Type Isomorphisms for Multiplicative-Additive Linear Logic

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─── **Abstract** ───

We characterize type isomorphisms in the multiplicative-additive fragment of linear logic (MALL), and thus for ⋆-autonomous categories with finite products, extending a result for the multiplicative fragment by Balat and Di Cosmo [2]. This yields a much richer equational theory involving distributivity and annihilation laws. The unit-free case is obtained by relying on the proof-net syntax introduced by Hughes and Van Glabbeek [10]. We then use the sequent calculus to extend our results to full MALL (including all units).

1 Introduction

The question of type isomorphisms consists in trying to understand when two types in a type system (or two formulas in a logic) are "the same". The general question can be described in category theory: two objects A and B are isomorphic ($A \simeq B$) if there exist morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$. f and g are the underlying isomorphisms. Given a (class of) category, the question is then to find equations characterizing when two objects A and B are isomorphic (in all instances of the class). The focus here is on pairs of isomorphic objects rather than on the isomorphisms themselves. For example, in the class of cartesian categories, one finds the following isomorphic objects: $A \times B \simeq B \times A$, $(A \times B) \times C \simeq A \times (B \times C)$ and $A \times \top \simeq A$. Regarding type systems and logics, one can instantiate the categorical notion. For instance in typed λ-calculi: two types A and B are isomorphic if there exist two λ-terms $M : A \to B$ and $N : B \to A$ such that $\lambda x : B.(M\ (N\ x)) =_{\beta\eta} \lambda x : B.x$ and $\lambda x : A.(N\ (M\ x)) =_{\beta\eta} \lambda x : A.x$ where $=_{\beta\eta}$ is $\beta\eta$-equality. This corresponds to isomorphic objects in the syntactic category generated by terms up to $=_{\beta\eta}$. Similarly, type isomorphisms can also be considered in logic, following what happens in the λ-calculus through the Curry-Howard correspondence: simply replace λ-terms with proofs, types with formulas, β-reduction with cut-elimination and η-expansion with axiom-expansion. In this way, type isomorphisms are studied in a wide range of theories, such as category theory [16], λ-calculus [4] and proof theory [2]. They may be used to develop practical tools, such as search in a library of a functional programming language [14].

Following the definition, it is usually easy to prove that the type-isomorphism relation is a congruence. It is then natural to look for an equational theory generating this congruence. Testing whether or not two types are isomorphic is then much easier. An equational theory $\mathcal{T}$ is called *sound* with respect to type isomorphisms if types equal up to $\mathcal{T}$ are isomorphic. It is called *complete* if it equates any pair of isomorphic types. Given a (class of) category, a type system or a logic, our goal is to find an associated sound and complete equational theory for type isomorphisms. This is not always possible as the induced theory may not be finitely axiomatisable (see for instance [6]).

Soundness is usually the easy direction as it is sufficient to exhibit pairs of terms corresponding to each equation. The completeness part is often harder, and there are in the literature two main approaches to solve this problem. The first is a semantic method, relying on the fact that if two types are isomorphic then they are isomorphic in all (denotational) models. One thus looks for a model in which isomorphisms can be computed (more easily than in the syntactic model) and are all included in the equational theory under consideration (this is the approach used in [16, 12] for example). Finding such a model simple enough for its isomorphisms to be computed, but still complex enough not to contain isomorphisms absent in the syntax is the difficulty. The second method is the syntactic one, which consists in studying isomorphisms directly in the syntax. The analysis of pairs of terms composing to the identity should provide information on their structure and then on their type so as to deduce the completeness of the equational theory (see for example [4, 2]). The easier the equality ($=_{\beta\eta}$ for example) between proof objects can be computed, the easier the analysis of isomorphisms will be.

We place ourselves in the framework of linear logic (LL) [7], the underlying question being "is there an equational theory corresponding to the isomorphisms between formulas in this logic?". LL is a very rich logic containing three classes of propositional connectives: multiplicative, additive and exponential ones. The multiplicative and additive families provide two copies of each classical propositional connective: two copies of conjunction ($\otimes$ and $\&$), of disjunction ($\bindnasrepma$ and $\oplus$), of true (1 and $\top$) and of false ($\bot$ and 0). The exponential family is constituted of two modalities ! and ? bridging the gap between multiplicatives and additives through four isomorphisms $!(A \& B) \simeq !A \otimes !B$, $?(A \oplus B) \simeq ?A \bindnasrepma ?B$, $!\top \simeq 1$ and $?0 \simeq \bot$. In the multiplicative fragment (MLL) of LL (using only $\otimes$, $\bindnasrepma$, 1 and $\bot$, and corresponding to $\star$-autonomous categories), the question of type isomorphisms was answered positively using a syntactic method based on proof-nets by Balat and Di Cosmo [2]: isomorphisms emerge from associativity and commutativity of the multiplicative connectives $\otimes$ and $\bindnasrepma$, as well as neutrality of the multiplicative units 1 and $\bot$. The question was also solved for the polarized fragment of LL by one of the authors using game semantics [12]. It is conjectured that isomorphisms in full LL correspond to those in its polarized fragment (Table 1 together with the four exponential equations above). As a step towards solving this conjecture, we prove the type isomorphisms in the multiplicative-additive fragment (MALL) of LL are generated by the equational theory of Table 1 (and this applies at the same time to the class of $\star$-autonomous categories with finite products).

This situation is much richer than in the multiplicative fragment since isomorphisms include not only associativity, commutativity and neutrality, but also the distributivity of the multiplicative connective $\otimes$ (resp. $\bindnasrepma$) over the additive $\oplus$ (resp. $\&$) as well as the associated annihilation laws for the additive unit 0 (resp. $\top$) over the multiplicative connective $\otimes$ (resp. $\bindnasrepma$). Using a semantic approach looks difficult as most of the known models of MALL immediately come with unwanted isomorphisms not valid in the syntax: $\top \otimes A \simeq \top$ in coherent spaces for example [7]. For this reason we use a syntactic method. We follow the

Table 1 Type isomorphisms in MALL.

Commutativity	$A \otimes B = B \otimes A$	$A \parr B = B \parr A$	$A \oplus B = B \oplus A$	$A \& B = B \& A$
Associativity	$A \otimes (B \otimes C) = (A \otimes B) \otimes C$		$A \parr (B \parr C) = (A \parr B) \parr C$	
	$A \oplus (B \oplus C) = (A \oplus B) \oplus C$		$A \& (B \& C) = (A \& B) \& C$	
Distributivity	$A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$		$A \parr (B \& C) = (A \parr B) \& (A \parr C)$	
Neutrality	$A \otimes 1 = A$	$A \parr \bot = A$	$A \oplus 0 = A$	$A \& \top = A$
Annihilation	$A \otimes 0 = 0$	$A \parr \top = \top$		

approach by Balat and Di Cosmo [2] based on proof-nets. Indeed, proof-nets provide a very good syntax for linear logic where studying composition of proofs by cut, cut-elimination and identity of proofs is very natural. However, already in [2] some trick had to be used to deal with units as proof-nets are working perfectly only in the unit-free multiplicative fragment of linear logic. If one puts units aside, there is a notion of proof-nets incorporating both multiplicative and additive connectives in such a way that cut-free proofs are represented in a *canonical* way, and cut-elimination can be dealt with in a parallel manner. This is the syntax of proof-nets introduced by Hughes & Van Glabbeek in [10].

Our proof of the completeness of the equational theory of Table 1 goes in two steps. First we adapt, in Section 3, the proof of Balat & Di Cosmo [2] to the setting of Hughes & Van Glabbeek's proof-nets [10]. This requires a precise analysis of the structure of proof-nets because of the richer structure induced by the presence of the additive connectives. The situation is much more complex than in the multiplicative setting since for example sub-formulas can be duplicated through distributivity equations, breaking a linearity property crucial in [2]. Once this is solved, it remains to add units (Section 4). By lack of a good-enough notion of proof-nets for MALL including units, we go back to the sequent calculus to deal with units on top of the results obtained for the unit-free fragment. This goes through a characterization of the equality of proofs up to cut-elimination and axiom-expansion by means of rule commutations. A result which is not surprising, but never proved before for MALL as far as we know, and rather tedious to settle. Using it, we analyse the behaviour of units inside isomorphisms to conclude that they can be replaced with fresh atoms, once formulas are simplified appropriately. We can conclude by means of the unit-free case. Finally, seeing MALL as a category, we extend our result to conclude that Table 1 (together with $A \multimap B \simeq A^\perp \parr B$, De Morgan's laws and involutivity of negation) provides the equational theory of isomorphisms valid in all $\star$-autonomous categories with finite products (Section 5).

2 Definitions and preliminary results

2.1 Multiplicative-Additive Linear Logic

The multiplicative-additive fragment of linear logic [7], denoted by MALL, has formulas given by the following grammar, where X belongs to a given enumerable set of *atoms*:

$$A, B \coloneqq X \mid X^\perp \mid A \otimes B \mid A \parr B \mid 1 \mid \bot \mid A \& B \mid A \oplus B \mid \top \mid 0$$

Orthogonality $(\cdot)^\perp$ expands into an involution on arbitrary formulas through $X^{\perp\perp} = X$ on an atom X, $1^\perp = \bot$, $\bot^\perp = 1$, $\top^\perp = 0$, $0^\perp = \top$ and De Morgan's laws $(A \otimes B)^\perp = B^\perp \parr A^\perp$, $(A \parr B)^\perp = B^\perp \otimes A^\perp$, $(A \& B)^\perp = B^\perp \oplus A^\perp$, $(A \oplus B)^\perp = B^\perp \& A^\perp$. The *non-commutative* De Morgan's laws are the good notion of duality, as shown in the context of cyclic linear logic where this leads to planar proof-nets [1]. This choice in our setting will often result in planar graphs on our illustrations, with axiom links not crossing each others.

Sequents are lists of formulas of the form $\vdash A_1, \ldots, A_n$. Sequent calculus rules are:[1]

$$\frac{}{\vdash A^\perp, A} \; ax \qquad \frac{\vdash \Gamma}{\vdash \sigma(\Gamma)} \; ex \qquad \frac{\vdash A, \Gamma \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta} \; cut$$

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash A \otimes B, \Gamma, \Delta} \; \otimes \qquad \frac{\vdash A, B, \Gamma}{\vdash A \,\mathbin{\rotatebox[origin=c]{180}{\&}}\, B, \Gamma} \; \mathbin{\rotatebox[origin=c]{180}{\&}} \qquad \frac{}{\vdash 1} \; 1 \qquad \frac{\vdash \Gamma}{\vdash \perp, \Gamma} \; \perp$$

$$\frac{\vdash A, \Gamma \qquad \vdash B, \Gamma}{\vdash A \,\&\, B, \Gamma} \; \& \qquad \frac{\vdash A, \Gamma}{\vdash A \oplus B, \Gamma} \; \oplus_1 \qquad \frac{\vdash B, \Gamma}{\vdash A \oplus B, \Gamma} \; \oplus_2 \qquad \frac{}{\vdash \top, \Gamma} \; \top$$

In practice we consider exchange rules as incorporated in the conclusion of the rule above, thus dealing with rules like: $\dfrac{\vdash A, B, \Gamma, \Delta}{\vdash \Gamma, A \,\mathbin{\rotatebox[origin=c]{180}{\&}}\, B, \Delta} \; \mathbin{\rotatebox[origin=c]{180}{\&}}$. In this spirit, when we write $\dfrac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, A \,\mathbin{\rotatebox[origin=c]{180}{\&}}\, B, \Delta} \; \mathbin{\rotatebox[origin=c]{180}{\&}}$ we mean that the appropriate permutation is also incorporated in the rule above.

The main difference with the multiplicative fragment of linear logic (MLL) is the &-rule, which introduces some sharing of the context Γ. From this comes the notion of a *slice* [7, 8] which is a partial proof missing some additive component. Slices are obtained by using the same rules as proofs except for the &-rule which is replaced by its two sliced versions:

$$\frac{\vdash A, \Gamma}{\vdash A \,\&\, B, \Gamma} \; \&_1 \qquad\qquad \frac{\vdash B, \Gamma}{\vdash A \,\&\, B, \Gamma} \; \&_2$$

By *unit-free* MALL, we mean the restriction of MALL to formulas not involving the units $1, \perp, \top$ and 0, and as such without the $1, \perp$ and $\top$-rules. When speaking of a *positive* formula, we mean a formula with main connective $\otimes$ or $\oplus$, a unit 1 or 0, or an atom X. A *negative* formula is one with main connective $\mathbin{\rotatebox[origin=c]{180}{\&}}$ or $\&$, a unit $\perp$ or $\top$, or a negated atom $X^\perp$.

2.2 Linear isomorphisms

▶ **Definition 1** (Isomorphism). *Two formulas A and B are* isomorphic, *denoted $A \simeq B$, if there exist proofs π of $\vdash A^\perp, B$ and π' of $\vdash B^\perp, A$ whose composition by cut over B (resp. A) is equal to the axiom on $\vdash A^\perp, A$ (resp. $\vdash B^\perp, B$) up to axiom-expansion and cut-elimination. (Axiom-expansion and cut-elimination for MALL are recalled in Appendix A.)*

Because of the analogy with the λ-calculus and since there will be no ambiguity, we use the notation $=_{\beta\eta}$ for equality of proofs up to cut-elimination (β) and axiom-expansion (η). Similarly, $=_\beta$ is equality up to cut-elimination only. We use the notations $\pi \overset{B}{\bowtie} \pi'$ for the proof obtained by adding a cut on B between π and π', and $A \overset{\pi, \pi'}{\simeq} B$ when π and π' define an isomorphism between A and B, that is when $\pi \overset{B}{\bowtie} \pi' =_{\beta\eta} \mathrm{id}_A$ and $\pi \overset{A}{\bowtie} \pi' =_{\beta\eta} \mathrm{id}_B$ (where id_A is the axiom-expansion of the proof of $\vdash A^\perp, A$ containing just an axiom rule).

We aim to prove that two MALL (resp. unit-free MALL) formulas are isomorphic if and only if they are equal in the equational theory $\mathcal{E}$ (resp. $\mathcal{E}^\dagger$) defined as follows.

▶ **Definition 2** (Equational theories). *We denote by $\mathcal{E}$ the equational theory given on Table 1 on Page 3, while $\mathcal{E}^\dagger$ denotes the part not involving units,* i.e. *with commutativity, associativity and distributivity only.*

Given an equational theory $\mathcal{T}$, the notation $A =_{\mathcal{T}} B$ means that formulas A and B are equal in the theory $\mathcal{T}$. As often, the soundness part is easy (but tedious) to prove.

▶ **Theorem 3** (Isomorphisms soundness, see Lemma 3 in [12]). *If $A =_{\mathcal{E}} B$ then $A \simeq B$.*

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[1] With A and B arbitrary formulas, Γ and Δ contexts (*i.e.* lists of formulas) and σ a permutation.

All the difficulty lies in the proof of the other implication, completeness, on which the rest of this work focuses.

2.3 Axiom-expansion

A first simplification is that we can reduce the definition of isomorphisms to proofs with expanded axioms only, no more using the η relation. Given an MALL proof π, we denote by $\eta(\pi)$ the η-normal form of π, *i.e.* the proof obtained by expanding iteratively all *ax*-rules in π (axiom-expansion is confluent and strongly normalizing).

▶ **Proposition 4** (Reduction to axiom-expanded proofs). *Let π and ϖ be MALL proofs such that $\pi =_{\beta\eta} \varpi$. Then $\eta(\pi) =_\beta \eta(\varpi)$ with, in this sequence, only proofs in η-normal form.*

Thus, we will from now on consider only proofs with expanded axioms, manipulated through composition by cut and cut-elimination. To prove completeness, we start with the unit-free case by using a syntactic approach based on the proof-nets from Hughes & Van Glabbeek [10], which are a more canonical representation of proofs [11].
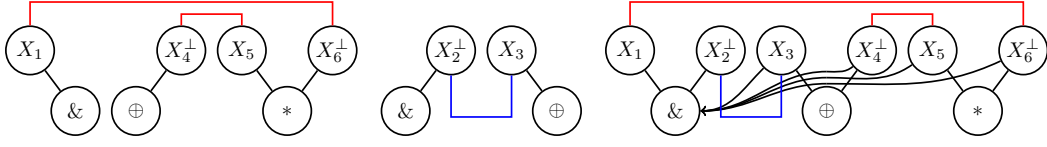
2.4 Proof-nets for unit-free MALL

We use the definition of unit-free MALL proof-net from [10]. Other definitions exist, see the original one from Girard [8], or others such as [5, 9]. Still, the definition we take is one of the most satisfactory, from the point of view of canonicity and cut-elimination for instance (see [10, 11], or the introduction of [9] for a comparison of alternative definitions). We recall here quickly this definition of proof-nets. Please refer to [10] for more details.

A sequent is seen as its syntactic forest with as internal vertices its connectives and as leaves the atoms of its formulas. We always identify a formula A with its syntactic tree $T(A)$. A *cut pair* is a formula $A * A^\perp$, for a formula A; the connective $*$ is unordered. A *cut sequent* $[\Sigma] \, \Gamma$ is composed of a list Σ of cut pairs and a sequent Γ. When $\Sigma = \emptyset$ is empty, we denote it simply by Γ. When we write a $\invamp\backslash\&$-vertex, we mean a $\invamp$- or $\&$-vertex (a *negative* vertex); similarly a $\otimes\backslash\oplus$-vertex is a $\otimes$- or $\oplus$-vertex (a *positive* vertex). An *additive resolution* of a cut sequent $[\Sigma] \, \Gamma$ is any result of deleting zero or more cut pairs from Σ and one argument subtree of each additive connective ($\&$ or $\oplus$) of $\Sigma \cup \Gamma$. A $\&$-*resolution* of a cut sequent $[\Sigma] \, \Gamma$ is any result of deleting one argument subtree of each $\&$-connective of $\Sigma \cup \Gamma$.

An *(axiom) link* on $[\Sigma] \, \Gamma$ is an unordered pair of complementary leaves in $\Sigma \cup \Gamma$ (labeled with X and $X^\perp$). A *linking* λ on $[\Sigma] \, \Gamma$ is a set of links on $[\Sigma] \, \Gamma$ such that the sets of the leaves of its links form a partition of the set of leaves of an additive resolution of $[\Sigma] \, \Gamma$, additive resolution which is denoted $[\Sigma] \, \Gamma \restriction \lambda$.

A set of linkings Λ on $[\Sigma] \, \Gamma$ *toggles* a $\&$-vertex W if both arguments (called *premises*) of W are in $[\Sigma] \, \Gamma \restriction \Lambda := \bigcup_{\lambda \in \Lambda}[\Sigma] \, \Gamma \restriction \lambda$. We say a link a *depends* on a $\&$-vertex W in Λ if there exist $\lambda, \lambda' \in \Lambda$ such that $a \in \lambda\backslash\lambda'$ and W is the only $\&$-vertex toggled by $\{\lambda; \lambda'\}$. The graph $\mathcal{G}_\Lambda$ is defined as $[\Sigma] \, \Gamma \restriction \Lambda$ with the edges from $\cup\Lambda$ and enriched with jump edges $l \to W$ for each leaf l and each $\&$-vertex W such that there exists $a \in \lambda \in \Lambda$, between l and some l', with a depending on W in Λ. When $\Lambda = \{\lambda\}$ is composed of a single linking, we shall simply denote $\mathcal{G}_\lambda = \mathcal{G}_{\{\lambda\}}$ (which is the graph $[\Sigma] \, \Gamma \restriction \lambda$ with the edges from λ and no jump edge).

A *switch edge* of a $\invamp\backslash\&$-vertex N is an in-edge of N, *i.e.* an edge between N and one of its premises or a jump to N. A *switching cycle* is a cycle with at most one switch edge of each $\invamp\backslash\&$-vertex. A $\invamp$-*switching* of a linking λ is any subgraph of $\mathcal{G}_\lambda$ obtained by deleting a switch edge of each $\invamp$-vertex; denoting by ϕ this choice of edges, the subgraph it yields is $\mathcal{G}_\phi$.

■ **Figure 1** Graphs from an example of a proof-net: from left to right $\mathcal{G}_{\lambda_1}$, $\mathcal{G}_{\lambda_2}$ and $\mathcal{G}_{\{\lambda_1;\lambda_2\}}$.

▶ **Definition 5** (Proof-net). *A* unit-free MALL proof-net *θ on a cut sequent $[\Sigma] \ \Gamma$ is a set of linkings satisfying:*

(P0) Cut: *Every cut pair of Σ has a leaf in θ.*
(P1) Resolution: *Exactly one linking of θ is on any given &-resolution of $[\Sigma] \ \Gamma$.*
(P2) MLL: *For every $\mathfrak{P}$-switching ϕ of every linking $\lambda \in \theta$, $\mathcal{G}_\phi$ is a tree.*
(P3) Toggling: *Every set $\Lambda \subseteq \theta$ of two or more linkings toggles a &-vertex that is in no switching cycle of $\mathcal{G}_\Lambda$.*

These conditions are called the *correctness criterion*. Condition (P0) is here to prevent unused $*$-vertices. A *cut-free* proof-net is one without $*$-vertices (it respects (P0) trivially). Condition (P1) is a correctness criterion for ALL proof-nets [10] and (P2) is the Danos-Regnier criterion for MLL proof-nets [3]. However, (P1) and (P2) together are insufficient for cut-free MALL proof-nets, hence the last condition (P3) taking into account interactions between the slices (see also [5] for a similar condition for example). Sets composed of a single linking λ are not considered in (P3), for by (P2) the graph $\mathcal{G}_\lambda$ has no switching cycle.

An example of proof-net, illustrated on Figure 1, is the following. On the cut sequent $[X_5 * X_6^\perp] \ X_1 \ \& \ X_2^\perp, X_3 \oplus X_4^\perp$ (where each X_i is an occurrence of the same atom X), set $\lambda_1 = \{(X_1, X_6^\perp), (X_4^\perp, X_5)\}$ and $\lambda_2 = \{(X_2^\perp, X_3)\}$. One can check $\{\lambda_1; \lambda_2\}$ is a proof-net.

In the particular setting of isomorphisms, we mainly consider proof-nets with two conclusions. This allows to define a notion of duality on leaves and connectives. Consider a cut sequent containing both A and $A^\perp$. For V a vertex in (the syntax tree $T(A)$ of) A, we denote by $V^\perp$ the corresponding vertex in $A^\perp$. As expected, $V^{\perp\perp} = V$. This also respects orthogonality for formulas on leaves: given a leaf l of A, labeled by a formula X, the label of $l^\perp$ is $X^\perp$. We can also define a notion of duality on premises: given a premise of a vertex $V \in T(A)$, the dual premise of $V^\perp$ is the corresponding premise in $T(A^\perp)$. In other words, if in $L - V - R$ we consider the premise L then in $R^\perp - V^\perp - L^\perp$ its dual premise is $L^\perp$.

▶ **Definition 6** (Composition). *Given proof-nets θ and ϑ of respective conclusions $[\Sigma] \ \Gamma, A$ and $[\Xi] \ \Delta, A^\perp$, the* composition *over A of θ and ϑ is the proof-net $\theta \overset{A}{\bowtie} \vartheta = \{\lambda \cup \mu \mid \lambda \in \theta, \mu \in \vartheta\}$, with conclusions $[\Sigma, \Xi, A * A^\perp] \ \Gamma, \Delta$.*

For example, see Figure 7 with a composition of the proof-nets on Figure 5.

▶ **Definition 7** (Cut-elimination). *Let θ be a set of linkings on a cut sequent $[\Sigma] \ \Gamma$, and $A * A^\perp$ a cut pair in Σ. Define the* elimination *of $A * A^\perp$ (or of the cut $*$ between A and $A^\perp$) as:*

(a) *If A is an atom, delete $A * A^\perp$ from Σ and replace any pair of links $(l, A), (A^\perp, m)$ (l and m being other occurrences of $A^\perp$ and A respectively) with the link (l, m).*
(b) *If $A = A_1 \otimes A_2$ and $A^\perp = A_2^\perp \ \mathfrak{P} \ A_1^\perp$ (or vice-versa), replace $A * A^\perp$ with two cut pairs $A_1 * A_1^\perp$ and $A_2 * A_2^\perp$. Retain all original linkings.*
(c) *If $A = A_1 \ \& \ A_2$ and $A^\perp = A_2^\perp \oplus A_1^\perp$ (or vice-versa), replace $A * A^\perp$ with two cut pairs $A_1 * A_1^\perp$ and $A_2 * A_2^\perp$. Delete all* inconsistent *linkings, namely those $\lambda \in \theta$ such that in $[\Sigma] \ \Gamma \upharpoonright \lambda$ the children & and $\oplus$ of the cut do not take dual premises. Finally, "garbage collect" by deleting any cut pair $B * B^\perp$ for which no leaf of $B * B^\perp$ is in any of the remaining linkings.*

See Figure 8 for a result on applying steps (b) and (c) to the proof-net of Figure 7.

▶ **Proposition 8** (Proposition 5.4 in [10]). *Eliminating a cut in a proof-net yields a proof-net.*

▶ **Theorem 9** (Theorem 5.5 in [10]). *Cut-elimination of proof-nets is strongly normalizing and confluent.*

A linking λ on a cut sequent $[\Sigma]\ \Gamma$ *matches* if, for every cut pair $A * A^\perp$ in Σ, any given leaf l of A is in $[\Sigma]\ \Gamma \upharpoonright \lambda$ if and only if $l^\perp$ of $A^\perp$ is in $[\Sigma]\ \Gamma \upharpoonright \lambda$. A linking matches if and only if, when cut-elimination is carried out, the linking never becomes inconsistent, and thus is never deleted. This allows defining *Turbo Cut-elimination* [10], eliminating a cut in a single step by removing inconsistent linkings.

3 Completeness for unit-free MALL

Our method relates closely to the one used by Balat and Di Cosmo in [2]. We work on proof-nets, as they highly simplify the problem by representing proofs up to rule commutations [11]. We start by transposing the study of unit-free MALL isomorphisms to proof-nets of a particular shape, called *bipartite full* (Sections 3.1 and 3.2). Then, we use the distributivity isomorphisms to reduce the problem to special formulas, called *distributed*, allowing to consider even more constrained proof-nets (Section 3.3). These are the key differences with the proof in MLL from [2], where some properties are given for free as there are no slice nor distributivity isomorphism. From this point the problem is similar to unit-free MLL, with commutativity and associativity only. We conclude as in [2]: restricting the problem to so-called *non-ambiguous* formulas, isomorphisms are easily characterized (Section 3.4).

3.1 Reduction to proof-nets

We desequentialize a unit-free MALL proof π (with expanded axioms) into a proof-net $\mathcal{R}(\pi)$ by induction on π using the steps detailed on Figure 2, following [10] with the notation $\theta \triangleright [\Sigma]\ \Gamma$ for "θ is a set of linkings on the cut sequent $[\Sigma]\ \Gamma$". As identified in Section 5.3.4 of [10], desequentializing with both *cut* and &-rules is complex, for cuts can be shared (or not) when translating a &-rule:
$$\frac{\theta \triangleright [\Sigma, \Xi]\ A, \Gamma \qquad \vartheta \triangleright [\Sigma, \Phi]\ B, \Gamma}{\theta \cup \vartheta \triangleright [\Sigma, \Xi, \Phi]\ A \& B, \Gamma}\ \& \quad .$$ We choose to never share cuts ($\Sigma = \emptyset$), thus desequentialization is a function. The cost being that the following $\& - cut$ commutation yields different proof-nets (contrary to the other commutations, see [11]).

$$\cfrac{\cfrac{\cfrac{\pi_1}{\vdash A, B, \Gamma} \quad \cfrac{\pi_2}{\vdash A, C, \Gamma}}{\vdash A, B \& C, \Gamma}\& \quad \cfrac{\pi_3}{\vdash A^\perp, \Delta}}{\vdash B \& C, \Gamma, \Delta}\ cut \quad \equiv \quad \cfrac{\cfrac{\cfrac{\pi_1}{\vdash A, B, \Gamma} \quad \cfrac{\pi_3}{\vdash A^\perp, \Delta}}{\vdash B, \Gamma, \Delta}\ cut \quad \cfrac{\cfrac{\pi_2}{\vdash A, C, \Gamma} \quad \cfrac{\pi_3}{\vdash A^\perp, \Delta}}{\vdash C, \Gamma, \Delta}\ cut}{\vdash B \& C, \Gamma, \Delta}\&$$

▶ **Theorem 10** (Sequentialization, Theorem 5.9 in [10]). *A set of linkings on a cut sequent is a translation of a MALL proof if and only if it is a proof-net.*

▶ **Definition 11** (Identity proof-net). *We call* identity proof-net *of a unit-free MALL formula A, the proof-net corresponding to the proof* id_A *(the axiom-expansion of* $\cfrac{}{\vdash A^\perp, A}\ ax$ *).*

▶ **Theorem 12** (Simulation Theorem). *Let π and ϖ be unit-free MALL proof trees (with expanded axioms). If $\pi =_\beta \varpi$, then $\mathcal{R}(\pi) =_\beta \mathcal{R}(\varpi)$.*

A notion of isomorphism $A \stackrel{\theta,\,\vartheta}{\simeq} B$ can be defined directly on proof-nets: θ and ϑ are two cut-free proof-nets of respective conclusions $A^\perp, B$ and $B^\perp, A$ such that $\theta \stackrel{B}{\bowtie} \vartheta$ and $\vartheta \stackrel{A}{\bowtie} \theta$ reduce by cut-elimination to identity proof-nets. Using the Simulation Theorem, we obtain:

$$\frac{}{\{\{(X, X^\perp)\}\} \rhd [\emptyset] \; X, X^\perp} \; ax \qquad \frac{\theta \rhd [\Sigma] \; \Gamma}{\theta \rhd [\Sigma] \; \sigma(\Gamma)} \; ex$$

$$\frac{\theta \rhd [\Sigma] \; A, \Gamma \qquad \vartheta \rhd [\Xi] \; A^\perp, \Delta}{\{\lambda \cup \mu \mid \lambda \in \theta, \mu \in \vartheta\} \rhd [\Sigma, \Xi, A * A^\perp] \; \Gamma, \Delta} \; cut$$

$$\frac{\theta \rhd [\Sigma] \; A, \Gamma \qquad \vartheta \rhd [\Xi] \; B, \Delta}{\{\lambda \cup \mu \mid \lambda \in \theta, \mu \in \vartheta\} \rhd [\Sigma, \Xi] \; A \otimes B, \Gamma, \Delta} \; \otimes \qquad \frac{\theta \rhd [\Sigma] \; A, B, \Gamma}{\theta \rhd [\Sigma] \; A \,⅋\, B, \Gamma} \;⅋$$

$$\frac{\theta \rhd [\Xi] \; A, \Gamma \qquad \vartheta \rhd [\Phi] \; B, \Gamma}{\theta \cup \vartheta \rhd [\Xi, \Phi] \; A \,\&\, B, \Gamma} \; \& \qquad \frac{\theta \rhd [\Sigma] \; A, \Gamma}{\theta \rhd [\Sigma] \; A \oplus B, \Gamma} \; \oplus_1 \qquad \frac{\theta \rhd [\Sigma] \; B, \Gamma}{\theta \rhd [\Sigma] \; A \oplus B, \Gamma} \; \oplus_2$$

We use the implicit tracking of formula occurrences downwards through the rules.

Figure 2 Inductive definition of the translation of unit-free MALL proof trees to sets of linkings.
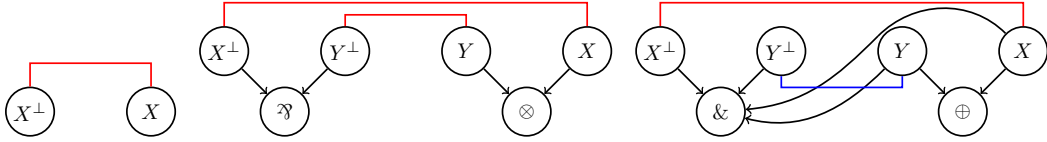
Figure 3 Identity proof-nets (from left to right: atoms, $⅋\backslash\otimes$ and $\&\backslash\oplus$).

▶ **Theorem 13** (Type isomorphisms in proof-nets). *Let A and B be two unit-free MALL formulas. If $A \simeq B$ then there exist two proof-nets θ and ϑ such that $A \overset{\theta,\vartheta}{\simeq} B$.*

3.2 Reduction to bipartite full proof-nets

▶ **Definition 14** (Full, *Ax*-unique, Bipartite proof-net). *A cut-free proof-net is called* full *if any of its leaves has (at least) one link on it. Furthermore, if for any leaf there exists a unique link on it (possibly shared among several linkings), then we call this proof-net* ax-*unique.*

A cut-free proof-net is bipartite *if it has two conclusions, A and B, and each of its links is between a leaf of A and a leaf of B (no link between leaves of A, or between leaves of B).*

We show identity proof-nets are bipartite *ax*-unique, and isomorphisms are bipartite full.

Using an induction on the formula A, we can prove the following results on the identity proof-net of A (see Figure 3 for a graphical intuition).

▶ **Proposition 15.**
  (i) *An identity proof-net is bipartite ax-unique.*
  (ii) *The axiom links of an identity proof-net are exactly the $(l, l^\perp)$, for any leaf l.*
  (iii) *In the identity proof-net of A, exactly one linking is on any given additive resolution of the conclusion A.*

Neither fullness, *ax*-uniqueness nor bipartiteness is preserved by cut anti-reduction. A counter-example is given on Figure 4, with a non bipartite proof-net and a non full one whose composition reduces to the identity proof-net (bipartite *ax*-unique by Proposition 15(i)).[2] However, if *both* compositions yield identity proof-nets, we get bipartiteness and fullness.

---

[2] This example gives a retraction between $(A \,⅋\, A^\perp) \otimes B$ and $((A \,⅋\, A^\perp) \otimes B) \oplus B$ in MALL which is not an isomorphism (as is the retraction between A and $(A \multimap A) \multimap A = (A \otimes A^\perp) \,⅋\, A$ in MLL).
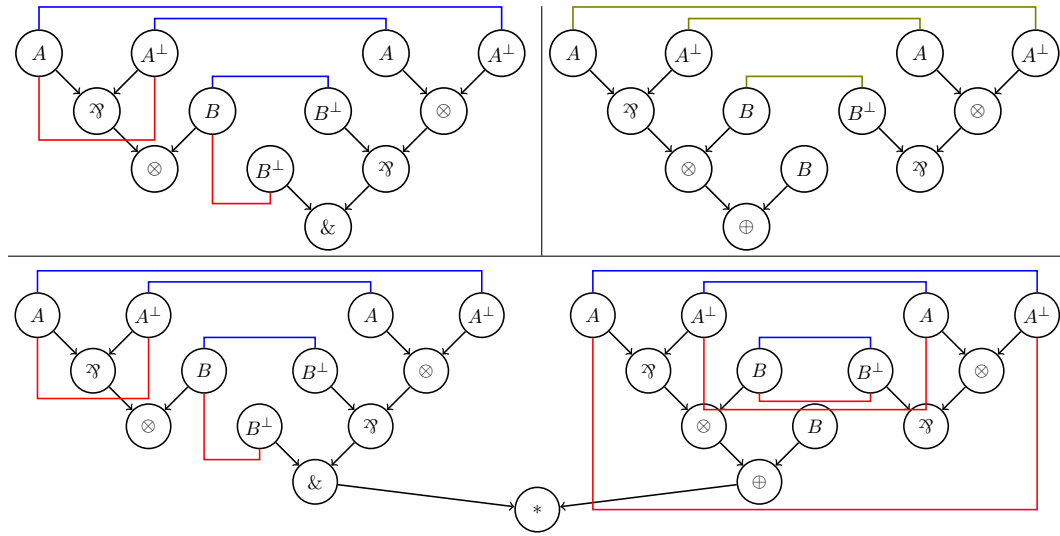
Figure 4 Non bipartite proof-net (top-left), non full proof-net (top-right) and one of their composition yielding the identity proof-net (bottom) (jump edges not represented).

▶ **Lemma 16.** *Let θ and θ' be cut-free proof-nets of respective conclusions $A^\perp, B$ and $B^\perp, A$, such that $\theta' \overset{A}{\bowtie} \theta$ reduces to the identity proof-net of B. For any linking $\lambda \in \theta$, there exists $\lambda' \in \theta'$ such that $\lambda \cup \lambda'$ matches in the composition over B of θ and θ', $\theta \overset{B}{\bowtie} \theta'$.*

Proof. Let us consider a linking $\lambda \in \theta$, and call $\mathcal{C}$ the choices of premise on additive connectives of B that λ makes. We search some $\lambda' \in \theta'$ making the dual choices of premise on additive connectives of $B^\perp$ compared to $\mathcal{C}$. Consider the composition of θ and θ' over A. It reduces to the identity proof-net of B by hypothesis. By Proposition 15(iii), there exists a unique linking in the identity proof-net of B corresponding to $\mathcal{C}$. Furthermore, the linkings of the identity proof-net are derived from the $\mu \cup \mu'$ for μ a linking of θ and μ' one of θ', with $\mu \cup \mu'$ matching for a cut over A: a linking in the identity proof-net is a linking of the form $\mu \cup \mu'$ where axiom links $(l, m) \in \mu$ and $(m^\perp, l^\perp) \in \mu'$ are replaced with $(l, l^\perp)$, with l a leaf of B and m one of $A^\perp$ (because an identity proof-net has only links of the form $(l, l^\perp)$ by Proposition 15(ii)). Therefore, there exist $\mu \in \theta$ and $\mu' \in \theta'$ such that μ makes the choices $\mathcal{C}$ on B and $\mu \cup \mu'$ matches for the composition of θ and θ' over both A and B. But λ makes the same choices $\mathcal{C}$ on B as μ: $\lambda \cup \mu'$ also matches for a cut over B. ◀

▶ **Corollary 17.** *Assuming $A \overset{\theta, \theta'}{\simeq} B$, θ and θ' are bipartite.*

Proof. We proceed by contradiction: *w.l.o.g.* there is a link a in some linking $\lambda \in \theta$ which is between leaves of $A^\perp$. By Lemma 16 there exists $\lambda' \in \theta'$ such that $\lambda \cup \lambda'$ matches for a cut over B. Whence a, which does not involve leaves of B, belongs to a linking of the composition where cuts have been eliminated (it belongs to the linking resulting from $\lambda \cup \lambda'$). But this reduction yields a bipartite proof-net by Proposition 15(i), a contradiction. ◀

▶ **Lemma 18.** *Assume θ and θ' are cut-free proof-nets of respective conclusions $A^\perp, B$ and $B^\perp, A$, and that their composition over B yields the identity proof-net of A. Then any leaf of $A^\perp$ (resp. A) has (at least) one axiom link on it in θ (resp. θ').*

▶ **Theorem 19.** *Assuming $A \overset{\theta, \theta'}{\simeq} B$, θ and θ' are bipartite full.*
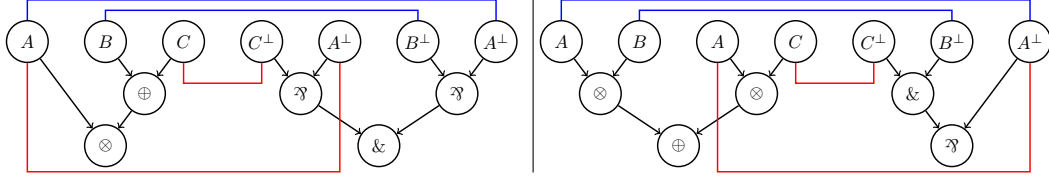
■ **Figure 5** Proof-nets for $A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$.

3.3 Distribution

In general, isomorphisms do not yield *ax*-unique proof-nets. A counter-example is distributivity: $A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$, see Figure 5. Nonetheless, distributivity equations are the only ones in $\mathcal{E}^\dagger$ not giving *ax*-unique proof-nets. We will restrict our study to so-called *distributed* formulas. Once formulas are distributed, distributivity isomorphisms can be ignored, and isomorphisms between distributed formulas happen to be bipartite *ax*-unique.

▶ **Definition 20** (Distributed formula). *An MALL formula is* distributed *if it does not have any sub-formula of the form* $A \otimes (B \oplus C)$, $(A \oplus B) \otimes C$, $A \otimes 1$, $1 \otimes A$, $A \oplus 0$, $0 \oplus A$, $A \otimes 0$, $0 \otimes A$ *or their duals* $(C \& B) \,\mathfrak{N}\, A$, $C \,\mathfrak{N}\, (B \& A)$, $\bot \,\mathfrak{N}\, A$, $A \,\mathfrak{N}\, \bot$, $\top \& A$, $A \& \top$, $\top \,\mathfrak{N}\, A$, $A \,\mathfrak{N}\, \top$ *(where A, B and C are any formulas).*

▶ Remark. This notion is stable by duality: if A is distributed, so is $A^\perp$.

▶ **Proposition 21.** *If $\mathcal{E}$ is complete for isomorphisms between distributed formulas, then it is complete for isomorphisms between arbitrary formulas.*

Proof. Up to equations of Table 1, any formula can be rewritten into a distributed one. ◀

We mostly use the correctness criterion through the fact we can sequentialize, *i.e.* recover a proof tree from a proof-net by Theorem 10. However, in order to prove *ax*-uniqueness, we make a direct use of the correctness criterion to deduce geometric properties of proof-nets. This part of the proof takes benefits from the specificities of this syntax. We begin with two preliminary results. For Λ a set of linkings and W a &-vertex, Λ^W denote the set of all linkings in Λ whose additive resolution does not contain the right argument of W.

▶ **Lemma 22** (Lemma 4.32 in [10], adapted). *Let ω be a jump-free switching cycle in a proof-net θ. There exists a subset of linkings $\Lambda \subseteq \theta$ such that $\omega \subseteq \mathcal{G}_\Lambda$, $\omega \not\subseteq \mathcal{G}_{\Lambda^W}$ and for any &-vertex W toggled by Λ, there exists an axiom link $a \in \omega$ depending on W in Λ.*

For U and V vertices in a tree, their *first common descendant* is the vertex of the tree which is a descendant of both U and V and which has no descendant respecting this property (with a tree represented with its root at the bottom, which is a *descendant* of the leaves).

▶ **Lemma 23.** *Let θ be a proof-net of conclusions Γ, A. If there is a jump edge $l \xrightarrow{j} W$ with $l, W \in T(A)$ and W not a descendant of l, then their first common descendant C is a $\mathfrak{N}$.*

Proof. As there is a jump $l \xrightarrow{j} W$, there exist linkings $\lambda, \lambda' \in \theta$ such that W is the only & toggled by $\{\lambda; \lambda'\}$, and a link $a \in \lambda \setminus \lambda'$ using the leaf l. In particular, the jump $l \xrightarrow{j} W$ is in $\mathcal{G}_{\{\lambda;\lambda'\}}$. For l and W are both in the additive resolution of λ, both premises of C are in the additive resolution of λ, thus C cannot be an additive connective, so not a & nor a $\oplus$-vertex.

Assume by contradiction that C is a $\otimes$. Call δ the path in $T(A)$ from W to C, and μ the one from C to l (see Figure 6). Then, $(l \xrightarrow{j} W)\delta\mu$ is a switching cycle in $\mathcal{G}_{\{\lambda;\lambda'\}}$. According to (P3), there exists a & toggled by $\{\lambda; \lambda'\}$ not in any switching cycle of $\mathcal{G}_{\{\lambda;\lambda'\}}$. A contradiction, for W is the only & toggled by $\{\lambda; \lambda'\}$. Whence, C can only be a $\mathfrak{N}$. ◀
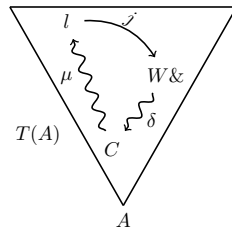
Figure 6 Illustration of the proof of Lemma 23.
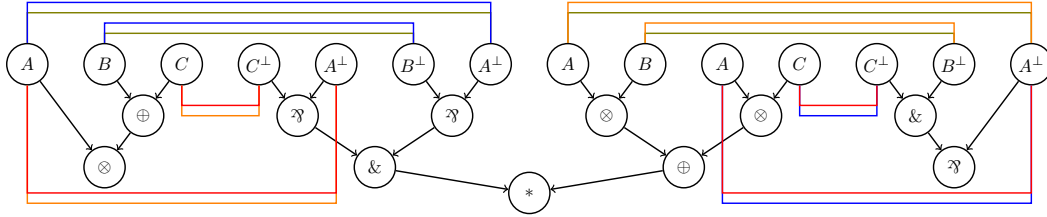
Figure 7 Proof-nets from Figure 5 composed by cut on $(A \otimes B) \oplus (A \otimes C)$.

Now, let us prove that isomorphisms of distributed formulas are bipartite ax-unique. We will consider proof-nets corresponding to an isomorphism that we cut and where we eliminate all cuts not involving atoms. To give some intuition, let us consider the non-ax-unique proof-nets of Figure 5. Composing them together by cut on $(A \otimes B) \oplus (A \otimes C)$ gives the proof-net illustrated on Figure 7. Reducing all cuts not involving atoms yields the proof-net on Figure 8, that we call an *almost reduced composition*. We stop there because of the switching cycle produced by the two links on A (dashed in blue on Figure 8), less visible in the non-reduced composition of Figure 7. However, reducing all cuts gives the identity proof-net, which has no switching cycle: during these reductions, both links on A are merged. By using almost reduced composition, we are going to prove that links preventing ax-uniqueness yield switching cycles, and moreover that these cycles are due to non-distributed formulas only.

▶ **Definition 24** (Almost reduced composition). *Take θ and θ' cut-free proof-nets of respective conclusions A, B and $B^\perp, C$. The* almost reduced composition *over B of θ and θ' is the proof-net resulting from the composition over B of θ and θ' where we repeatedly reduce all cuts not involving atoms (*i.e. *not applying step (a) of Definition 7).*

Let us fix A and B two unit-free MALL (not necessarily distributed yet) formulas as well as θ and θ' such that $A \overset{\theta, \theta'}{\simeq} B$. By Theorem 19, θ and θ' are bipartite full. We denote by ϑ the almost reduced composition over B of θ and θ'. Here, we can extend our duality on vertices and premises (defined in Section 2.4) to links.
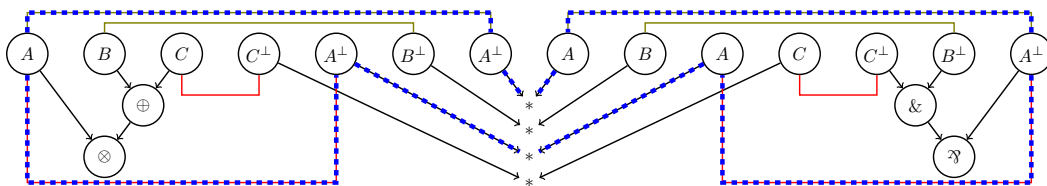
Figure 8 An almost reduced composition of the proof-nets on Figure 5.
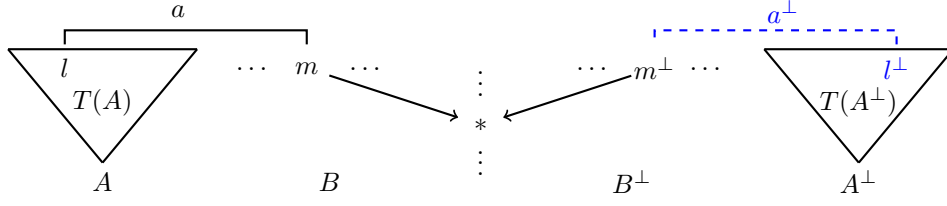
■ **Figure 9** Illustration of Lemma 25.

▶ **Lemma 25.** *Given l a leaf of A (resp. $A^\perp$) and m one of $B^\perp$ (resp. B), there is an axiom link $a = (l, m)$ in some linking $\lambda \in \vartheta$ if and only if there is an axiom link $(l^\perp, m^\perp)$ in the same linking λ, that we will denote $a^\perp = (l^\perp, m^\perp)$ (see Figure 9).*

Proof. By symmetry, assume $(l, m) \in \lambda \in \vartheta$. As the cut $m * m^\perp$ belongs to the additive resolution of λ (for m is inside), $m^\perp$ is a leaf in this resolution. Thus, there is a link $(m^\perp, l') \in \lambda$ for some leaf l', which necessarily belongs to A by bipartiteness of θ'. It stays to prove $l' = l^\perp$. If we were to eliminate all cuts in ϑ, we would get the identity proof-net on A by hypothesis. But eliminating the cut $m * m^\perp$ yields a link (l, l'), which is not modified by the elimination of the other atomic cuts. By Proposition 15(ii), $l' = l^\perp$ follows.   ◀

▶ **Lemma 26.** *Let λ be a linking of ϑ, and V an additive vertex in its additive resolution. Then $V^\perp$ is also inside, with as premise kept the dual premise of the one kept for V.*

▶ **Lemma 27.** *Let W and P be respectively a $\&$-vertex and a $\oplus$-vertex in ϑ, with W an ancestor of P. Then for any axiom link a depending on W in ϑ, a also depends on $P^\perp$ in ϑ.*

Proof. There exist linkings $\lambda, \lambda' \in \vartheta$ such that W is the only $\&$ toggled by $\{\lambda; \lambda'\}$ and $a \in \lambda \backslash \lambda'$. We consider a linking $\lambda_{P^\perp}$ defined by taking an arbitrary $\&$-resolution of λ where we choose the other premise for $P^\perp$ (and arbitrary premises for $\&$-vertices introduced this way): by (P1), there exists a unique linking on it. By Lemma 26, the additive resolutions of λ and $\lambda_{P^\perp}$ (resp. λ and λ') differ exactly on ancestors of P and $P^\perp$ (resp. W and $W^\perp$). Thus, the additive resolutions of λ' and $\lambda_{P^\perp}$ also differ exactly on ancestors of P and $P^\perp$, for W is an ancestor of P. In particular, $\{\lambda; \lambda_{P^\perp}\}$, as well as $\{\lambda'; \lambda_{P^\perp}\}$, toggles only $P^\perp$. If $a \in \lambda_{P^\perp}$, then a depends on $P^\perp$ in $\{\lambda'; \lambda_{P^\perp}\}$. Otherwise, a depends on $P^\perp$ in $\{\lambda; \lambda_{P^\perp}\}$.   ◀

The key result to use distributivity is that a positive vertex "between" a leaf l and a $\&$-vertex W in the same tree prevents them from interacting, *i.e.* there is no jump $l \xrightarrow{j} W$.

▶ **Lemma 28.** *Let $l \xrightarrow{j} W$ be a jump edge in ϑ, with l not an ancestor of W and $l, W \in T(A^\perp)$ (resp. $T(A)$). Denoting by N the first common descendant of l and W, there is no positive vertex in the path between N and W in $T(A^\perp)$ (resp. $T(A)$).*

Proof. Let P be a vertex on the path between N and W in $T(A^\perp)$. By Lemma 23, N is a $\invamp$-vertex. We prove by contradiction that P can neither be a $\oplus$ nor a $\otimes$-vertex.

Suppose P is a $\oplus$-vertex. By Lemma 27, a depends on $P^\perp$, and so does $a^\perp$ through Lemma 25: there is a jump edge $l^\perp \xrightarrow{j} P^\perp$. Applying Lemma 23, the first common descendant of $l^\perp$ and $P^\perp$, which is $N^\perp$, is a $\invamp$-vertex: a contradiction as it is a $\otimes$-vertex.

Assume now P to be a $\otimes$-vertex. As there is a jump $l \xrightarrow{j} W$, there exist linkings $\lambda, \lambda' \in \vartheta$ and a leaf m of B such that W is the only $\&$ toggled by $\{\lambda; \lambda'\}$ and $a = (l, m) \in \lambda \backslash \lambda'$. For P is a $\otimes$, there is a leaf p which is an ancestor of P in the additive resolution of λ, from a different
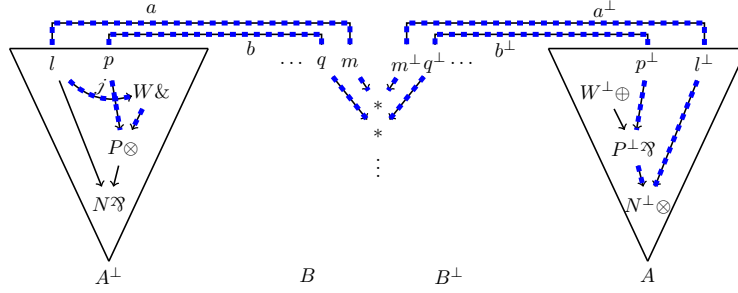
Figure 10 Switching cycle containing W if P is a $\otimes$-vertex in the proof of Lemma 28.
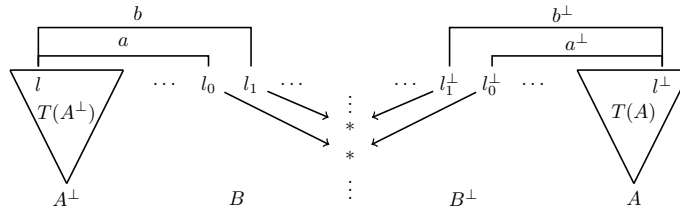
Figure 11 Almost reduced composition ϑ of θ and θ' by cut over B in the proof of Theorem 29.

premise of P than W; it is used by a link $b = (p, q) \in \lambda^3$ (see Figure 10). Then the switching cycle $l \xrightarrow{j} W \to P \leftarrow p \xrightarrow{b} q \to * \leftarrow q^\perp \xrightarrow{b^\perp} p^\perp \to P^\perp \to N^\perp \leftarrow l^\perp \xrightarrow{a^\perp} m^\perp \to * \leftarrow m \xrightarrow{a} l$ (dashed in blue on Figure 10) belongs to $\mathcal{G}_{\{\lambda;\lambda'\}}$. Contradiction: W, the only $\&$ toggled by $\{\lambda;\lambda'\}$, cannot be in any switching cycle of $\mathcal{G}_{\{\lambda;\lambda'\}}$ by (P3). ◀

▶ **Theorem 29.** *Assuming $A \overset{\theta,\theta'}{\simeq} B$ with A and B distributed, θ and θ' are bipartite ax-unique.*

Proof. We already know that θ and θ' are bipartite full thanks to Theorem 19. We reason by contradiction and assume *w.l.o.g.* that θ is not *ax*-unique: there exist a leaf l of $A^\perp$ and two distinct leaves l_0 and l_1 of B with links $a = (l, l_0)$ and $b = (l, l_1)$ in θ. We consider ϑ the almost reduced composition of θ and θ' over B, depicted on Figure 11. By Lemma 16, a and b are also links in ϑ (for the linkings they belong to in θ have matching linkings in θ', and we did not eliminate atomic cuts). Using Lemma 25, we have in $\mathcal{G}_\vartheta$ a switching cycle $\omega = l \xrightarrow{a} l_0 \to * \leftarrow l_0^\perp \xrightarrow{a^\perp} l^\perp \xrightarrow{b^\perp} l_1^\perp \to * \leftarrow l_1 \xrightarrow{b} l$.

Let Λ be a set of linkings given by Lemma 22 applied to ω. As there are two distinct links on l in $\omega \subseteq \mathcal{G}_\Lambda$, Λ contains at least two linkings. By (P3), there exists W a $\&$ toggled by Λ that is not in any switching cycle of $\mathcal{G}_\Lambda$. By Lemma 22, a, $a^\perp$, b or $b^\perp$ depends on W. So a or b depends on W by Lemma 25; *w.l.o.g.* a depends on W. The vertex W belongs to either $T(A)$ or $T(A^\perp)$: up to considering $a^\perp$ instead of a, W is in $T(A^\perp)$. Remark l is not an ancestor of W: if it were, by symmetry assume it is a left-ancestor. Whence a and b belong to Λ^W, so $a^\perp$ and $b^\perp$ too (Lemma 25); thus $\omega \subseteq \mathcal{G}_{\Lambda^W}$, contradicting Lemma 22. By Lemma 23, the first common descendant N of l and W in $T(A^\perp)$ is a $\mathfrak{N}$. There is a $\otimes \backslash \oplus$ on the path between the $\mathfrak{N}$ N and its ancestor the $\&$ W in $T(A^\perp)$, for there is no sub-formula of the shape $- \mathfrak{N} (- \& -)$ in the distributed $A^\perp$. This contradicts Lemma 28. ◀

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3 With $q \neq m$, as a and b are two distinct links in the same linking λ.

3.4 Non-ambiguous formulas & Completeness for unit-free MALL

Once our study is restricted to bipartite *ax-unique* proof-nets, we can also restrict formulas.

▶ **Definition 30** (Non-ambiguous formula). *A formula A is said* non-ambiguous *if each atom in A occurs at most once positive and once negative.*

▶ **Remark.** This means all leaves in A are distinct. If A is non-ambiguous, so is $A^\perp$.

For instance, $X \,\&\, X^\perp$ is non-ambiguous, whereas $(A \otimes B) \oplus (A \otimes C)$ is ambiguous. The reduction to non-ambiguous formulas requires to restrict to distributed formulas first: in $(A \otimes B) \oplus (A \otimes C) \simeq A \otimes (B \oplus C)$ we need the two occurrences of A to factorize. The two following results are a direct adaptation of Section 3 in [2].

▶ **Corollary 31** (Reduction to distributed non-ambiguous formulas). *The set of couples of distributed formulas A and B such that $A \overset{\theta,\vartheta}{\simeq} B$ is the set of instances (by a substitution on atoms) of couples of distributed* non-ambiguous *formulas A' and B' such that $A' \overset{\rho',\vartheta'}{\simeq} B'$.*

▶ **Corollary 32.** *Let A and B be non-ambiguous formulas. If there exist bipartite proof-nets θ and ϑ of respective conclusions $A^\perp, B$ and $B^\perp, A$, then $A \overset{\theta,\vartheta}{\simeq} B$.*

We then prove the completeness of $\mathcal{E}^\dagger$ for unit-free MALL by reasoning as in Section 4 of [2] (with some more technicalities for we reorder not only $\mathfrak{N}$-vertices but also &-vertices).

▶ **Theorem 33** (Isomorphisms completeness for unit-free MALL). *Given A and B two unit-free MALL formulas, if $A \simeq B$, then $A =_{\mathcal{E}^\dagger} B$.*

4 Completeness for MALL with units

We now consider full MALL, with units, and show how to reduce it to the unit-free case. We solve this addition purely in sequent calculus showing that, for distributed formulas, multiplicative and additive units can be replaced by fresh atoms.

A key property of proof-nets is to define a quotient of sequent calculus proofs up to rule commutations [11] (see Appendix A for rule commutations in MALL). Because no such notion of proof-nets exist with units, we are forced to stay in the sequent calculus, meaning that we have to deal with possible rule commutations. As a key example, cut-elimination in proof-nets is confluent and leads to a unique normal form. This is not true in the sequent calculus and we need to relate the different possible cut-free proofs obtained by cut-elimination.

▶ **Theorem 34** (Confluence up to rule commutations). *If π_1 and π_2 are cut-free proofs obtained by cut-elimination from the same proof π, then π_1 and π_2 are equal up to rule commutations.*

This result is not surprising but has not already been proved as far as we know for it is rather tedious to establish. It is an important general result about sequent calculus which we are convinced should hold for full linear logic. It can be lifted to $\beta\eta$-equality of proofs.

▶ **Theorem 35.** *Let π and ϖ be $\beta\eta$-equal MALL proofs. Then, letting π' (resp. ϖ') be a result of expanding all axioms and then eliminating all cuts in π (resp. ϖ), π' is equal to ϖ' up to rule commutations.*

After these general properties, let us move to the question of type isomorphisms. We need to analyse the behaviour of units in proofs equal to id_A up to rule commutations. We only do so for a *distributed* formula A as we have already seen it is enough in Section 3.3.

▶ **Proposition 36.** *Let π be a proof equal, up to rule commutations, to id_A with A distributed. The $\top$-rules of π are of the shape* $\dfrac{}{\vdash \top, 0} \top$ *(with $\top$ in A being the dual of 0 in $A^\perp$, or vice-versa) and $\perp$-rules and 1-rules come by pairs separated with $\oplus_i$-rules only, called a*

$1/\oplus/\perp$-pattern: $\dfrac{\dfrac{\dfrac{}{\vdash 1} 1}{\vdash F} \rho}{\vdash \perp, F} \perp$ *where ρ is a sequence of $\oplus_i$-rules (with $\perp$ in A being the dual of 1 in $A^\perp$, or vice-versa).*

Proof. The key idea is to find properties of id_A preserved by all rule commutations and ensuring the properties described in the statement. For any sequent S in the proof:

(1) the formulas of S are distributed;

(2) if $\top$ is a formula of S, then $S = {\vdash} \top, 0$;

(3) if $\perp$ is a formula of S, then $S = {\vdash} \perp, F$ with F given by the following grammar $F \;:=\; 1 \mid F \oplus D \mid D \oplus F$, where the distinguished 1 is the dual of $\perp$ in $A^\perp$ if $\perp$ a sub-formula of A (or vice-versa), D is any formula, and the sub-proof of π above S is a sequence of $\oplus_i$ rules leading to the distinguished 1;

(4) if $B \& C$ is a formula of S, then $S = {\vdash} B \& C, F$ with F given by the following grammar $F \;:=\; C^\perp \oplus B^\perp \mid F \oplus D \mid D \oplus F$, where the distinguished $C^\perp \oplus B^\perp$ is the dual of $B \& C$ in $A^\perp$ if $B \& C$ a sub-formula of A (or vice-versa), D is any formula, and in the sub-proof of π above S the $\oplus$-rules of the distinguished $C^\perp \oplus B^\perp$ are a $\oplus_2$-rule in the left-branch of the $\&$-rule of $B \& C$, and a $\oplus_1$-rule in its right branch;

(5) if S contains several negative formulas or several positive formulas, then its negative formulas are $\invamp$-formulas. ◀

These properties are preserved by cut anti-reduction.

▶ **Lemma 37.** *If $A \overset{\pi, \pi'}{\simeq} B$ with π and π' cut-free then all $\top$-rules in π and π' are of the form* $\dfrac{}{\vdash \top, 0} \top$ *and all $\perp$-rules and 1-rules belong to $1/\oplus/\perp$-patterns.*

Moving each $\perp$-rule up to the associated 1-rule (which can be done up to $\beta\eta$-equality) allows us to consider units as fresh atoms introduced by ax-rules and to apply Theorem 33.

▶ **Theorem 38** (Isomorphisms completeness with units)**.** *If $A \simeq B$ then $A =_{\mathcal{E}} B$.*

5 Star-autonomous categories with finite products

We prove here that the equational theory $\mathcal{E}$ (along $A \multimap B \simeq A^\perp \invamp B$, De Morgan's laws and involutivity of negation) also corresponds to the isomorphisms present in all $\star$-autonomous categories with finite products. For the historical result of how linear logic can be seen as a category, see [15].

We establish this result from the one on MALL, first proving that MALL (with proofs considered up to $\beta\eta$-equality) defines a $\star$-autonomous category with finite products (Section 5.1). Then, we conclude using a semantic method (Section 5.2).

5.1 MALL as a star-autonomous category with finite products

The logic MALL, with proofs taken up to $\beta\eta$-equality, defines a $\star$-autonomous category with finite products, that we will call $\mathbb{MALL}$. Indeed, we can define it as follows.

Objects of $\mathbb{MALL}$ are formulas of MALL, while its morphisms from A to B are proofs of $\vdash A^\perp, B$, considered up to $\beta\eta$-equality.[4] One can check that a proof of MALL is an isomorphism if and only if, when seen as a morphism, it is an isomorphism in $\mathbb{MALL}$.

We define a bifunctor $\otimes$ on $\mathbb{MALL}$, associating to formulas (*i.e.* objects) A and B the formula $A \otimes B$ and to proofs (*i.e.* morphisms) π_0 and π_1 respectively of $\vdash A_0^\perp, B_0$ and $\vdash A_1^\perp, B_1$ the following proof of $\vdash (A_0 \otimes A_1)^\perp, B_0 \otimes B_1$:

$$\cfrac{\cfrac{\cfrac{\pi_0}{\vdash A_0^\perp, B_0} \quad \cfrac{\pi_1}{\vdash A_1^\perp, B_1}}{\vdash A_1^\perp, A_0^\perp, B_0 \otimes B_1} \otimes}{\vdash A_1^\perp \,\mathfrak{N}\, A_0^\perp, B_0 \otimes B_1} \mathfrak{N}$$

One can check that $(\mathbb{MALL}, \otimes, 1, \alpha, \lambda, \rho, \gamma)$ forms a symmetric monoidal category, where 1 is the 1-formula, α are isomorphisms of MALL associated to $(A \otimes B) \otimes C \simeq A \otimes (B \otimes C)$ seen as a natural isomorphism of $\mathbb{MALL}$, and similarly for λ with $1 \otimes A \simeq A$, ρ with $A \otimes 1 \simeq A$, and γ with $A \otimes B \simeq B \otimes A$.

Furthermore, define $A \multimap B := A^\perp \,\mathfrak{N}\, B$ and $ev_{A,B}$ as the following morphism from $(A \multimap B) \otimes A$ to B (*i.e.* a proof of $\vdash A^\perp \,\mathfrak{N}\, (B^\perp \otimes A), B$):

$$\cfrac{\cfrac{\cfrac{}{\vdash B^\perp, B} ax \quad \cfrac{}{\vdash A^\perp, A} ax}{\vdash A^\perp, B^\perp \otimes A, B} \otimes}{\vdash A^\perp \,\mathfrak{N}\, (B^\perp \otimes A), B} \mathfrak{N}$$

It can be checked that $\mathbb{MALL}$ is a symmetric monoidal closed category with as exponential object $(A \multimap B, ev_{A,B})$ for objects A and B.

Moreover, one can also check that $\perp$ is a dualizing object for this category, making $\mathbb{MALL}$ a $\star$-autonomous category. This relies on the following morphism from $(A \multimap \perp) \multimap \perp$ to A (which is an inverse of the curryfication of $ev_{A,\perp}$):

$$\cfrac{\cfrac{}{\vdash 1} 1 \quad \cfrac{\cfrac{\cfrac{\cfrac{}{\vdash A^\perp, A} ax}{\vdash A^\perp, \perp, A} \perp}{\vdash A^\perp \,\mathfrak{N}\, \perp, A} \mathfrak{N}}{}}{\vdash 1 \otimes (A^\perp \,\mathfrak{N}\, \perp), A} \otimes$$

Finally, $\top$ is a terminal object of $\mathbb{MALL}$, and $A \,\&\, B$ is the product of objects A and B, with as projections π_A and π_B the following morphisms respectively from $A \,\&\, B$ to A and from $A \,\&\, B$ to B:

$$\cfrac{\cfrac{}{\vdash A^\perp, A} ax}{\vdash B^\perp \oplus A^\perp, A} \oplus_2 \qquad \text{and} \qquad \cfrac{\cfrac{}{\vdash B^\perp, B} ax}{\vdash B^\perp \oplus A^\perp, B} \oplus_1$$

Therefore, $\mathbb{MALL}$ is a $\star$-autonomous category with finite products [15].

5.2   Isomorphisms of star-autonomous categories with finite products

We take the same notations as in the previous section ($\&$ for product, ...). One can easily check that isomorphisms in a $\star$-autonomous category with finite products form a congruence (as all binary connectives define bifunctors), and that $\mathcal{E}$ is sound (*i.e.* that equations defining

---

[4] We recall that $(\cdot)^\perp$ is defined by induction, making it an involution.

Table 2 De Morgan's isomorphisms.

$$
\begin{array}{rclcrcl}
A \multimap B & \simeq & A^{\perp} \mathbin{\bindnasrepma} B & \qquad & X^{\perp\perp} & \simeq & X \\
(A \otimes B)^{\perp} & \simeq & B^{\perp} \mathbin{\bindnasrepma} A^{\perp} & & (A \mathbin{\bindnasrepma} B)^{\perp} & \simeq & B^{\perp} \otimes A^{\perp} \\
1^{\perp} & \simeq & \perp & & \perp^{\perp} & \simeq & 1 \\
(A \mathbin{\&} B)^{\perp} & \simeq & B^{\perp} \oplus A^{\perp} & & (A \oplus B)^{\perp} & \simeq & B^{\perp} \mathbin{\&} A^{\perp} \\
\top^{\perp} & \simeq & 0 & & 0^{\perp} & \simeq & \top
\end{array}
$$

$\mathcal{E}$ in Table 1 on Page 3 are isomorphisms in any $\star$-autonomous category with finite products). Moreover the isomorphisms of Table 2 (which are equalities in $\mathbb{MALL}$) also hold in any $\star$-autonomous category with finite products.

Completeness follows by Theorem 38 (isomorphisms in $\mathbb{MALL}$ are exactly those given by $\mathcal{E}$) and from the fact that two objects definable in the language of $\star$-autonomous categories with finite products are equal in $\mathbb{MALL}$ if and only if they are related by the equational theory generated by Table 2. For example, one can deduce $(A \multimap \perp) \multimap \perp \simeq (A^{\perp} \mathbin{\bindnasrepma} \perp)^{\perp} \mathbin{\bindnasrepma} \perp \simeq (A^{\perp} \mathbin{\bindnasrepma} \perp)^{\perp} \simeq 1 \otimes A^{\perp\perp} \simeq A^{\perp\perp} \simeq A$ (the last equation being derivable by induction on A). Henceforth, isomorphisms valid in all $\star$-autonomous categories with finite products are included in $\mathcal{E}$ enriched with Table 2.

▶ **Theorem 39** (Isomorphisms in $\star$-autonomous categories with finite products)**.** *$\mathcal{E}$ enriched with Table 2 is a sound and complete equational theory for isomorphisms in $\star$-autonomous categories with finite products.*

6 Conclusion

Extending the result of Balat and Di Cosmo in [2], we give an equational theory characterising type isomorphisms in multiplicative-additive linear logic with units as well as in $\star$-autonomous categories with finite products: the one described on Table 1 on Page 3 (together with Table 2 for $\star$-autonomous categories). Looking at the proof, we get as a sub-result that isomorphisms for ALL (resp. unit-free ALL) are given by the equational theory $\mathcal{E}$ (resp. $\mathcal{E}^{\dagger}$) restricted to ALL formulas (and more generally this applies to any fragment of MALL, thanks to the sub-formula property). Proof-nets were a major tool to prove completeness, as notions like fullness and ax-uniqueness are much harder to define and manipulate in sequent calculus. However, we could not use them for taking care of the (additive) units, because there is no known appropriate notion of proof-nets. We have thus been forced to develop (some parts of) the theory of cut-elimination, axiom-expansion and rule commutations for the sequent calculus of MALL with units.

The immediate question to address is the extension of our results to the characterization of type isomorphisms for full propositional linear logic, thus including the exponential connectives. This is clearly not immediate since the interaction between additive and exponential connectives is not well described in proof-nets.

A more general problem is the study of type retractions (where only one of the two compositions yields an identity) which is also much more difficult (see for example [13]). The question is mostly open in the case of linear logic. Even in multiplicative linear logic (where there is for example a retraction between A and $(A \multimap A) \multimap A = (A \otimes A^{\perp}) \mathbin{\bindnasrepma} A$ which is not an isomorphism, and where the associated proof-nets are not bipartite), no characterization is known. In the multiplicative-additive fragment, the problem looks even harder, with more retractions; for instance the one depicted on Figure 4, but there also is a retraction between A and $A \oplus A$.

─────── **References** ───────

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A   Transformations of sequent calculus proofs in MALL

▶ **Definition 40.** *In the sequent calculus of MALL, we call* axiom-expansion *the rewriting system* $\overset{\eta}{\longrightarrow}$ *described on Table 3.*

▶ **Definition 41.** *In the sequent calculus of MALL, we call* cut-elimination *the rewriting system* $\overset{\beta}{\longrightarrow}$ *described on Tables 4 and 5 (up to commuting the two branches of a cut-rule).*

▶ **Definition 42.** *In the sequent calculus of MALL, we call* rule commutation *the equational theory* $=_c$ *described on Tables 6 and 7. This corresponds to rule commutations in cut-free MALL; in particular, in a* $\top - \otimes$ *permutation we assume the created or erased sub-proof to be cut-free.*

▪ **Table 3** Axiom-expansion in the sequent calculus of MALL.

$\mathfrak{V} - \otimes$	$\dfrac{}{\vdash A \otimes B, B^\perp \,\mathfrak{V}\, A^\perp}\ ax \qquad \xrightarrow{\eta}$	$\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}\ ax \qquad \dfrac{}{\vdash B^\perp, B}\ ax}{\vdash A \otimes B, A^\perp, B^\perp}\ \otimes}{\vdash A \otimes B, B^\perp \,\mathfrak{V}\, A^\perp}\ \mathfrak{V}$
$\& - \oplus$	$\dfrac{}{\vdash A \oplus B, B^\perp \,\&\, A^\perp}\ ax \qquad \xrightarrow{\eta}$	$\dfrac{\dfrac{\dfrac{}{\vdash B, B^\perp}\ ax}{\vdash A \oplus B, B^\perp}\ \oplus_2 \qquad \dfrac{\dfrac{}{\vdash A, A^\perp}\ ax}{\vdash A \oplus B, A^\perp}\ \oplus_1}{\vdash A \oplus B, B^\perp \,\&\, A^\perp}\ \& $
$\perp - 1$	$\dfrac{}{\vdash 1, \perp}\ ax \qquad \xrightarrow{\eta}$	$\dfrac{\dfrac{}{\vdash 1}\ 1}{\vdash 1, \perp}\ \perp$
$\top - 0$	$\dfrac{}{\vdash 0, \top}\ ax \qquad \xrightarrow{\eta}$	$\dfrac{}{\vdash 0, \top}\ \top$

▪ **Table 4** Cut-elimination in sequent calculus (key cases).

ax	$\dfrac{\dfrac{}{\vdash A^\perp, A}\ ax \qquad \dfrac{\pi}{\vdash A, \Gamma}}{\vdash A, \Gamma}\ cut \qquad \xrightarrow{\beta}$	$\dfrac{\pi}{\vdash A, \Gamma}$
$\mathfrak{V} - \otimes$	$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A, \Gamma} \quad \dfrac{\pi_2}{\vdash B, \Delta}}{\vdash A \otimes B, \Gamma, \Delta}\ \otimes \quad \dfrac{\dfrac{\pi_3}{\vdash B^\perp, A^\perp, \Sigma}}{\vdash B^\perp \,\mathfrak{V}\, A^\perp, \Sigma}\ \mathfrak{V}}{\vdash \Gamma, \Delta, \Sigma}\ cut \quad \xrightarrow{\beta}$	$\dfrac{\dfrac{\pi_1}{\vdash A, \Gamma} \quad \dfrac{\dfrac{\pi_2}{\vdash B, \Delta} \quad \dfrac{\pi_3}{\vdash B^\perp, A^\perp, \Sigma}}{\vdash A^\perp, \Delta, \Sigma}\ cut}{\vdash \Gamma, \Delta, \Sigma}\ cut$
$\& - \oplus_1$	$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A_1, \Gamma} \quad \dfrac{\pi_2}{\vdash A_2, \Gamma}}{\vdash A_1 \,\&\, A_2, \Gamma}\ \& \quad \dfrac{\dfrac{\pi_3}{\vdash A_2^\perp, \Delta}}{\vdash A_2^\perp \oplus A_1^\perp, \Delta}\ \oplus_1}{\vdash \Gamma, \Delta}\ cut \quad \xrightarrow{\beta}$	$\dfrac{\dfrac{\pi_2}{\vdash A_2, \Gamma} \quad \dfrac{\pi_3}{\vdash A_2^\perp, \Delta}}{\vdash \Gamma, \Delta}\ cut$
$\& - \oplus_2$	$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A_1, \Gamma} \quad \dfrac{\pi_2}{\vdash A_2, \Gamma}}{\vdash A_1 \,\&\, A_2, \Gamma}\ \& \quad \dfrac{\dfrac{\pi_3}{\vdash A_1^\perp, \Delta}}{\vdash A_2^\perp \oplus A_1^\perp, \Delta}\ \oplus_2}{\vdash \Gamma, \Delta}\ cut \quad \xrightarrow{\beta}$	$\dfrac{\dfrac{\pi_1}{\vdash A_1, \Gamma} \quad \dfrac{\pi_3}{\vdash A_1^\perp, \Delta}}{\vdash \Gamma, \Delta}\ cut$
$\perp - 1$	$\dfrac{\dfrac{}{\vdash 1}\ 1 \quad \dfrac{\dfrac{\pi}{\vdash \Gamma}}{\vdash \Gamma, \perp}\ \perp}{\vdash \Gamma}\ cut \quad \xrightarrow{\beta}$	$\dfrac{\pi}{\vdash \Gamma}$
	(No $\top - 0$ key case as there are no rule for 0.)	

■ **Table 5** Cut-elimination in sequent calculus (commutative cases).

$\mathfrak{N} - cut$	$$\cfrac{\cfrac{\cfrac{\pi_1}{\vdash A,B,C,\Gamma}}{\vdash A,B\,\mathfrak{N}\,C,\Gamma}\mathfrak{N} \quad \cfrac{\pi_2}{\vdash A^\perp,\Delta}}{\cfrac{\vdash B\,\mathfrak{N}\,C,\Gamma,\Delta}{}cut}$$	$\xrightarrow{\beta}$ $$\cfrac{\cfrac{\cfrac{\pi_1}{\vdash A,B,C,\Gamma}\quad\cfrac{\pi_2}{\vdash A^\perp,\Delta}}{\vdash B,C,\Gamma,\Delta}cut}{\vdash B\,\mathfrak{N}\,C,\Gamma,\Delta}\mathfrak{N}$$
$\otimes - cut - 1$	$$\cfrac{\cfrac{\cfrac{\pi_1}{\vdash A,B,\Gamma}\quad\cfrac{\pi_2}{\vdash C,\Delta}}{\vdash A,B\otimes C,\Gamma,\Delta}\otimes \quad \cfrac{\pi_3}{\vdash A^\perp,\Sigma}}{\vdash B\otimes C,\Gamma,\Delta,\Sigma}cut$$	$\xrightarrow{\beta}$ $$\cfrac{\cfrac{\cfrac{\pi_1}{\vdash A,B,\Gamma}\quad\cfrac{\pi_3}{\vdash A^\perp,\Sigma}}{\vdash B,\Gamma,\Sigma}cut\quad\cfrac{\pi_2}{\vdash C,\Delta}}{\vdash B\otimes C,\Gamma,\Delta,\Sigma}\otimes$$
$\otimes - cut - 2$	$$\cfrac{\cfrac{\cfrac{\pi_1}{\vdash B,\Gamma}\quad\cfrac{\pi_2}{\vdash A,C,\Delta}}{\vdash A,B\otimes C,\Gamma,\Delta}\otimes \quad \cfrac{\pi_3}{\vdash A^\perp,\Sigma}}{\vdash B\otimes C,\Gamma,\Delta,\Sigma}cut$$	$\xrightarrow{\beta}$ $$\cfrac{\cfrac{\pi_1}{\vdash B,\Gamma}\quad\cfrac{\cfrac{\pi_2}{\vdash A,C,\Delta}\quad\cfrac{\pi_3}{\vdash A^\perp,\Sigma}}{\vdash C,\Delta,\Sigma}cut}{\vdash B\otimes C,\Gamma,\Delta,\Sigma}\otimes$$
$\& - cut$	$$\cfrac{\cfrac{\cfrac{\pi_1}{\vdash A,B,\Gamma}\quad\cfrac{\pi_2}{\vdash A,C,\Gamma}}{\vdash A,B\,\&\,C,\Gamma}\& \quad \cfrac{\pi_3}{\vdash A^\perp,\Delta}}{\vdash B\,\&\,C,\Gamma,\Delta}cut$$	$\xrightarrow{\beta}$ $$\cfrac{\cfrac{\cfrac{\pi_1}{\vdash A,B,\Gamma}\,\cfrac{\pi_3}{\vdash A^\perp,\Delta}}{\vdash B,\Gamma,\Delta}cut\;\cfrac{\cfrac{\pi_2}{\vdash A,C,\Gamma}\,\cfrac{\pi_3}{\vdash A^\perp,\Delta}}{\vdash C,\Gamma,\Delta}cut}{\vdash B\,\&\,C,\Gamma,\Delta}\&$$
$\oplus_i - cut$	$$\cfrac{\cfrac{\cfrac{\pi_1}{\vdash A,B_i,\Gamma}}{\vdash A,B_1\oplus B_2,\Gamma}\oplus_i \quad \cfrac{\pi_2}{\vdash A^\perp,\Delta}}{\vdash B_1\oplus B_2,\Gamma,\Delta}cut$$	$\xrightarrow{\beta}$ $$\cfrac{\cfrac{\cfrac{\pi_1}{\vdash A,B_i,\Gamma}\quad\cfrac{\pi_2}{\vdash A^\perp,\Delta}}{\vdash B_i,\Gamma,\Delta}cut}{\vdash B_1\oplus B_2,\Gamma,\Delta}\oplus_i$$
$\perp - cut$	$$\cfrac{\cfrac{\cfrac{\pi_1}{\vdash A,\Gamma}}{\vdash A,\perp,\Gamma}\perp \quad \cfrac{\pi_2}{\vdash A^\perp,\Delta}}{\vdash \perp,\Gamma,\Delta}cut$$	$\xrightarrow{\beta}$ $$\cfrac{\cfrac{\cfrac{\pi_1}{\vdash A,\Gamma}\quad\cfrac{\pi_2}{\vdash A^\perp,\Delta}}{\vdash \Gamma,\Delta}cut}{\vdash \perp,\Gamma,\Delta}\perp$$
$\top - cut$	$$\cfrac{\cfrac{}{\vdash A,\top,\Gamma}\top \quad \cfrac{\pi}{\vdash A^\perp,\Delta}}{\vdash \top,\Gamma,\Delta}cut$$	$\xrightarrow{\beta}$ $$\cfrac{}{\vdash \top,\Gamma,\Delta}\top$$
$cut - cut$	$$\cfrac{\cfrac{\cfrac{\pi_1}{\vdash A,B,\Gamma}\quad\cfrac{\pi_2}{\vdash B^\perp,\Delta}}{\vdash A,\Gamma,\Delta}cut \quad \cfrac{\pi_3}{\vdash A^\perp,\Sigma}}{\vdash \Gamma,\Delta,\Sigma}cut$$	$\xrightarrow{\beta}$ $$\cfrac{\cfrac{\cfrac{\pi_1}{\vdash A,B,\Gamma}\quad\cfrac{\pi_3}{\vdash A^\perp,\Sigma}}{\vdash B,\Gamma,\Sigma}cut\quad\cfrac{\pi_2}{\vdash B^\perp,\Delta}}{\vdash \Gamma,\Delta,\Sigma}cut$$

(No $ax - cut$ nor $1 - cut$ nor $0 - cut$ commutative cases as the ax and 1-rules have no context and there are no rule for 0.)

■ **Table 6** Rule commutations involving a unit rule.

$\cfrac{}{\vdash A_1\,\mathfrak{N}\,A_2,\top,\Gamma}\top$	$\begin{smallmatrix}c^{\mathfrak{N}}_\top\\\longrightarrow\\\longleftarrow\\c^\top_{\mathfrak{N}}\end{smallmatrix}$	$\cfrac{\cfrac{}{\vdash A_1,A_2,\top,\Gamma}\top}{\vdash A_1\,\mathfrak{N}\,A_2,\top,\Gamma}\mathfrak{N}$	$\cfrac{\cfrac{\cfrac{\pi_1}{\vdash A_1,A_2,\Gamma}}{\vdash A_1\,\mathfrak{N}\,A_2,\Gamma}\mathfrak{N}}{\vdash A_1\,\mathfrak{N}\,A_2,\perp,\Gamma}\perp$	$\begin{smallmatrix}c^{\mathfrak{N}}_\perp\\\longrightarrow\\\longleftarrow\\c^\perp_{\mathfrak{N}}\end{smallmatrix}$	$\cfrac{\cfrac{\cfrac{\pi_1}{\vdash A_1,A_2,\Gamma}}{\vdash A_1,A_2,\perp,\Gamma}\perp}{\vdash A_1\,\mathfrak{N}\,A_2,\perp,\Gamma}\mathfrak{N}$
$\cfrac{}{\vdash A_1\otimes A_2,\top,\Gamma,\Delta}\top$	$\begin{smallmatrix}c^{\otimes}_\top\\\longrightarrow\\\longleftarrow\\c^\top_{\otimes}\end{smallmatrix}$	$\cfrac{\cfrac{}{\vdash A_1,\top,\Gamma}\top\quad\cfrac{\pi}{\vdash A_2,\Delta}}{\vdash A_1\otimes A_2,\top,\Gamma,\Delta}\otimes$	$\cfrac{\cfrac{\cfrac{\pi_1}{\vdash A_1,\Gamma}\quad\cfrac{\pi_2}{\vdash A_2,\Delta}}{\vdash A_1\otimes A_2,\Gamma,\Delta}\otimes}{\vdash A_1\otimes A_2,\perp,\Gamma,\Delta}\perp$	$\begin{smallmatrix}c^{\otimes}_\perp\\\longrightarrow\\\longleftarrow\\c^\perp_{\otimes}\end{smallmatrix}$	$\cfrac{\cfrac{\cfrac{\pi_1}{\vdash A_1,\Gamma}}{\vdash A_1,\perp,\Gamma}\perp\quad\cfrac{\pi_2}{\vdash A_2,\Delta}}{\vdash A_1\otimes A_2,\perp,\Gamma,\Delta}\otimes$
$\cfrac{}{\vdash A_1\otimes A_2,\top,\Gamma,\Delta}\top$	$\begin{smallmatrix}c^{\otimes}_\top\\\longrightarrow\\\longleftarrow\\c^\top_{\otimes}\end{smallmatrix}$	$\cfrac{\cfrac{\pi}{\vdash A_1,\Gamma}\quad\cfrac{}{\vdash A_2,\top,\Delta}\top}{\vdash A_1\otimes A_2,\top,\Gamma,\Delta}\otimes$	$\cfrac{\cfrac{\cfrac{\pi_1}{\vdash A_1,\Gamma}\quad\cfrac{\pi_2}{\vdash A_2,\Delta}}{\vdash A_1\otimes A_2,\Gamma,\Delta}\otimes}{\vdash A_1\otimes A_2,\perp,\Gamma,\Delta}\perp$	$\begin{smallmatrix}c^{\otimes}_\perp\\\longrightarrow\\\longleftarrow\\c^\perp_{\otimes}\end{smallmatrix}$	$\cfrac{\cfrac{\pi_1}{\vdash A_1,\Gamma}\quad\cfrac{\cfrac{\pi_2}{\vdash A_2,\Delta}}{\vdash A_2,\perp,\Delta}\perp}{\vdash A_1\otimes A_2,\perp,\Gamma,\Delta}\otimes$
$\cfrac{}{\vdash A_1\,\&\,A_2,\top,\Gamma}\top$	$\begin{smallmatrix}c^{\&}_\top\\\longrightarrow\\\longleftarrow\\c^\top_{\&}\end{smallmatrix}$	$\cfrac{\cfrac{}{\vdash A_1,\top,\Gamma}\top\quad\cfrac{}{\vdash A_2,\top,\Gamma}\top}{\vdash A_1\,\&\,A_2,\top,\Gamma}\&$	$\cfrac{\cfrac{\cfrac{\pi_1}{\vdash A_1,\Gamma}\quad\cfrac{\pi_2}{\vdash A_2,\Gamma}}{\vdash A_1\,\&\,A_2,\Gamma}\&}{\vdash A_1\,\&\,A_2,\perp,\Gamma}\perp$	$\begin{smallmatrix}c^{\&}_\perp\\\longrightarrow\\\longleftarrow\\c^\perp_{\&}\end{smallmatrix}$	$\cfrac{\cfrac{\cfrac{\pi_1}{\vdash A_1,\Gamma}}{\vdash A_1,\perp,\Gamma}\perp\;\cfrac{\cfrac{\pi_2}{\vdash A_2,\Gamma}}{\vdash A_2,\perp,\Gamma}\perp}{\vdash A_1\,\&\,A_2,\perp,\Gamma}\&$
$\cfrac{}{\vdash A_1\oplus A_2,\top,\Gamma}\top$	$\begin{smallmatrix}c^{\oplus_i}_\top\\\longrightarrow\\\longleftarrow\\c^\top_{\oplus_i}\end{smallmatrix}$	$\cfrac{\cfrac{}{\vdash A_i,\top,\Gamma}\top}{\vdash A_1\oplus A_2,\top,\Gamma}\oplus_i$	$\cfrac{\cfrac{\cfrac{\pi}{\vdash A_i,\Gamma}}{\vdash A_1\oplus A_2,\Gamma}\oplus_i}{\vdash A_1\oplus A_2,\perp,\Gamma}\perp$	$\begin{smallmatrix}c^{\oplus_i}_\perp\\\longrightarrow\\\longleftarrow\\c^\perp_{\oplus_i}\end{smallmatrix}$	$\cfrac{\cfrac{\cfrac{\pi}{\vdash A_i,\Gamma}}{\vdash A_i,\perp,\Gamma}\perp}{\vdash A_1\oplus A_2,\perp,\Gamma}\oplus_i$
$\cfrac{}{\vdash \top_0,\top_1,\Gamma}\top_0$	$\begin{smallmatrix}c^{\top}_\top\\\longrightarrow\end{smallmatrix}$	$\cfrac{}{\vdash \top_0,\top_1,\Gamma}\top_1$	$\cfrac{\cfrac{\cfrac{\pi}{\vdash \Gamma}}{\vdash \perp_0,\Gamma}\perp_0}{\vdash \perp_0,\perp_1,\Gamma}\perp_1$	$\begin{smallmatrix}c^{\perp}_\perp\\\longrightarrow\end{smallmatrix}$	$\cfrac{\cfrac{\cfrac{\pi}{\vdash \Gamma}}{\vdash \perp_1,\Gamma}\perp_1}{\vdash \perp_0,\perp_1,\Gamma}\perp_0$
$\cfrac{}{\vdash \top,\perp,\Gamma}\top$	$\begin{smallmatrix}c^{\perp}_\top\\\longrightarrow\\\longleftarrow\\c^\top_{\perp}\end{smallmatrix}$	$\cfrac{\cfrac{}{\vdash \top,\Gamma}\top}{\vdash \top,\perp,\Gamma}\perp$			

(No commutation with ax, 1 nor 0 as the ax and 1-rules have no context and there are no rule for 0.)

Table 7 Rule commutations not involving a unit rule.

$\dfrac{\dfrac{\dfrac{\pi}{\vdash A_1, A_2, B_1, B_2, \Gamma}}{\vdash A_1 \parr A_2, B_1, B_2, \Gamma}\,\parr}{\vdash A_1 \parr A_2, B_1 \parr B_2, \Gamma}\,\parr$	$\xrightarrow{c_\parr^\parr}$	$\dfrac{\dfrac{\dfrac{\pi}{\vdash A_1, A_2, B_1, B_2, \Gamma}}{\vdash A_1, A_2, B_1 \parr B_2, \Gamma}\,\parr}{\vdash A_1 \parr A_2, B_1 \parr B_2, \Gamma}\,\parr$
$\dfrac{\dfrac{\pi_1}{\vdash A_1, \Gamma}\quad \dfrac{\dfrac{\pi_2}{\vdash A_2, B_1, \Delta}\quad \dfrac{\pi_3}{\vdash B_2, \Sigma}}{\vdash A_2, B_1 \otimes B_2, \Delta, \Sigma}\,\otimes}{\vdash A_1 \otimes A_2, B_1 \otimes B_2, \Gamma, \Delta, \Sigma}\,\otimes$	$\xrightarrow{c_\otimes^\otimes}$	$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A_1, \Gamma}\quad \dfrac{\pi_2}{\vdash A_2, B_1, \Delta}}{\vdash A_1 \otimes A_2, B_1, \Gamma, \Delta}\,\otimes \quad \dfrac{\pi_3}{\vdash B_2, \Sigma}}{\vdash A_1 \otimes A_2, B_1 \otimes B_2, \Gamma, \Delta, \Sigma}\,\otimes$
$\dfrac{\dfrac{\pi_1}{\vdash A_1, \Gamma}\quad \dfrac{\dfrac{\pi_2}{\vdash B_1, \Delta}\quad \dfrac{\pi_3}{\vdash A_2, B_2, \Sigma}}{\vdash A_2, B_1 \otimes B_2, \Delta, \Sigma}\,\otimes}{\vdash A_1 \otimes A_2, B_1 \otimes B_2, \Gamma, \Delta, \Sigma}\,\otimes$	$\xrightarrow{c_\otimes^\otimes}$	$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A_1, \Gamma}\quad \dfrac{\pi_3}{\vdash A_2, B_2, \Sigma}}{\vdash A_1 \otimes A_2, B_2, \Gamma, \Sigma}\,\otimes \quad \dfrac{\pi_2}{\vdash B_1, \Delta}}{\vdash A_1 \otimes A_2, B_1 \otimes B_2, \Gamma, \Delta, \Sigma}\,\otimes$
$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A_1, B_1, \Gamma}\quad \dfrac{\pi_2}{\vdash B_2, \Delta}}{\vdash A_1, B_1 \otimes B_2, \Gamma, \Delta}\,\otimes \quad \dfrac{\pi_3}{\vdash A_2, \Sigma}}{\vdash A_1 \otimes A_2, B_1 \otimes B_2, \Gamma, \Delta, \Sigma}\,\otimes$	$\xrightarrow{c_\otimes^\otimes}$	$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A_1, B_1, \Gamma}\quad \dfrac{\pi_3}{\vdash A_2, \Sigma}}{\vdash A_1 \otimes A_2, B_1, \Gamma, \Sigma}\,\otimes \quad \dfrac{\pi_2}{\vdash B_2, \Delta}}{\vdash A_1 \otimes A_2, B_1 \otimes B_2, \Gamma, \Delta, \Sigma}\,\otimes$
$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash B_1, \Gamma}\quad \dfrac{\pi_2}{\vdash A_1, B_2, \Delta}}{\vdash A_1, B_1 \otimes B_2, \Gamma, \Delta}\,\otimes \quad \dfrac{\pi_3}{\vdash A_2, \Sigma}}{\vdash A_1 \otimes A_2, B_1 \otimes B_2, \Gamma, \Delta, \Sigma}\,\otimes$	$\xrightarrow{c_\otimes^\otimes}$	$\dfrac{\dfrac{\pi_1}{\vdash B_1, \Gamma}\quad \dfrac{\dfrac{\pi_2}{\vdash A_1, B_2, \Delta}\quad \dfrac{\pi_3}{\vdash A_2, \Sigma}}{\vdash A_1 \otimes A_2, B_2, \Delta, \Sigma}\,\otimes}{\vdash A_1 \otimes A_2, B_1 \otimes B_2, \Gamma, \Delta, \Sigma}\,\otimes$
$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A_1, B_1, \Gamma}\quad \dfrac{\pi_2}{\vdash A_2, B_1, \Gamma}}{\vdash A_1 \& A_2, B_1, \Gamma}\,\& \quad \dfrac{\dfrac{\pi_3}{\vdash A_1, B_2, \Gamma}\quad \dfrac{\pi_4}{\vdash A_2, B_2, \Gamma}}{\vdash A_1 \& A_2, B_2, \Gamma}\,\&}{\vdash A_1 \& A_2, B_1 \& B_2, \Gamma}\,\&$	$\xrightarrow{c_\&^\&}$	$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A_1, B_1, \Gamma}\quad \dfrac{\pi_3}{\vdash A_1, B_2, \Gamma}}{\vdash A_1, B_1 \& B_2, \Gamma}\,\& \quad \dfrac{\dfrac{\pi_2}{\vdash A_2, B_1, \Gamma}\quad \dfrac{\pi_4}{\vdash A_2, B_2, \Gamma}}{\vdash A_2, B_1 \& B_2, \Gamma}\,\&}{\vdash A_1 \& A_2, B_1 \& B_2, \Gamma}\,\&$
$\dfrac{\dfrac{\dfrac{\pi}{\vdash A_i, B_j, \Gamma}}{\vdash A_1 \oplus A_2, B_j, \Gamma}\,\oplus_i}{\vdash A_1 \oplus A_2, B_1 \oplus B_2, \Gamma}\,\oplus_j$	$\xrightarrow{c_{\oplus_j}^{\oplus_i}}$	$\dfrac{\dfrac{\dfrac{\pi}{\vdash A_i, B_j, \Gamma}}{\vdash A_i, B_1 \oplus B_2, \Gamma}\,\oplus_j}{\vdash A_1 \oplus A_2, B_1 \oplus B_2, \Gamma}\,\oplus_i$
$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A_1, A_2, B_1, \Gamma}}{\vdash A_1 \parr A_2, B_1, \Gamma}\,\parr \quad \dfrac{\pi_2}{\vdash B_2, \Delta}}{\vdash A_1 \parr A_2, B_1 \otimes B_2, \Gamma, \Delta}\,\otimes$	$\underset{\xleftarrow{c_\parr^\otimes}}{\xrightarrow{c_\otimes^\parr}}$	$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A_1, A_2, B_1, \Gamma}\quad \dfrac{\pi_2}{\vdash B_2, \Delta}}{\vdash A_1, A_2, B_1 \otimes B_2, \Gamma, \Delta}\,\otimes}{\vdash A_1 \parr A_2, B_1 \otimes B_2, \Gamma, \Delta}\,\parr$
$\dfrac{\dfrac{\pi_1}{\vdash B_1, \Gamma}\quad \dfrac{\dfrac{\pi_2}{\vdash A_1, A_2, B_2, \Delta}}{\vdash A_1 \parr A_2, B_2, \Delta}\,\parr}{\vdash A_1 \parr A_2, B_1 \otimes B_2, \Gamma, \Delta}\,\otimes$	$\underset{\xleftarrow{c_\parr^\otimes}}{\xrightarrow{c_\otimes^\parr}}$	$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash B_1, \Gamma}\quad \dfrac{\pi_2}{\vdash A_1, A_2, B_2, \Delta}}{\vdash A_1, A_2, B_1 \otimes B_2, \Gamma, \Delta}\,\otimes}{\vdash A_1 \parr A_2, B_1 \otimes B_2, \Gamma, \Delta}\,\parr$
$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A_1, A_2, B_1, \Gamma}}{\vdash A_1 \parr A_2, B_1, \Gamma}\,\parr \quad \dfrac{\dfrac{\pi_1}{\vdash A_1, A_2, B_2, \Gamma}}{\vdash A_1 \parr A_2, B_2, \Gamma}\,\parr}{\vdash A_1 \parr A_2, B_1 \& B_2, \Gamma}\,\&$	$\underset{\xleftarrow{c_\parr^\&}}{\xrightarrow{c_\&^\parr}}$	$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A_1, A_2, B_1, \Gamma}\quad \dfrac{\pi_1}{\vdash A_1, A_2, B_2, \Gamma}}{\vdash A_1, A_2, B_1 \& B_2, \Gamma}\,\&}{\vdash A_1 \parr A_2, B_1 \& B_2, \Gamma}\,\parr$
$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A_1, A_2, B_i, \Gamma}}{\vdash A_1 \parr A_2, B_i, \Gamma}\,\parr}{\vdash A_1 \parr A_2, B_1 \oplus B_2, \Gamma}\,\oplus_i$	$\underset{\xleftarrow{c_\parr^{\oplus_i}}}{\xrightarrow{c_{\oplus_i}^\parr}}$	$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A_1, A_2, B_i, \Gamma}}{\vdash A_1, A_2, B_1 \oplus B_2, \Gamma}\,\oplus_i}{\vdash A_1 \parr A_2, B_1 \oplus B_2, \Gamma}\,\parr$
$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A_1, \Gamma}\quad \dfrac{\pi_2}{\vdash A_2, B_1, \Delta}}{\vdash A_1 \otimes A_2, B_1, \Gamma, \Delta}\,\otimes \quad \dfrac{\dfrac{\pi_1}{\vdash A_1, \Gamma}\quad \dfrac{\pi_3}{\vdash A_2, B_2, \Delta}}{\vdash A_1 \otimes A_2, B_2, \Gamma, \Delta}\,\otimes}{\vdash A_1 \otimes A_2, B_1 \& B_2, \Gamma, \Delta}\,\&$	$\underset{\xleftarrow{c_\otimes^\&}}{\xrightarrow{c_\&^\otimes}}$	$\dfrac{\dfrac{\pi_1}{\vdash A_1, \Gamma}\quad \dfrac{\dfrac{\pi_2}{\vdash A_2, B_1, \Delta}\quad \dfrac{\pi_3}{\vdash A_2, B_2, \Delta}}{\vdash A_2, B_1 \& B_2, \Delta}\,\&}{\vdash A_1 \otimes A_2, B_1 \& B_2, \Gamma, \Delta}\,\otimes$
$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A_1, B_1, \Gamma}\quad \dfrac{\pi_2}{\vdash A_2, \Delta}}{\vdash A_1 \otimes A_2, B_1, \Gamma, \Delta}\,\otimes \quad \dfrac{\dfrac{\pi_3}{\vdash A_1, B_2, \Gamma}\quad \dfrac{\pi_2}{\vdash A_2, \Delta}}{\vdash A_1 \otimes A_2, B_2, \Gamma, \Delta}\,\otimes}{\vdash A_1 \otimes A_2, B_1 \& B_2, \Gamma, \Delta}\,\&$	$\underset{\xleftarrow{c_\otimes^\&}}{\xrightarrow{c_\&^\otimes}}$	$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A_1, B_1, \Gamma}\quad \dfrac{\pi_3}{\vdash A_1, B_2, \Gamma}}{\vdash A_1, B_1 \& B_2, \Gamma}\,\& \quad \dfrac{\pi_2}{\vdash A_2, \Delta}}{\vdash A_1 \otimes A_2, B_1 \& B_2, \Gamma, \Delta}\,\otimes$
$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A_1, B_i, \Gamma}\quad \dfrac{\pi_2}{\vdash A_2, \Delta}}{\vdash A_1 \otimes A_2, B_i, \Gamma, \Delta}\,\otimes}{\vdash A_1 \otimes A_2, B_1 \oplus B_2, \Gamma, \Delta}\,\oplus_i$	$\underset{\xleftarrow{c_\otimes^{\oplus_i}}}{\xrightarrow{c_{\oplus_i}^\otimes}}$	$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A_1, B_i, \Gamma}}{\vdash A_1, B_1 \oplus B_2, \Gamma}\,\oplus_i \quad \dfrac{\pi_2}{\vdash A_2, \Delta}}{\vdash A_1 \otimes A_2, B_1 \oplus B_2, \Gamma, \Delta}\,\otimes$
$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A_1, \Gamma}\quad \dfrac{\pi_2}{\vdash A_2, B_i, \Delta}}{\vdash A_1 \otimes A_2, B_i, \Gamma, \Delta}\,\otimes}{\vdash A_1 \otimes A_2, B_1 \oplus B_2, \Gamma, \Delta}\,\oplus_i$	$\underset{\xleftarrow{c_\otimes^{\oplus_i}}}{\xrightarrow{c_{\oplus_i}^\otimes}}$	$\dfrac{\dfrac{\pi_1}{\vdash A_1, \Gamma}\quad \dfrac{\dfrac{\pi_2}{\vdash A_2, B_i, \Delta}}{\vdash A_2, B_1 \oplus B_2, \Delta}\,\oplus_i}{\vdash A_1 \otimes A_2, B_1 \oplus B_2, \Gamma, \Delta}\,\otimes$
$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A_1, B_i, \Gamma}\quad \dfrac{\pi_2}{\vdash A_2, B_i, \Gamma}}{\vdash A_1 \& A_2, B_i, \Gamma}\,\&}{\vdash A_1 \& A_2, B_1 \oplus B_2, \Gamma}\,\oplus_i$	$\underset{\xleftarrow{c_{\oplus_i}^\&}}{\xrightarrow{c_\&^{\oplus_i}}}$	$\dfrac{\dfrac{\dfrac{\pi_1}{\vdash A_1, B_i, \Gamma}}{\vdash A_1, B_1 \oplus B_2, \Gamma}\,\oplus_i \quad \dfrac{\dfrac{\pi_2}{\vdash A_2, B_i, \Gamma}}{\vdash A_2, B_1 \oplus B_2, \Gamma}\,\oplus_i}{\vdash A_1 \& A_2, B_1 \oplus B_2, \Gamma}\,\&$
(No commutation with ax, 1 nor 0 as the ax and 1-rule have no context and there are no rule for 0.)		