7 Conclusion

In this paper, we explain how our original project of categorifying the rule algebra $(\mathcal{R}, \star)$ associated to a compositional rewriting theory brought us to formulate a very general notion of convolution product $* : \widehat{\mathbb{D}} \times \widehat{\mathbb{D}} \to \widehat{\mathbb{D}}$ for vertical presheaves over a double category $\mathbb{D}$. The convolution product is only oplax associative in general, and we thus investigate in the paper sufficient conditions on the double category $\mathbb{D}$ for the convolution product to be strongly associative. We start by establishing that the convolution product is strongly associative in the case of framed bicategories, but not necessarily strongly unital. We then extend this result by formulating a more general cylindrical decomposition property for double categories (as an instance of the more general notion of relative opfibration) which, we show, implies that the convolution product is strongly associative under the assumption of n-CDP for all $n > 0$. The question of the strong unitality of the convolution product appears to be very subtle and interesting: it fails for the framed bicategory $\mathbb{S}\mathsf{pan}$ (Example 4.5), it holds for $\mathbb{T}\mathsf{RS}[\Sigma]$ as a consequence of 0-CDP, and it holds for $\mathbb{D}\mathsf{PO}$ *despite* the failure of 0-CDP.

One main achievement of the paper is to justify the view that the convolution product $* : \widehat{\mathbb{D}} \times \widehat{\mathbb{D}} \to \widehat{\mathbb{D}}$ categorifies the product $\star : \mathcal{R} \otimes_{\mathbf{k}} \mathcal{R} \to \mathcal{R}$ of the rule algebra, thanks to formulas (30) and (31) which play the role of formula (1). We see this as a foundation for developing a deeper understanding of the rule algebra representation $\rho : \mathcal{R} \to \mathit{Endo}_{\mathbf{k}}(\mathcal{S})$ defined by formula (2) in the introduction, as well as formula (3). A strong benefit of categorification which we will clarify in future work is that it unifies, thanks to the Yoneda embedding, the rule algebra $\mathcal{R}$ with its action on states in $\mathcal{S}$ through the representation ρ, following a healthy analogy with the well-known principle of Cayley theorem in algebra.

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