

Planar #CSP Equality Corresponds to Quantum Isomorphism – A Holant Viewpoint

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Abstract

Recently, Mančinska and Roberson proved [11] that two graphs G and G' are *quantum isomorphic* if and only if they admit the same number of homomorphisms from all *planar* graphs. We extend this result to planar #CSP with any pair of sets \mathcal{F} and \mathcal{F}' of real-valued, arbitrary-arity constraint functions. Graph homomorphism is the special case where each of \mathcal{F} and \mathcal{F}' contains a single symmetric 0-1-valued binary constraint function. Our treatment uses the framework of planar Holant problems. To prove that quantum isomorphic constraint function sets give the same value on any planar #CSP instance, we apply a novel form of *holographic transformation* of Valiant [13], using the quantum permutation matrix \mathcal{U} defining the quantum isomorphism. Due to the noncommutativity of \mathcal{U} 's entries, it turns out that this form of holographic transformation is only applicable to planar Holant. To prove the converse, we introduce the quantum automorphism group $\text{Qut}(\mathcal{F})$ of a set of constraint functions/tensors \mathcal{F} , and characterize the intertwiners of $\text{Qut}(\mathcal{F})$ as the signature matrices of planar Holant($\mathcal{F} \mid \mathcal{EQ}$) quantum gadgets. Then we define a new notion of (projective) connectivity for constraint functions and reduce arity while preserving the quantum automorphism group. Finally, to address the challenges posed by generalizing from 0-1 valued to real-valued constraint functions, we adapt a technique of Lovász [9] in the classical setting for isomorphisms of real-weighted graphs to the setting of quantum isomorphisms.

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1 Introduction

Graph Homomorphism and #CSP

A homomorphism from graph K to graph X is an edge-preserving map from the vertex set $V(K)$ of K to the vertex set $V(X)$ of X . A well-studied problem in complexity theory is to count the number of distinct homomorphisms from K to X , which can be expressed as

$$\sum_{\sigma: V(K) \rightarrow V(X)} \prod_{(u,v) \in E(K)} (A_X)_{\sigma(u), \sigma(v)},$$

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the value of the *partition function* of X evaluated on K , where A_X is the adjacency matrix of X . From this perspective, graph homomorphism naturally generalizes to a counting constraint satisfaction problem (#CSP) by replacing $\{A_X\}$ with a set \mathcal{F} of \mathbb{R} or \mathbb{C} -valued *constraint functions* on one or more inputs from a finite domain $V(\mathcal{F})$, and replacing K with a set of constraints and variables, where each constraint applies a constraint function to a sequence of variables. The problem is to compute the partition function, which is the sum over all variable assignments of the product of the constraint function evaluations. Letting $V(\mathcal{F}) = V(X)$ and the constraint and variable sets be $E(K)$ and $V(K)$, respectively, with each edge/constraint applying A_X to its two endpoints, we recover the special case of counting homomorphisms from K to X .

Bulatov [2] proved that every problem $\#CSP(\mathcal{F})$, parameterized by a finite set \mathcal{F} of 0-1-valued constraint functions, is either (1) solvable in polynomial-time or (2) #P-complete. Dyer and Richerby [6] proved that this *complexity dichotomy* has a decidable criterion. This dichotomy was further extended to nonnegative real-valued, and then to all complex-valued constraint functions [5, 4]. When we restrict to planar #CSP instances (for which the bipartite constraint-variable incidence graph is planar), a further *complexity trichotomy* is known for the Boolean domain (where $V(\mathcal{F}) = \{0, 1\}$) [7], that there are exactly three classes: (1) polynomial-time solvable; (2) #P-hard for general instances but solvable in polynomial-time over planar structures; and (3) #P-hard over planar structures. Furthermore, Valiant's holographic algorithm with matchgates [13] is universal for all problems in class (2): *Every #P-hard #CSP problem that is solvable in polynomial-time in the planar setting is solvable by this one algorithmic strategy.* However, for planar #CSP on domains of size greater than 2, a full complexity classification is open.

Holant Problems

We carry out much of our work in the planar Holant framework from counting complexity, which we find natural to this theory, and of which planar #CSP itself is a special case. Like a #CSP problem, a Holant problem is parameterized by a set \mathcal{F} of constraint functions. The input to a planar Holant problem is a *signature grid*, a planar graph where each edge represents a variable and every vertex is assigned a constraint function from \mathcal{F} . A vertex's constraint function is applied to its incident edges. This is dual to the #CSP view of graph homomorphism, where each edge is a (necessarily binary) constraint and each vertex is a variable. As with #CSP, the computational problem is to compute the Holant value – the sum over all variable (edge) assignments, of the product of the evaluations of the constraint functions. A (planar) *gadget* is a (planar) Holant signature grid with a number of *dangling* edges, representing external variables. Each gadget has an associated *signature matrix*, which stores the Holant value for each fixed assignment to the dangling edges. The study of Holant problems is motivated by Valiant's *holographic transformations* [13], which are certain Holant value-preserving transformations of the constraint functions by invertible matrices.

Classical and Quantum Isomorphism

As suggested above, one can view a q -vertex real-weighted graph X , via its adjacency matrix $A_X \in \mathbb{R}^{q \times q}$, as an \mathbb{R} -valued binary (i.e. two input variables) constraint function. Two q -vertex graphs X and Y are isomorphic if one can apply a permutation to the rows and columns of A_X to obtain A_Y . Equivalently, if we convert A_X and A_Y to vectors $a_X, a_Y \in \mathbb{R}^{q^2}$, there is a permutation matrix P satisfying $P^{\otimes 2} a_X = a_Y$. For n -ary constraint functions $F, G \in \mathbb{R}^{[q]^n}$, where $[q] = \{1, 2, \dots, q\}$, a natural generalization applies. F and G are isomorphic if there is a permutation matrix P satisfying $P^{\otimes n} f = g$, where $f, g \in \mathbb{R}^{q^n}$ are the vector versions of F and G .

Quantum isomorphism of (undirected, unweighted) graphs, introduced in [1], is a relaxation of classical isomorphism. Graphs X and Y are *quantum isomorphic* if there is a perfect winning strategy in a two-player *graph isomorphism game* in which the players share and can perform measurements on an entangled quantum state. This condition is equivalent to the existence of a *quantum permutation matrix* matrix \mathcal{U} – a relaxation of a permutation matrix whose entries do not necessarily commute – satisfying $\mathcal{U}^{\otimes 2} a_X = a_Y$ [10]. Analogously to classical isomorphism, in this work we define n -ary constraint functions F and G to be *quantum isomorphic* if there is a quantum permutation matrix \mathcal{U} satisfying $\mathcal{U}^{\otimes n} f = g$. Sets \mathcal{F} and \mathcal{G} of constraint functions of equal cardinality are quantum isomorphic if there is a single quantum permutation matrix defining a quantum isomorphism between every pair of corresponding functions in \mathcal{F} and \mathcal{G} .

In [8], Lovász proved that two graphs are isomorphic if and only if they admit the same number of homomorphisms from every graph. Fifty years later, Mančinska and Roberson [11] proved that two graphs are *quantum isomorphic* if and only if they admit the same number of homomorphisms from all *planar* graphs. We generalize this result to #CSP and sets of constraint functions. We achieve this via graph combinatorics, results from quantum group theory, and a novel form of holographic transformation, establishing new connections between planar Holant, #CSP, quantum permutation matrices, and quantum isomorphism.

While quantum permutation matrices, quantum isomorphism, and other quantum constructions in this paper are somewhat abstract and technical, we believe it is precisely these concepts' abstractness that makes the connections we develop between them and the very concrete, combinatorial concept of planarity so fascinating and potentially fruitful. Our result that quantum isomorphism exactly captures planarity could lead to entirely novel, algebraic methods of studying the complexity of planar #CSP and Holant.

Our Results

Our main result is the following theorem, a broad extension of the main result of Mančinska and Roberson [11], recast into the well-studied Holant and #CSP frameworks.

► **Theorem** (Theorem 9, informal). *Sets \mathcal{F} and \mathcal{G} of \mathbb{R} -valued constraint functions are quantum isomorphic iff the partition function of every planar #CSP(\mathcal{F}) instance is preserved upon replacing every constraint function in \mathcal{F} with the corresponding function in \mathcal{G} .*

Our general constraint functions add significant complexity relative to the graph homomorphism special case in [11], since, unlike unweighted graph adjacency matrices, they can be (1) asymmetric (i.e. permuting the argument order affects their value), (2) n -ary, for $n > 2$, and (3) arbitrary real-valued. Each of these three extensions adds intricacies and challenges not present in [11], which we address with novel approaches that reveal new, deeper connections between quantum permutation matrices and planar graphs.

First, in Subsection 3.1 we give a procedure for decomposing any planar Holant signature grid corresponding to a planar #CSP instance into a small set of simple gadgets. Here arise the first new complications associated with higher-arity signatures. The dangling edges of simple gadgets extracted from the signature grid may not be oriented correctly, so we must use certain other gadgets to pivot them to the correct orientation, respecting planarity.

With some preparation in Subsection 3.2, we prove the quantum Holant theorem in Subsection 3.3. The forward direction of Theorem 9 is a direct corollary, giving a more graphical and more intuitive proof than that of the graph homomorphism special case in [11]. The gadget decomposition gives an expression for the Holant value as a product of the component gadgets' signature matrices. So, assuming \mathcal{F} and \mathcal{G} are quantum isomorphic, we

use the quantum permutation matrix \mathcal{U} defining the quantum isomorphism as a *quantum holographic transformation*, inserting tensor powers of \mathcal{U} and its inverse between every pair of signature matrices in the product without changing the Holant value. Then a sequence of these holographic transformations converts every signature in \mathcal{F} to the corresponding signature in \mathcal{G} . The quantum holographic transformation does not work on general signature grids, since viewing \mathcal{U} itself as a constraint function in the signature grid is not in general well-defined, as \mathcal{U} 's entries do not commute and the partition function does not specify an order to multiply the constraint function evaluations. However, the planarity of the signature grid and the resulting gadget decomposition and matrix product expression for the Holant value implicitly provide a multiplication order. Quantum holographic transformations apply to the planar version of the general Holant problem parameterized by a set \mathcal{F} of constraint functions (not just the special case of #CSP), and should be of independent interest.

The success of the quantum holographic transformation for asymmetric signatures is also dependent on the fact that the holographic transformation action of a quantum permutation matrix is invariant under gadget rotations and reflections. The asymmetry and rotation and reflection issues are only relevant in the context of planar signature grids, since in nonplanar grids, one can simply cross and twist the incident edges to achieve the desired input order. Hence this is another interesting connection between quantum permutation matrices and the structural properties of planar graphs.

In Section 4, to prove the reverse direction of Theorem 9, we turn to the theory of quantum groups [15, 14]. We introduce the quantum automorphism group $\text{Qut}(\mathcal{F})$ of a set \mathcal{F} of signatures, an abstraction of the classical automorphism group satisfying many of the same properties. Using the planar gadget decomposition, we prove that the signature matrices of planar Holant gadgets in the context of #CSP(\mathcal{F}), a very concrete, combinatorial concept, exactly capture the abstract *intertwiner space* of $\text{Qut}(\mathcal{F})$. A natural approach to the rest of the proof breaks down for constraint functions of arity > 2 . Hence we introduce a method to reduce a constraint function's arity while maintaining its inclusion in the original intertwiner space. Then we say a constraint function is *projectively connected* if this procedure yields a connected graph upon reaching arity 2. Finally, we show that if \mathcal{F} and \mathcal{G} are projectively connected and the quantum automorphism group of the disjoint union of \mathcal{F} and \mathcal{G} maps a "vertex" of \mathcal{F} to a "vertex" of \mathcal{G} , then \mathcal{F} and \mathcal{G} are quantum isomorphic (analogous to the familiar classical fact for graphs). For 0-1 valued functions, Mančinska and Roberson [11] ensured connectivity by taking complements. However, for real-valued functions F and G this method does not work: we cannot take the complement to assume they are projectively connected. Instead, we adapt to the quantum setting a technique of Lovász [9] in the classical setting for real-weighted graphs, and extract a quantum isomorphism to complete the proof.

All of the above results extend to sets of constraint functions over \mathbb{C} that are closed under conjugation and for which the quantum isomorphism respects conjugation (both properties are trivially satisfied by constraint functions over \mathbb{R}). In the full version, our proof is carried out in this setting. In this extended abstract, we specialize to constraint functions over \mathbb{R} .

In Appendix A, we give an alternate approach for enforcing constraint function connectivity due to Roberson [12], which adds new binary connected constraint functions to \mathcal{F} and \mathcal{G} rather than modify the existing constraint functions to be projectively connected. We explore two further topics in the full version. First, we extend the connection between quantum isomorphism and nonlocal games. We define graph isomorphism games for real-weighted directed graphs and prove the following generalization of a result in [11]: real-weighted graphs F and G admit the same number of homomorphisms from all planar graphs if and only if there is a perfect quantum commuting strategy for the (F, G) -isomorphism game. Second,

we discuss how pivoting dangling edges around a gadget and horizontally reflecting gadgets, graphical manipulations that arise naturally throughout our work, correspond to the dual and adjoint operations in the pivotal dagger category of gadgets.

We hope that our results, in particular the quantum holographic transformation technique in Theorem 18, will lead to further applications of quantum group theory in the study of planar #CSP and Holant complexity.

2 Preliminaries

Constraint functions and #CSP

► **Definition 1** (Constraint function, $V(F)$, $V(\mathcal{F})$). A tensor $F \in \mathbb{R}^{[q]^n}$, for $q, n \geq 1$, is a constraint function of domain size q and arity n . For $\mathbf{x} \in [q]^n$, we write $F_{\mathbf{x}} = F_{x_1, \dots, x_n} = F(x_1, \dots, x_n) \in \mathbb{R}$. We write $V(F)$ for $[q]$, thus $F \in \mathbb{R}^{V(F)^n}$. Whenever we specify a set \mathcal{F} of constraint functions, it is assumed that all $F \in \mathcal{F}$ have the same domain, which we call $V(\mathcal{F})$, with $|V(\mathcal{F})| = q$.

► **Definition 2** (#CSP, Z). A #CSP problem $\#CSP(\mathcal{F})$ is parameterized by a set \mathcal{F} of constraint functions. A #CSP(\mathcal{F}) instance K is defined by a pair (V, C) , where V is a set of variables and C is a multiset of constraints. Each constraint $c = (F^c, v_{c_1}, \dots, v_{c_{n_F}})$ consists of a constraint function $F^c \in \mathcal{F}$ and an ordered tuple of variables to which F is applied. The partition function Z , on input #CSP(\mathcal{F}) instance K , outputs

$$Z(K) = \sum_{\sigma: V \rightarrow V(\mathcal{F})} \prod_{(F^c, v_{c_1}, \dots, v_{c_{n_F}}) \in C} F^c(\sigma(v_{c_1}), \dots, \sigma(v_{c_{n_F}})).$$

► **Definition 3** (Compatible constraint function sets, $K_{\mathcal{F} \rightarrow \mathcal{G}}$). Let $\mathcal{F} = \{F_i\}_{i \in [t]}$, $\mathcal{G} = \{G_i\}_{i \in [t]}$ be two sets of constraint functions on the same domain $[q]$. \mathcal{F} and \mathcal{G} are compatible if, for all $i \in [t]$, F_i and G_i have common arity n_i . Call F_i and G_i corresponding constraint functions.

For compatible \mathcal{F} and \mathcal{G} and any #CSP(\mathcal{F}) instance K , define a #CSP(\mathcal{G}) instance $K_{\mathcal{F} \rightarrow \mathcal{G}}$ by replacing every constraint $(F_i, v_{i_1}, \dots, v_{i_{n_i}})$ of K with the corresponding constraint $(G_i, v_{i_1}, \dots, v_{i_{n_i}})$.

Often it will be useful to “flatten” a constraint function F into a matrix:

► **Definition 4** ($F^{m,d}$, f). For $F \in \mathbb{R}^{[q]^n}$ and any $m, d \geq 0$, $m + d = n$, let $F^{m,d} \in \mathbb{R}^{[q^m] \times [q^d]}$ be the $q^m \times q^d$ matrix defined by $F_{x_1 \dots x_m, x_{m+1} \dots x_{m+d}}^{m,d} = F(x_1, \dots, x_n)$, where $x_1 \dots x_m \in \mathbb{N}$ is the base- q integer with the most significant digit x_1 , and similarly for $x_{m+1} \dots x_{m+d}$ (in decreasing index). We write $f = F^{n,0} \in \mathbb{R}^{q^n}$; it is called the signature vector of F .

Quantum permutation matrices and quantum isomorphism

A core construction in this work is the *quantum permutation matrix*, a generalization of classical permutation matrix, whose entries come from an arbitrary C^* -algebra rather than $\{0, 1\}$. For the purposes of this work, one can view a C^* -algebra as simply an abstraction of \mathbb{C} , equipped with an involution $*$ analogous to conjugation, and whose elements, critically, do not necessarily commute. More generally, one can think of a C^* -algebra as the algebra of bounded operators on a Hilbert space.

► **Definition 5** (Quantum permutation matrix). A matrix $U = (u_{ij})$ with entries from a C^* -algebra with unit element $\mathbf{1}$ is called a quantum permutation matrix if it satisfies the following conditions for all i, j :

- $u_{ij}^2 = u_{ij} = u_{ij}^*$;
- $\sum_j u_{ij} = \sum_i u_{ij} = \mathbf{1}$.

If the C^* -algebra in question is \mathbb{C} , then the first condition implies \mathcal{U} is a 0-1 matrix, and then the second condition implies \mathcal{U} is a classical permutation matrix. Hence the abstraction of \mathbb{C} to an arbitrary C^* -algebra is one of the many abstractions from “classical” to “quantum” constructions throughout this work.

Recall that graphs X and Y with adjacency matrices $A_X, A_Y \in \{0, 1\}^{[q] \times [q]}$ are classically isomorphic if and only if $PA_X = A_Y P$ for some classical permutation matrix P . Hence we say X and Y are *quantum isomorphic* ($X \cong_{qc} Y$) [1, 10] if there is a quantum permutation matrix \mathcal{U} satisfying $\mathcal{U}A_X = A_Y\mathcal{U}$. Equivalently, $\mathcal{U}^{\otimes 2}a_X = a_Y$, where $a_X, a_Y \in \{0, 1\}^{q^2}$ are the signature vectors of A_X and A_Y . Hence the following definition is a generalization of quantum graph isomorphism to higher-arity constraint functions over \mathbb{R} .

► **Definition 6** (\cong_{qc}). $F, G \in \mathbb{R}^{[q]^n}$ are quantum isomorphic ($F \cong_{qc} G$) if there is a $q \times q$ quantum permutation matrix \mathcal{U} satisfying $\mathcal{U}^{\otimes n}f = g$. Compatible sets \mathcal{F} and \mathcal{G} of constraint functions are quantum isomorphic ($\mathcal{F} \cong_{qc} \mathcal{G}$) if there is a $q \times q$ quantum permutation matrix \mathcal{U} satisfying $\mathcal{U}^{\otimes \text{arity}(F_i)}f_i = g_i$ for every i .

Holant, gadgets and signature matrices

A Holant problem $\text{Holant}(\mathcal{F})$, like a #CSP problem, is parameterized by a set \mathcal{F} of constraint functions, usually called *signatures*. The input to $\text{Holant}(\mathcal{F})$ is a *signature grid* Ω , which consists of an underlying multigraph X with vertex set V and edge set E . Each vertex $v \in V$ is assigned a signature $F_v \in \mathcal{F}$ of arity $\deg(v)$. The incident edges $E(v) = (e_1^v, \dots, e_{\deg(v)}^v)$ of v are input variables to F_v taking values in $V(\mathcal{F})$. We use $\text{Pl-Holant}(\mathcal{F})$ to specify that input signature grids must have planar underlying multigraphs. For planar Holant, the input variables of F_v are labeled in cyclic order starting with one particular edge, labeled with a diamond. The output on input Ω is

$$\text{Holant}_\Omega(\mathcal{F}) = \sum_{\sigma: E \rightarrow V(\mathcal{F})} \prod_{v \in V} F_v(\sigma|_{E(v)}), \quad (1)$$

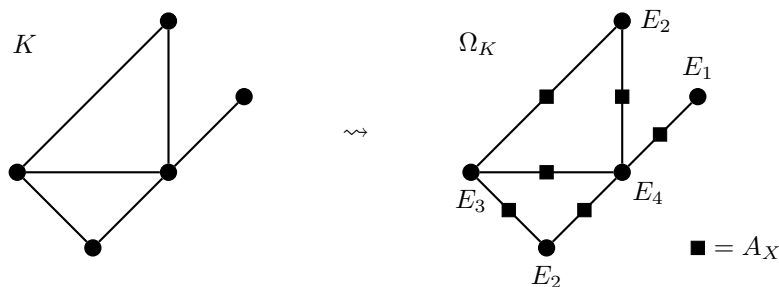
where $F_v(\sigma|_{E(v)}) = (F_v)(\sigma(e_1^v), \dots, \sigma(e_{\deg(v)}^v))$. For sets \mathcal{F} and \mathcal{G} of signatures, define the problem $\text{Holant}(\mathcal{F} | \mathcal{G})$ as follows. A signature grid in the context of $\text{Holant}(\mathcal{F} | \mathcal{G})$ has a bipartite underlying multigraph with bipartition $V = V_1 \sqcup V_2$ such that the vertices in V_1 and V_2 are assigned signatures from \mathcal{F} and \mathcal{G} , respectively.

The Holant problems in this work always include the following set of signatures.

► **Definition 7** (E_n, \mathcal{EQ}). For fixed q , define the 0-1-valued equality constraint function $E_n \in \mathbb{R}^{[q]^n}$ by $E_n(x_1, \dots, x_n) = 1$ iff $x_1 = \dots = x_n$. Define $\mathcal{EQ} = \bigcup_n E_n$.

To each #CSP(\mathcal{F}) instance $K = (V, C)$ we associate a signature grid Ω_K in the context of $\text{Holant}(\mathcal{F} | \mathcal{EQ})$ defined as follows: For every constraint $c \in C$, if c applies function F of arity n , create a degree- n vertex assigned F , called a *constraint vertex*. For each variable $v \in V$, if v appears in the multiset of constraints $C_v \subseteq C$, create a degree- $|C_v|$ vertex assigned $E_{|C_v|} \in \mathcal{EQ}$, called an *equality vertex*, and edges (v, c) for every $c \in C_v$ such that the cyclic order of edges incident to each constraint vertex matches the order of variables in the constraint. Any edge assignment σ must assign all edges incident to an equality vertex the same value (or else the term corresponding to σ is 0), so we can view σ as #CSP variable assignment. Hence $Z(K) = \text{Holant}_{\Omega_K}(\mathcal{F} | \mathcal{EQ})$.

For example, to compute the number of homomorphisms $K \rightarrow X$, consider a $\#CSP(A_X)$ instance where the vertices of K are variables and each edge of K is a constraint applying function $A_X \in \mathbb{R}^{V(X)^2}$ (X 's adjacency matrix) to the edge's two endpoints. The corresponding Holant signature grid Ω_K starts with underlying graph K , with K 's vertices assigned the appropriate equality signature from \mathcal{EQ} , and we subdivide each of K 's edges by placing degree-2 constraint vertices, assigned signature A_X , connected to the labeled equality vertices. See Figure 1. We always depict equality and constraint vertices as circles and squares, respectively.



■ **Figure 1** A graph K and the corresponding Holant($A_X \mid \mathcal{EQ}$) signature grid Ω_K for computing the number of homomorphisms from K to X . Square vertices are assigned signature A_X .

Generalizing graph homomorphism to $\#CSP$ entails replacing A_X with an arbitrary set \mathcal{F} of constraint functions, and replacing the degree-2 vertices assigned A_X with arbitrary-degree vertices assigned signatures from \mathcal{F} .

► **Definition 8** (Planar $\#CSP$ instance). *A $\#CSP$ instance K is planar if the underlying multigraph of the corresponding Holant signature grid Ω_K is planar.*

We now have the notation to state our main theorem.

► **Theorem 9** (Main result). *Let \mathcal{F}, \mathcal{G} be compatible sets of constraint functions. Then $\mathcal{F} \cong_{qc} \mathcal{G}$ if and only if $Z(K) = Z(K_{\mathcal{F} \rightarrow \mathcal{G}})$ for every planar $\#CSP(\mathcal{F})$ instance K .*

If $\mathcal{F} = \{A_X\}$ and $\mathcal{G} = \{A_Y\}$, then Theorem 9 specializes to the result of [11]: graphs X and Y are quantum isomorphic iff they admit the same number of homomorphisms from every planar graph K .

► **Definition 10** (Gadget). *A gadget is a Holant signature grid equipped with an ordered set of dangling edges (edges with only one endpoint), defining external variables.*

Our gadgets will be in the context of $\text{Pl-Holant}(\mathcal{F} \mid \mathcal{EQ})$, the Holant problem equivalent to $\#CSP(\mathcal{F})$. In this case, we specify that all dangling edges must be attached to equality vertices (vertices assigned signatures in \mathcal{EQ}).

See Figures 2 and 3 for examples of gadgets in the context of $\text{Pl-Holant}(\mathcal{F} \mid \mathcal{EQ})$. We draw dangling edges lighter and thinner than internal edges.

► **Definition 11** ($M(\mathbf{K})$). *Let \mathbf{K} be a gadget with n dangling edges and containing signatures of domain size q . For any $m, d \geq 0, m + d = n$, define \mathbf{K} 's (m, d) -signature matrix $M(\mathbf{K}) \in \mathbb{R}^{q^m \times q^d}$ by letting $M(\mathbf{K})_{\mathbf{x}, \mathbf{y}}$ be the Holant value when the first m dangling edges (called output dangling edges) are assigned x_1, \dots, x_m and the last d dangling edges (called input dangling edges) are assigned y_d, \dots, y_1 . We draw the output/input dangling edges to the left/right of the gadget.*

► **Definition 12.** *Gadget \mathbf{K} is planar if the underlying multigraph has an embedding with no edges (dangling or not) crossing, and the dangling edges are in cyclic order in the outer face.*

For a plane embedding of a planar gadget, we draw its output and input dangling edges on the left in order from top to bottom and on the right in order from bottom to top, respectively. For a gadget’s signature matrix, observe that we consider the input dangling edges in reverse order, so from top to bottom. This definition preserves planarity of \circ :

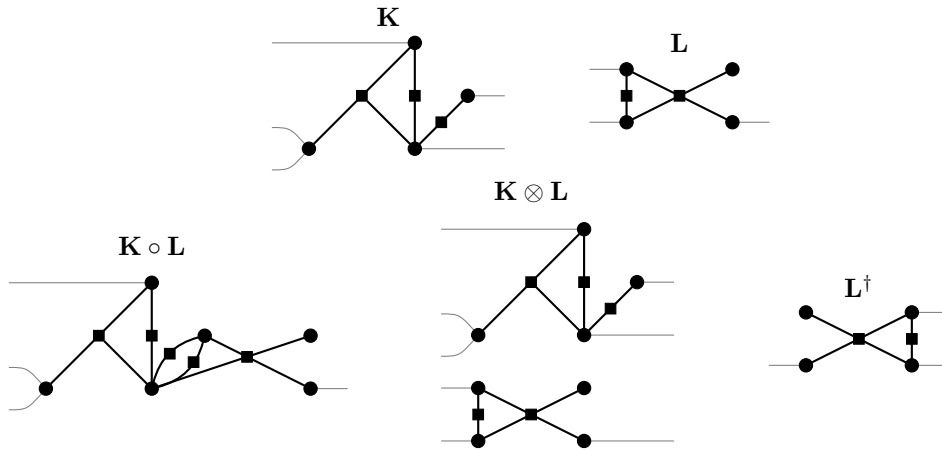
► **Definition 13** (Gadget \circ, \otimes, \dagger). For a gadget \mathbf{K} with $m+d$ dangling edges, write $\mathbf{K} \in \mathfrak{G}(m, d)$ to mean we consider \mathbf{K} with m output and d input dangling edges.

■ For $\mathbf{K} \in \mathfrak{G}(m, d), \mathbf{L} \in \mathfrak{G}(d, w)$, define the composition $\mathbf{K} \circ \mathbf{L} \in \mathfrak{G}(m, w)$ by connecting each input dangling edge of \mathbf{K} with the corresponding output dangling edge of \mathbf{L} . If \circ creates adjacent vertices assigned $E_a, E_b \in \mathcal{EQ}$, we contract the edge between them, merging them into a single vertex assigned E_{a+b-2} . This does not change the Holant value.

■ For $\mathbf{K} \in \mathfrak{G}(m_1, d_1), \mathbf{L} \in \mathfrak{G}(m_2, d_2)$, define the tensor product $\mathbf{K} \otimes \mathbf{L} \in \mathfrak{G}(m_1+m_2, d_1+d_2)$ by placing \mathbf{K} above \mathbf{L} .

■ For $\mathbf{K} \in \mathfrak{G}(m, d)$, define the (conjugate) transpose $\mathbf{K}^\dagger \in \mathfrak{G}(d, m)$ by reflecting \mathbf{K} ’s underlying multigraph horizontally.

See Figure 2. It is well known that applying the \circ, \otimes, \dagger operations to gadgets corresponds to applying these operations to their signature matrices. See e.g. [3].



■ **Figure 2** Operations on gadgets \mathbf{K} and \mathbf{L} in the context of $\text{Pl-Holant}(\mathcal{F} \mid \mathcal{EQ})$.

3 Quantum Isomorphism Implies Planar #CSP Equivalence

3.1 The Planar Gadget Decomposition

Throughout, let $F \in \mathbb{R}^{[q]^n}$ denote a constraint function in \mathcal{F} , a set of constraint functions.

► **Definition 14** ($\mathcal{P}_{\mathcal{F}}, \mathcal{P}_{\mathcal{F}}(m, d)$). Let $\mathcal{P}_{\mathcal{F}}$ be the collection of all planar gadgets in the context of $\text{Holant}(\mathcal{F} \mid \mathcal{EQ})$. Recall that all dangling edges of such a gadget are attached to vertices assigned signatures in \mathcal{EQ} .

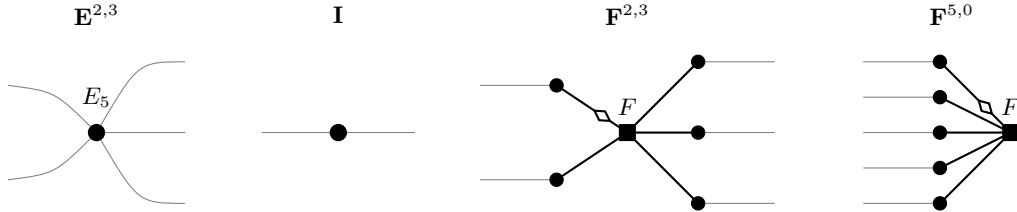
Let $\mathcal{P}_{\mathcal{F}}(m, d) \subseteq \mathcal{P}_{\mathcal{F}}$ be the subset of gadgets with m output and d input dangling edges.

The discussion after Definition 7 constructs a $\text{Pl-Holant}(\mathcal{F} \mid \mathcal{EQ})$ instance modelling any given planar $\#CSP(\mathcal{F})$ instance. One can easily invert this construction to produce a planar $\#CSP(\mathcal{F})$ instance modelling any given $\text{Pl-Holant}(\mathcal{F} \mid \mathcal{EQ})$ instance. Hence the signature grids underlying $\mathcal{P}_{\mathcal{F}}(m, d)$ are exactly the set of signature grids Ω_K corresponding to planar $\#CSP(\mathcal{F})$ instances.

We next introduce two families of fundamental gadgets in $\mathcal{P}_{\mathcal{F}}$. See Figure 3.

► **Definition 15** ($\mathbf{E}^{m,d}, \mathbf{F}^{m,d}$). For $m, d \geq 0$, let $\mathbf{E}^{m,d}$ be the gadget consisting of a single vertex, assigned E_{m+d} , with m output and d input dangling edges.

For $m, d \geq 0$ and $(m+d)$ -ary signature function F , let $\mathbf{F}^{m,d}$ be the gadget consisting of a central degree- $(m+d)$ vertex assigned F , and m left and d right “arms”, each with a vertex assigned E_2 with an output or input dangling edge, respectively. Define $\mathbf{I} = \mathbf{E}^{1,1}$.

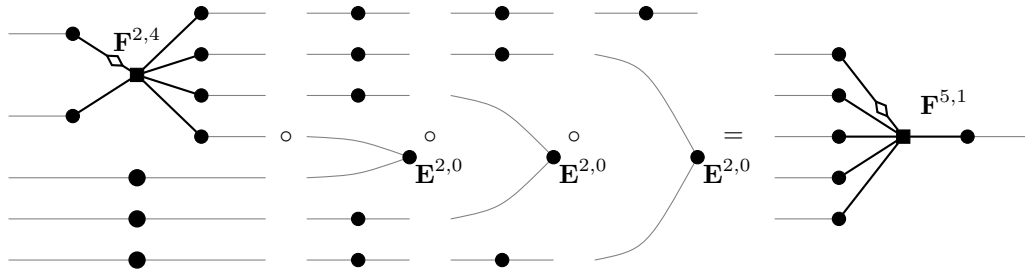


■ **Figure 3** Examples of the fundamental gadgets $\mathbf{E}^{m,d}$ and $\mathbf{F}^{m,d}$. The diamond indicates the first input to asymmetric F .

Observe that $M(\mathbf{E}^{m,d}) = E^{m,d}$ and $M(\mathbf{F}^{m,d}) = F^{m,d}$; in particular $M(\mathbf{F}^{n,0}) = f$.

The next lemma addresses a key new issue raised by viewing our higher-arity constraint functions as explicit vertices in the signature grid. Section 4 requires all \mathbf{F} gadgets to be in the form $\mathbf{F}^{m+d,0}$, but the decomposition procedure below produces gadgets $\mathbf{F}^{m,d}$ for arbitrary m and d (see Figure 5). Hence we must pivot $\mathbf{F}^{m,d}$'s dangling edges between input and output. $\mathbf{E}^{2,0}, \mathbf{E}^{0,2}, \mathbf{I} \in \langle \mathbf{E}^{1,0}, \mathbf{E}^{1,2} \rangle_{\circ, \otimes, \dagger}$ (the closure of $\{\mathbf{E}^{1,0}, \mathbf{E}^{1,2}\}$ under \circ, \otimes, \dagger), so we apply a procedure like the one in Figure 4.

► **Lemma 16.** Let F be an n -ary constraint function. Then $\mathbf{F}^{m,d} \in \langle \mathbf{E}^{1,0}, \mathbf{E}^{1,2}, \mathbf{F}^{n,0} \rangle_{\circ, \otimes, \dagger}$ for all $m+d = n$.

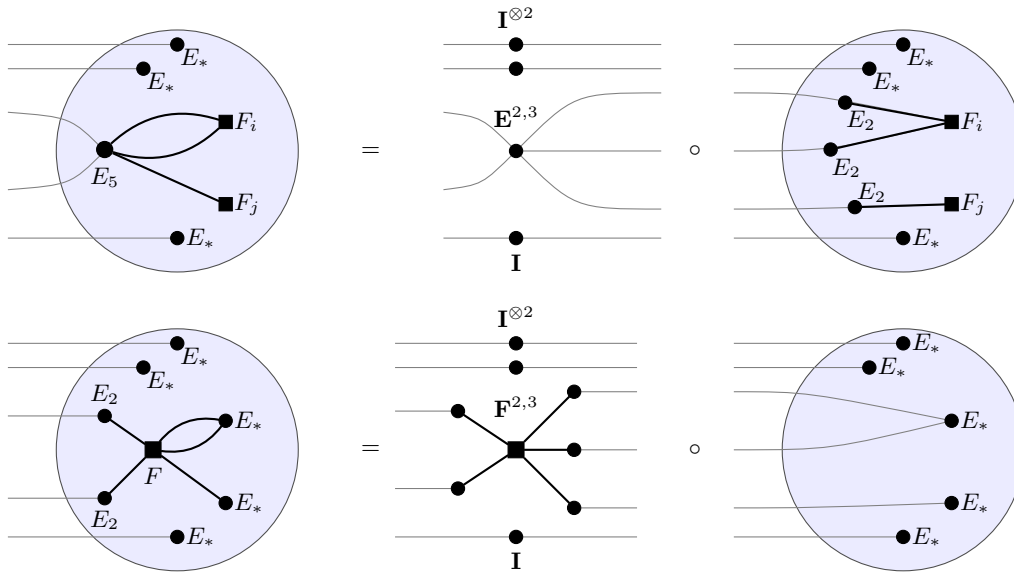


■ **Figure 4** $(\mathbf{F}^{2,4} \otimes \mathbf{I}^{\otimes 3}) \circ (\mathbf{I}^{\otimes 3} \otimes \mathbf{E}^{2,0} \otimes \mathbf{I}^{\otimes 2}) \circ (\mathbf{I}^{\otimes 2} \otimes \mathbf{E}^{2,0} \otimes \mathbf{I}) \circ (\mathbf{I} \otimes \mathbf{E}^{2,0}) = \mathbf{F}^{5,1}$.

Next we show that any planar Holant($\mathcal{F} \mid \mathcal{EQ}$) gadget can be decomposed into the $\mathbf{F}^{\text{arity}(F),0}$ gadgets containing the signatures in \mathcal{F} , and two small equality gadgets.

► **Theorem 17.** $\mathcal{P}_{\mathcal{F}} = \langle \mathbf{E}^{1,0}, \mathbf{E}^{1,2}, \{\mathbf{F}^{\text{arity}(F),0} \mid F \in \mathcal{F}\} \rangle_{\circ, \otimes, \dagger}$.

The reverse inclusion follows from the fact that \circ, \otimes, \dagger preserve planarity. The idea for the forward inclusion is to decompose an arbitrary $\mathbf{K} \in \mathcal{P}_{\mathcal{F}}$ into a composition of copies of $\mathbf{E}^{m,d}$ and appropriate $\mathbf{F}^{m,d}$, tensored with copies of \mathbf{I} . We use Lemma 16 to convert each $\mathbf{F}^{m,d}$ to $\mathbf{F}^{n,0}$. To extract an equality or constraint vertex, we apply one of the two extraction procedures shown in Figure 5, or their horizontal reflections. The extraction procedures guarantee that remaining gadget is still planar, bipartite, and has all dangling edges incident to equality vertices (i.e. is in $\mathcal{P}_{\mathcal{F}}$), so we apply induction.



■ **Figure 5** Extracting an equality (top) or constraint vertex from a planar Holant($\mathcal{F} \mid \mathcal{EQ}$) gadget.

3.2 Gadgets and quantum permutation matrices

The quantum Holant theorem, proved in Subsection 3.3, stems from viewing the quantum permutation matrix \mathcal{U} itself as a signature in a Holant signature grid, indicated by a triangle vertex \blacktriangle . An immediate corollary of this theorem is one half of our main result Theorem 9: Planar #CSP instances with quantum isomorphic signature sets have the same partition function value. The proof of this result via the quantum Holant theorem is graphical and more intuitive than the proof in [11] of the graph homomorphism special case, and ties quantum isomorphism into the well-studied Holant framework. Furthermore, the graphical calculus of the quantum Holant theorem nicely highlights one of the key new difficulties of our generalization: unlike constraint functions derived from homomorphisms to undirected graphs (the case considered in [11]), general constraint functions F can be *asymmetric*: F 's value is not necessarily preserved under permutation of its inputs. By planarity, permutations that cross F 's input edges are not allowed, but the dihedral group actions – rotations and reflection – are allowed. Define the rotated constraint function $F^{(r)}$ for $r \in [\text{arity}(F)]$ by $F^{(r)}(x_1, \dots, x_n) = F(x_{r+1}, \dots, x_n, x_1, \dots, x_r)$ and the reflected constraint function F^\dagger by $F^\dagger(x_1, \dots, x_n) = F(x_n, \dots, x_1)$. Assuming F and G are quantum isomorphic, that is, $\mathcal{U}^{\otimes n} f = g$, it is not *a priori* obvious (due to noncommutativity), but true, that

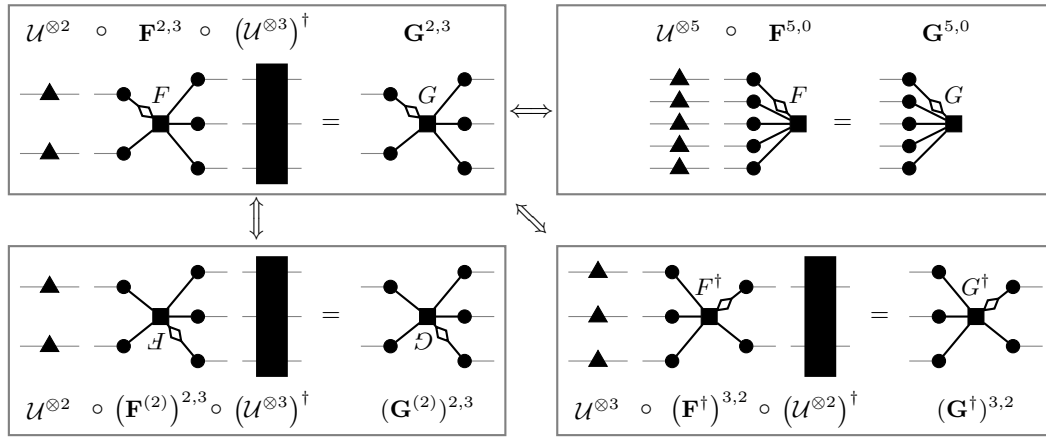
$$\mathcal{U}^{\otimes n} f = g \iff \mathcal{U}^{\otimes n} f^{(r)} = g^{(r)} \text{ and } \mathcal{U}^{\otimes n} f = g \iff \mathcal{U}^{\otimes n} f^\dagger = g^\dagger. \quad (2)$$

The quantum Holant theorem uses \mathcal{U} to transform each F in the Holant signature grid into G (via the assumption $\mathcal{U}^{\otimes n} f = g$), and since the signatures $F^{(r)}$ and F^\dagger may appear in the signature grid in place of F , the theorem's success is dependent on the identities (2).

The bottom two panes of Figure 6 below illustrate a case of (2) for 5-ary f and g . The fact that Figure 6 uses $\mathbf{F}^{2,3}$ and $\mathbf{G}^{2,3}$ (with signature matrices $F^{2,3}$ and $G^{2,3}$) in place of $\mathbf{F}^{5,0}$ and $\mathbf{G}^{5,0}$ (with signature vectors f and g) is due to the additional useful identity

$$\mathcal{U}^{\otimes n} f = g \iff \mathcal{U}^{\otimes m} F^{m,d} (\mathcal{U}^{\otimes d})^\dagger = G^{m,d} \text{ for any } m + d = n, \quad (3)$$

illustrated in the upper right pane of Figure 6.



■ **Figure 6** The “action” of \mathcal{U} transforming F to G is preserved under (clockwise from top right) edge pivoting, signature reflection, and signature rotation. $(\mathcal{U}^{\otimes n})^\dagger$ is drawn as a black box, since, due to noncommutativity, $(\mathcal{U}^{\otimes n})^\dagger \neq (\mathcal{U}^\dagger)^{\otimes n}$ in general.

3.3 The Quantum Holant Theorem

The quantum Holant theorem gives a quantum version of *holographic transformation*. A holographic transformation [13] transforms one Holant signature grid Ω into another Ω' resulting in the same Holant value. For a set \mathcal{F} of signatures and an invertible $T \in \mathbb{C}^{q \times q}$, write $T\mathcal{F} = \{T^{\otimes k} f \mid f \in \mathcal{F} \text{ has arity } k\}$. Define $\mathcal{G}T$ similarly. Valiant’s Holant Theorem in [13] states that $\text{Holant}_\Omega(\mathcal{F} \mid \mathcal{G}) = \text{Holant}_{\Omega'}(T\mathcal{F} \mid \mathcal{G}T^{-1})$, for any Ω and sets of signatures \mathcal{F}, \mathcal{G} , where Ω' is constructed from Ω by replacing every signature in \mathcal{F} or \mathcal{G} with the corresponding transformed signature.

The Holant signature grids Ω_K and $\Omega_{K_{\mathcal{F} \rightarrow \mathcal{G}}}$ satisfying $Z(K) = \text{Pl-Holant}_{\Omega_K}(\mathcal{F} \mid \mathcal{EQ})$ and $Z(K_{\mathcal{F} \rightarrow \mathcal{G}}) = \text{Pl-Holant}_{\Omega_{K_{\mathcal{F} \rightarrow \mathcal{G}}}}(\mathcal{G} \mid \mathcal{EQ})$ are the same, up to replacement of every signature $F \in \mathcal{F}$ by the corresponding signature $G \in \mathcal{G}$. Assuming $\mathcal{F} \cong_{qc} \mathcal{G}$, there is a quantum permutation matrix \mathcal{U} satisfying $\mathcal{U}^{\otimes n} f = g$ for every $F \in \mathcal{F}$ and corresponding $G \in \mathcal{G}$. This suggests we perform a *quantum* holographic transformation using \mathcal{U} . A calculation shows that $E^{0,n} = E^{0,n} \mathcal{U}^{\otimes n}$ for every n and any quantum permutation matrix \mathcal{U} . As $\mathcal{U}^{-1} = \mathcal{U}^\dagger$ is a quantum permutation matrix, $(\mathcal{EQ})\mathcal{U}^{-1} = \mathcal{EQ}$. Then the Holant theorem setting T to \mathcal{U} seems to give

$$Z(K) = \text{Holant}_{\Omega_K}(\mathcal{F} \mid \mathcal{EQ}) = \text{Holant}_{\Omega_{K_{\mathcal{F} \rightarrow \mathcal{G}}}}(\mathcal{G} \mid \mathcal{EQ}) = Z(K_{\mathcal{F} \rightarrow \mathcal{G}}) \tag{4}$$

for any, not necessarily planar, $\#\text{CSP}(\mathcal{F})$ instance K . However, this cannot be true. If $\mathcal{F} = \{F\}$ and $\mathcal{G} = \{G\}$, where F and G are symmetric, binary, and 0-1 valued, (4) implies that the graphs with adjacency matrices F and G admit the same number of homomorphisms from any graph, giving $F \cong G$, a classical result of Lovász [8]. In other words, any quantum isomorphic graphs are classically isomorphic. But this is known to be false – see e.g. [1]. The main reason for this failure is the noncommutativity of \mathcal{U} ’s entries.

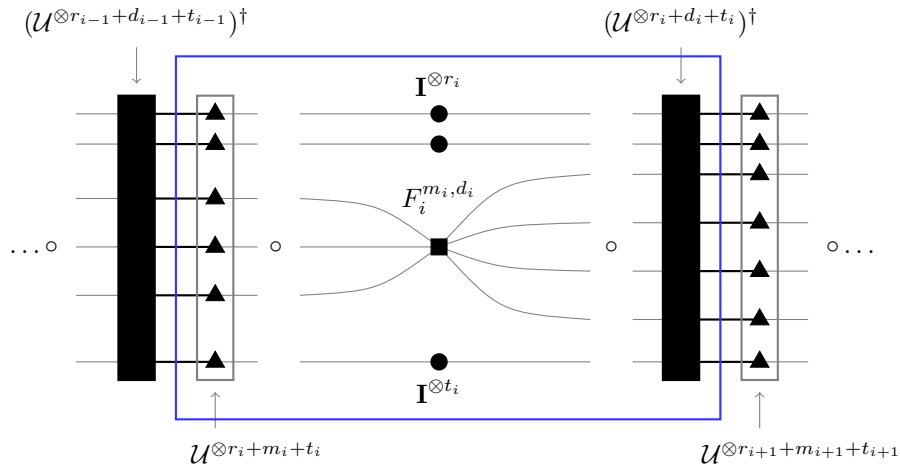
When Ω_K is planar, however, this can be rescued by using the decomposition procedure for Ω_K in the proof of Theorem 17, which produces a sequence of gadgets whose signature matrices multiply to the Holant value. This defines an order of Ω_K ’s vertices, giving a globally consistent way in which \mathcal{U} ’s entries are multiplied, for the partition function (sum of product expression). We will use \mathcal{U} as a “quantum holographic transformation” by inserting $\mathcal{U}^{\otimes k}$ and its inverse $(\mathcal{U}^{\otimes k})^\dagger$ between every pair of these gadgets, converting every $F \in \mathcal{F}$ to the corresponding $G \in \mathcal{G}$ and preserving \mathcal{EQ} .

► **Theorem 18** (Quantum Holant Theorem). *Let \mathcal{U} be a $q \times q$ quantum permutation matrix, and let \mathcal{F} and \mathcal{UF} be compatible sets of domain- q real-valued signatures. Then for every Pl-Holant(\mathcal{F}) signature grid Ω ,*

$$\text{Pl-Holant}_{\Omega}(\mathcal{F}) = \text{Pl-Holant}_{\Omega'}(\mathcal{UF}),$$

where Ω' is constructed from Ω by replacing every signature in \mathcal{F} with the corresponding signature in \mathcal{UF} .

We omit its proof in this extended abstract. The main idea is to perform successive quantum holographic transformations using \mathcal{U} ; the “pushing through” a set of \mathcal{U} ’s as tensor powers is illustrated in Figure 7. Since the central gadget may be $(\mathbf{F}_i^{(r)})^{m_i, d_i}$ or $(\mathbf{F}_i^{\dagger})^{m_i, d_i}$ (depending on the orientation of F in the signature grid) rather than simply $\mathbf{F}^{m_i+d_i, 0}$, the identities (2) and (3) illustrated in Figure 6 are necessary.



■ **Figure 7** A visualization of a step in the proof of the quantum Holant theorem. We insert factors of the form $I = (\mathcal{U}^{\otimes k})^{\dagger} \mathcal{U}^{\otimes k}$ between every pair of gadgets $\mathbf{I}^{\otimes r_i} \otimes \mathbf{F}_i^{m_i, d_i} \otimes \mathbf{I}^{\otimes t_i}$, then apply gadget composition associativity. The resulting gadget in the blue box is $\mathbf{I}^{\otimes r_i} \otimes \mathbf{G}_i^{m_i, d_i} \otimes \mathbf{I}^{\otimes t_i}$.

Observe that, unlike other results in this work, the quantum Holant theorem is not only applicable to $\text{Pl-Holant}(\mathcal{F} \mid \mathcal{EQ})$, but more generally to $\text{Pl-Holant}(\mathcal{F})$ for *any* signature set \mathcal{F} . Since holographic transformations are an indispensable tool for the study of Holant problems, and planar Holant is the subject of much active research, the quantum Holant theorem should be of independent interest.

► **Corollary 19.** *Let \mathcal{F} and \mathcal{G} be compatible sets of constraint functions. If $\mathcal{F} \cong_{qc} \mathcal{G}$, then $Z(K) = Z(K_{\mathcal{F} \rightarrow \mathcal{G}})$ for every planar #CSP(\mathcal{F}) instance K .*

4 Planar #CSP Equivalence Implies Quantum Isomorphism

The Quantum Automorphism Group

The most abstract construction in this work is the quantum automorphism group $\text{Qut}(\mathcal{F})$ of a constraint function set \mathcal{F} . We omit the full formal definition here, as understanding it is not relevant to the rest of the work.

► **Definition 20** ($\text{Qut}(\mathcal{F})$). For a set \mathcal{F} of constraint functions with $|V(\mathcal{F})| = q$, the quantum automorphism group $\text{Qut}(\mathcal{F})$ of \mathcal{F} is defined by the universal² C^* -algebra $C(\text{Qut}(\mathcal{F}))$ generated by the entries of a $q \times q$ matrix $U = (u_{ij})$ subject to the relations specifying that U is a quantum permutation matrix and $U^{\otimes \text{arity}(F)} f = f$ for every $F \in \mathcal{F}$.

Observe that, just as $U^{\otimes \text{arity}(F)} f = g$ defines a quantum isomorphism between F and G , $U^{\otimes \text{arity}(F)} f = f$ defines quantum automorphisms of F . It is helpful, and sufficient for our purposes, to identify $\text{Qut}(\mathcal{F})$ with U and $C(\text{Qut}(\mathcal{F}))$, thought of as the algebra of continuous functionals on $\text{Qut}(\mathcal{F})$. Indeed, if U 's entries commute, then $C(\text{Qut}(\mathcal{F}))$ concretizes as the algebra of continuous functionals on the classical automorphism group $\text{Aut}(\mathcal{F})$. Thus the relationship between $\text{Qut}(\mathcal{F})$ and $\text{Aut}(\mathcal{F})$ is analogous to the relationship between quantum and classical permutation matrices: the former is an abstraction of the latter, sharing many familiar properties and constructions.

One such construction is the *orbits* of $\text{Qut}(\mathcal{F})$ on the domain $V(\mathcal{F})$ [10]. If U is the quantum permutation matrix defining $\text{Qut}(\mathcal{F})$, then $x, y \in V(\mathcal{F})$ are in the same orbit of $\text{Qut}(\mathcal{F})$ if and only if $u_{xy} \neq 0$ (one can draw an analogy with the orbits of $\text{Aut}(\mathcal{F})$ on $V(\mathcal{F})$ by viewing u_{xy} as corresponding to the automorphisms mapping x to y).

Another such construction is the *intertwiner space* of $\text{Qut}(\mathcal{F})$.

► **Definition 21** ($C_{\text{Qut}(\mathcal{F})}(m, d)$, $C_{\text{Qut}(\mathcal{F})}$). $C_{\text{Qut}(\mathcal{F})}(m, d) = \{M \in \mathbb{C}^{q^m \times q^d} \mid U^{\otimes m} M = M U^{\otimes d}\}$ is the space of (m, d) -intertwiners of $\text{Qut}(\mathcal{F})$, and $C_{\text{Qut}(\mathcal{F})} = \bigcup_{m,d} C_{\text{Qut}(\mathcal{F})}(m, d)$.

Observe that the equation $U^{\otimes m} M = M U^{\otimes d}$ resembles, after multiplying by $(U^{\otimes d})^{-1} = (U^{\otimes d})^\dagger$, the equation $U^{\otimes m} F^{m,d} (U^{\otimes d})^\dagger = G^{m,d}$ characterizing quantum isomorphism of F and G (recall (3)). Hence it is reasonable to assume that $C_{\text{Qut}(\mathcal{F})}$ consists of gadget signature matrices. The next lemma, proved in the full version using techniques from quantum group theory, is a step towards this conclusion.

► **Lemma 22.** $C_{\text{Qut}(\mathcal{F})} = \langle E^{1,0}, E^{1,2}, \{f \mid F \in \mathcal{F}\} \rangle_{+, \circ, \otimes, \dagger}$.

Note that the RHS of Lemma 22 is the linear span of the signature matrices of the gadgets in the RHS of Theorem 17. This observation motivates the following definition.

► **Definition 23** ($\mathfrak{Q}_{\mathcal{F}}^{\mathcal{P}}(m, d)$). A planar (m, d) -quantum \mathcal{F} -gadget is a finite linear combination of gadgets in $\mathcal{P}_{\mathcal{F}}(m, d)$ with coefficients in \mathbb{C} . Let $\mathfrak{Q}_{\mathcal{F}}^{\mathcal{P}}(m, d)$ be the collection of all planar (m, d) -quantum \mathcal{F} -gadgets. Extend the signature matrix function M linearly to $\mathfrak{Q}_{\mathcal{F}}^{\mathcal{P}}(m, d)$.

Applying M to quantum \mathcal{F} -gadgets composed of gadgets on the RHS of Theorem 17 yields the RHS of Lemma 22, so we have the following key connection between the planar gadget decomposition and the intertwiners of $\text{Qut}(\mathcal{F})$.

► **Theorem 24.** $C_{\text{Qut}(\mathcal{F})}(m, d) = \{M(\mathbf{Q}) \mid \mathbf{Q} \in \mathfrak{Q}_{\mathcal{F}}^{\mathcal{P}}(m, d)\}$ for every $m, d \in \mathbb{N}$.

Say a #CSP instance $K = (V, C)$ is 1-labeled if there is a distinguished labeled variable $v \in V$. For $x \in V(\mathcal{F})$, let Z^x be the partition function on 1-labeled instances, defined identically to Z , except only summing over those $\sigma : V \rightarrow V(\mathcal{F})$ such that $\sigma(v) = x$. A 1-labeled #CSP(\mathcal{F}) instance K with labeled variable v corresponds to a $\text{Holant}(\mathcal{F} \mid \mathcal{EQ})$ gadget \mathbf{K} with a single dangling edge incident to the equality vertex constructed from v . For $x \in V(\mathcal{F})$, $M(\mathbf{K})_x = Z^x(K)$. Then the connection between intertwiners and gadget signature matrices established in Theorem 24 yields the following lemma, a quantum analogue of several similar classical results for graph homomorphism, including [9, Lemma 2.4].

► **Lemma 25.** $x, y \in V(\mathcal{F})$ are in the same orbit of $\text{Qut}(\mathcal{F})$ if and only if $Z^x(K) = Z^y(K)$ for all 1-labeled planar #CSP(\mathcal{F}) instances K .

² A “universal” C^* -algebra is roughly analogous to the generator-and-relation presentation of a group.

Arity Reduction and Projective Connectivity

For n -ary constraint functions, the remaining results require n -dimensional generalizations of orbits. However, such higher-dimensional orbits are not known to exist for $n > 2$. This is another new difficulty posed by our extension of binary graph homomorphism to higher-arity functions. We overcome it by the following arity-reduction technique.

► **Lemma 26.** *Let F be an n -ary constraint function with $n > 2$ and let \mathcal{U} define $\text{Qut}(\{F\})$, so $\mathcal{U}^{\otimes n} f = f$. Define an arity- $(n-1)$ constraint function F' by $F'_{x_2, \dots, x_n} = \sum_{x_1} F_{x_1, x_2, \dots, x_n}$. Then $\mathcal{U}^{\otimes n-1} f' = f'$ (where f' is the vectorization of F').*

After $n-2$ applications of Lemma 26, the resulting binary constraint function is the adjacency matrix of a \mathbb{R} -weighted graph. We would like this \mathbb{R} -weighted graph to be *connected*, meaning the transitive closure of the relation \sim on $V(X)$ defined by $x \sim y \iff X_{xy} \neq 0$ has only a single equivalence class. We define connectivity for higher-arity tensors motivated by Lemma 26. For $n \geq 2$, an n -ary constraint function F is *projectively connected* if the \mathbb{R} -weighted graph X defined by $X_{uv} = \sum_{x_1, \dots, x_{n-2}} F_{x_1, \dots, x_{n-2}, u, v}$ is connected.

► **Definition 27** (\oplus). *Let $F \in \mathbb{R}^{V(F)^n}$, $G \in \mathbb{R}^{V(G)^n}$ be constraint functions of the same arity n . The direct sum $F \oplus G \in \mathbb{R}^{(V(F) \sqcup V(G))^n}$ of F and G is defined for all $\mathbf{x} \in (V(F) \sqcup V(G))^n$ by setting $(F \oplus G)_{\mathbf{x}}$ to be $F_{\mathbf{x}}$ or $G_{\mathbf{x}}$ if $\mathbf{x} \in V(F)^n$ or $\mathbf{x} \in V(G)^n$, respectively, and 0 otherwise. For constraint function sets \mathcal{F} and \mathcal{G} of size s , define $\mathcal{F} \oplus \mathcal{G} = \{F_i \oplus G_i \mid i \in [s]\}$.*

For $n = 2$ and 0-1 valued F and G , the direct sum is the disjoint union of the graphs whose adjacency matrices are F and G . Projective connectivity is desirable due to the following lemma, an extension of [10, Theorem 4.5] to higher arity, and an analogue of the fact that, for connected graphs X and Y , if there exist vertices $x \in V(X)$ and $y \in V(Y)$ in the same orbit of the automorphism group of the disjoint union of X and Y , then $X \cong Y$.

► **Lemma 28.** *Let \mathcal{F} and \mathcal{G} be constraint function sets with domains $V(\mathcal{F})$ and $V(\mathcal{G})$, respectively, and further assume that \mathcal{F} and \mathcal{G} contain a pair of corresponding projectively connected constraint functions. If there are some $\hat{x} \in V(\mathcal{F})$, $\hat{y} \in V(\mathcal{G})$ in the same orbit of $\text{Qut}(\mathcal{F} \oplus \mathcal{G})$, then $\mathcal{F} \cong_{qc} \mathcal{G}$.*

The proof of Lemma 28 proceeds roughly as follows. Let \mathcal{U} define $\text{Qut}(\mathcal{F} \oplus \mathcal{G})$, and let F and G be the corresponding projectively connected constraint functions in \mathcal{F} and \mathcal{G} , respectively. Summing out all but two indices of $F \oplus G$ (as in Lemma 26), we obtain a \mathbb{R} -weighted graph Z whose subgraphs induced by $V(F)$ and $V(G)$ are connected, and such that $\mathcal{U}^{\otimes 2} z = z$. Then, following the proof of [10, Theorem 4.5], we extract from Z enough information about \mathcal{U} to show that its quarter submatrix $(u_{xy})_{x \in V(X), y \in V(Y)}$ is itself a quantum permutation matrix, defining a quantum isomorphism between \mathcal{F} and \mathcal{G} .

Finally, we come to the proof of the reverse implication of Theorem 9. Lemma 28 assumes \mathcal{F} contains a projectively constraint function, which in general is not true. For (unweighted) graphs, one can take the complement to assume the graphs are connected, but this trick does not apply to our \mathbb{R} -weighted constraint functions. Instead, we adapt an idea from Lovász's proof of [9, Corollary 2.6], which is roughly the *classical* case of Theorem 9 restricted to positive-real-weighted graphs. The idea is to add a new vertex to each graph, each adjacent to all other vertices by edges of the same nonzero weight. The new vertices are the targets of a result analogous to Lemma 25, each graph is now connected, and by symmetry of the new vertices, their addition preserves isomorphism. Somewhat remarkably, the same idea applies to quantum isomorphism of higher-arity constraint functions.

► **Lemma 29.** *Let \mathcal{F} and \mathcal{G} be compatible sets of constraint functions. If $Z(K) = Z(K_{\mathcal{F} \rightarrow \mathcal{G}})$ for every planar $\#CSP(\mathcal{F})$ instance K , then $\mathcal{F} \cong_{qc} \mathcal{G}$.*

We now present a proof sketch of Lemma 29. See the full version for a full proof. Let 0_F and 0_G be new domain elements. For each $F \in \mathcal{F}$, $G \in \mathcal{G}$, define constraint functions F' and G' on $V(\mathcal{F}) \sqcup \{0_F\}$ and $V(\mathcal{G}) \sqcup \{0_G\}$, by letting

$$F'_x = \begin{cases} F_x & \mathbf{x} \in V(\mathcal{F})^n \\ \gamma & \mathbf{x} = (0_F, 0_F, \dots, 0_F, c), c \neq 0_F \\ 0 & \text{otherwise} \end{cases}; \quad G'_x = \begin{cases} G_x & \mathbf{x} \in V(\mathcal{G})^n \\ \gamma & \mathbf{x} = (0_G, 0_G, \dots, 0_G, c), c \neq 0_G \\ 0 & \text{otherwise} \end{cases}$$

for some fixed $\gamma \in \mathbb{R} \setminus \{0\}$. Let $\mathcal{F}' = \{F' \mid F \in \mathcal{F}\}$, $\mathcal{G}' = \{G' \mid G \in \mathcal{G}\}$, with $V(\mathcal{F}') = V(\mathcal{F}) \sqcup \{0_F\}$ and $V(\mathcal{G}') = V(\mathcal{G}) \sqcup \{0_G\}$.

F' and G' are designed to simultaneously satisfy three properties. First, defining \mathbb{R} -weighted graphs X' and Y' from F' and G' by summing the first $n-2$ indices as in Lemma 26, we have $X'_{0_F v} = \gamma \neq 0$ for every $v \in V(X') \setminus \{0_F\}$, and similarly for Y' . Thus X' and Y' are connected, so F' and G' are projectively connected.

Second, we wish to obtain $Z^{0_F}(K) = Z^{0_G}(K)$ for every planar 1-labeled $\#CSP(\mathcal{F}' \oplus \mathcal{G}')$ instance $K = (V, C)$. Then Lemma 25 asserts that 0_F and 0_G are in the same orbit of $\text{Qut}(\mathcal{F}' \oplus \mathcal{G}')$. Hence, since F' and G' are projectively connected, $0_F \in V(\mathcal{F}')$, and $0_G \in V(\mathcal{G}')$, Lemma 28 gives $\mathcal{F}' \cong_{qc} \mathcal{G}'$. Let $K = (V, C)$. To obtain $Z^{0_F}(K) = Z^{0_G}(K)$, consider the following. Let v_0 be the labeled variable in V . When v_0 takes value $0_F \in V(\mathcal{F}')$, all variables must take values in $V(\mathcal{F}')$ (otherwise the assignment contributes 0 to the partition function, as $F' \oplus G'$ takes value 0 unless its inputs are all in $V(\mathcal{F}')$ or all in $V(\mathcal{G}')$). Furthermore, by construction of \mathcal{F}' , any constraint function applied to variable v_0 evaluates to 0 unless most of its other arguments also take value 0_F . This fixes more variables to 0_F , and the effect cascades to other constraint functions applied to those variables and so on. Any nonzero constraint with a variable fixed to 0_F always evaluates to γ . Let D be the set of such constraints. Remaining constraints in $C \setminus D$ apply some F' to inputs in only $V(\mathcal{F})$, so F' , by construction, acts as the original F . Therefore, the sub-instance of K induced by constraints $C \setminus D$ is effectively a planar $\#CSP(\mathcal{F})$ instance, so $Z^{0_F}(K)$ is expressible as $\gamma^{|D|}$ times the sum of $Z(K')$ for various planar $\#CSP(\mathcal{F})$ instances K' . Similarly, by the symmetry of \mathcal{F}' and \mathcal{G}' , $Z^{0_G}(K)$ is expressible as $\gamma^{|D|}$ times the sum of $Z(K'')$ for matching planar $\#CSP(\mathcal{G})$ instances K'' , and by assumption each $Z(K') = Z(K'')$. Hence $Z^{0_F}(K) = Z^{0_G}(K)$.

Third, upon obtaining the quantum isomorphism \mathcal{U} between \mathcal{F}' and \mathcal{G}' , we must recover a quantum isomorphism between the original \mathcal{F} and \mathcal{G} . Define the matrix $\widehat{\mathcal{U}} = (u_{uv})_{u \in V(\mathcal{G}), v \in V(\mathcal{F})}$ (in other words, we eliminate from \mathcal{U} the row and column corresponding to the new vertices 0_G and 0_F , respectively). \mathcal{F}' and \mathcal{G}' were constructed so that, if γ is chosen to be sufficiently large, then $\widehat{\mathcal{U}}$ is a quantum permutation matrix, and furthermore defines a quantum isomorphism between \mathcal{F} and \mathcal{G} .

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A Appendix: An alternate approach to connectivity

The proof of Lemma 28 makes use of the 2-dimensional orbits, or *orbitals*, of $\text{Qut}(\mathcal{F} \oplus \mathcal{G})$. This construction, as mentioned earlier, does not extend to dimensions higher than 2. This is why we, via projective connectivity, effectively require that \mathcal{F} and \mathcal{G} contain a *binary* connected constraint function in the hypothesis of Lemma 28. To satisfy this hypothesis, we ensure that the modified constraint functions F' and G' in the proof of Lemma 29 are projectively connected. In this appendix, we present a different construction, due to Roberson [12], which, rather than modify the existing constraint functions, adds new binary connected constraint functions to \mathcal{F} and \mathcal{G} , while preserving quantum isomorphism. This removes the need for projective connectivity entirely, simplifies the proof of Lemma 28, and could simplify the proof of Lemma 29, since it is no longer necessary that F' and G' be projectively connected (though we still need $Z^{0_{F'}}(K) = Z^{0_{G'}}(K)$ for all planar 1-labeled K). Additionally, the alternate construction makes use of two lemmas which should be of independent interest.

First, we extend Definition 3 to gadgets.

► **Definition 30** ($\mathbf{K}_{\mathcal{F} \rightarrow \mathcal{G}}$). For compatible constraint function sets \mathcal{F} and \mathcal{G} and $\mathbf{K} \in \mathcal{P}_{\mathcal{F}}$, let $\mathbf{K}_{\mathcal{F} \rightarrow \mathcal{G}} \in \mathcal{P}_{\mathcal{G}}$ be the corresponding gadget formed by replacing each constraint signature $F_i \in \mathcal{F}$ with the corresponding $G_i \in \mathcal{G}$. Extend this mapping linearly to $\mathcal{Q}_{\mathcal{F}}^{\mathcal{P}}$.

The first lemma shows that, viewing intertwiners themselves as constraint functions, we may add “corresponding” pairs of intertwiners (the signature matrices of corresponding quantum \mathcal{F} and \mathcal{G} -gadgets – recall Theorem 24) to \mathcal{F} and \mathcal{G} , while preserving equivalence on planar #CSP instances.

► **Lemma 31** ([12]). *Let \mathcal{F} and \mathcal{G} be compatible constraint function sets. Let $M_F \in C_{\text{Qut}(\mathcal{F})}(m, d)$ and $M_G \in C_{\text{Qut}(\mathcal{G})}(m, d)$ such that $M_F = M(\mathbf{Q})$ and $M_G = M(\mathbf{Q}_{\mathcal{F} \rightarrow \mathcal{G}})$ for some quantum \mathcal{F} -gadget $\mathbf{Q} \in \mathfrak{Q}_{\mathcal{F}}^{\mathcal{P}}(m, d)$. Let F and G be the constraint functions satisfying $F^{m, d} = M_F$ and $G^{m, d} = M_G$, and let $\mathcal{F}' = \mathcal{F} \cup \{F\}$ and $\mathcal{G}' = \mathcal{G} \cup \{G\}$. Then $Z(K) = Z(K_{\mathcal{F} \rightarrow \mathcal{G}})$ for all planar $\#CSP(\mathcal{F})$ instances K if and only if $Z(K) = Z(K_{\mathcal{F}' \rightarrow \mathcal{G}'})$ for all planar $\#CSP(\mathcal{F}')$ instances K .*

Proof. The backward direction is immediate. Let K be a planar $\#CSP(\mathcal{F}')$ instance, and let Ω_K be the corresponding Pl-Holant($\mathcal{F}' \mid \mathcal{EQ}$) instance. Create a “quantum signature grid” $\widehat{\Omega}_K \in \mathfrak{Q}_{\mathcal{F}}^{\mathcal{P}}(0, 0)$ by replacing every instance of a vertex v assigned F in Ω_K by the equivalent quantum gadget $\mathbf{Q} \in \mathfrak{Q}_{\mathcal{F}}^{\mathcal{P}}$, matching the cyclically-ordered dangling edges of each gadget to the cyclically-ordered incident edges of v (and contracting the edges between the adjacent equality vertices to preserve bipartiteness). Similarly create $\widehat{\Omega}_{K_{\mathcal{F}' \rightarrow \mathcal{G}'}} \in \mathfrak{Q}_{\mathcal{G}}^{\mathcal{P}}(0, 0)$ by replacing every instance of a vertex assigned G in $\Omega_{K_{\mathcal{F}' \rightarrow \mathcal{G}'}}$ with $\mathbf{Q}_{\mathcal{F} \rightarrow \mathcal{G}} \in \mathfrak{Q}_{\mathcal{G}}^{\mathcal{P}}$. Then the index- α summand of $\widehat{\Omega}_K$ or $\widehat{\Omega}_{K_{\mathcal{F}' \rightarrow \mathcal{G}'}}$ is a planar $\#CSP(\mathcal{F})$ instance $K_{\mathcal{F}}^{\alpha}$ or planar $\#CSP(\mathcal{G})$ instance $K_{\mathcal{G}}^{\alpha}$, respectively, and furthermore $K_{\mathcal{G}}^{\alpha} = (K_{\mathcal{F}}^{\alpha})_{\mathcal{F} \rightarrow \mathcal{G}}$. Thus

$$\begin{aligned} Z(K) &= \text{Pl-Holant}_{\Omega_K}(\mathcal{F}' \mid \mathcal{EQ}) \\ &= \text{Pl-Holant}_{\widehat{\Omega}_K}(\mathcal{F} \mid \mathcal{EQ}) \\ &= \text{Pl-Holant}_{\widehat{\Omega}_{K_{\mathcal{F}' \rightarrow \mathcal{G}'}}}(\mathcal{G} \mid \mathcal{EQ}) \\ &= \text{Pl-Holant}_{\Omega_{K_{\mathcal{F}' \rightarrow \mathcal{G}'}}}(\mathcal{G}' \mid \mathcal{EQ}) \\ &= Z(K_{\mathcal{F}' \rightarrow \mathcal{G}'}). \end{aligned} \quad \blacktriangleleft$$

An alternate proof would also need the following lemma, which is equivalent to Lemma 31 once Theorem 9 is proved.

► **Lemma 32** ([12]). *Suppose \mathcal{F} , \mathcal{G} , F , G , \mathcal{F}' , and \mathcal{G}' satisfy the hypotheses of Lemma 31. Then $\mathcal{F} \cong_{qc} \mathcal{G} \iff \mathcal{F}' \cong_{qc} \mathcal{G}'$.*

Proof. The backward direction is immediate. Let \mathcal{U} be the quantum permutation matrix defining the quantum isomorphism between \mathcal{F} and \mathcal{G} . It suffices to show $\mathcal{U}^{\otimes m+d} f = g$, or equivalently, by (3), $\mathcal{U}^{\otimes m} M_F (\mathcal{U}^{\otimes d})^{\dagger} = M_G$. This identity follows from the proof of the quantum Holant theorem. Indeed, while the statement in Theorem 18 only applies to signature grids (gadgets in $\mathfrak{Q}_{\mathcal{F}}^{\mathcal{P}}(0, 0)$), the proof may be easily modified to apply to $\mathbf{Q} \in \mathfrak{Q}_{\mathcal{F}}^{\mathcal{P}}(m, d)$ as follows. After inserting $(\mathcal{U}^{\otimes k})^{\dagger} \mathcal{U}^{\otimes k}$ between each pair of gadgets in the Theorem 17 decomposition of (each summand of) \mathbf{Q} and reassociating to convert every \mathcal{F} signature to the corresponding \mathcal{G} signature and fix the internal equality vertices, we must effectively apply an additional \mathcal{U} or \mathcal{U}^{\dagger} to each original dangling edge of \mathbf{Q} to fully transform each equality vertex with a dangling edge back to itself via $\mathcal{U}^{\otimes a} E^{a, b} (\mathcal{U}^{\otimes b})^{\dagger} = E^{a, b}$. Thus $\mathcal{U}^{\otimes m} \mathbf{Q} (\mathcal{U}^{\otimes d})^{\dagger} = \mathbf{Q}_{\mathcal{F} \rightarrow \mathcal{G}}$, and applying M to both sides gives the result. \blacktriangleleft

Together, Lemmas 31 and 32 show that, to prove Theorem 9, we may assume that \mathcal{F} and \mathcal{G} contain (the constraint functions created from) any intertwiners M_F and M_G which are the signature matrices of corresponding quantum \mathcal{F} and \mathcal{G} -gadgets. In particular, we trivially have $(\mathbf{E}^{1,0} \circ \mathbf{E}^{0,1})_{\mathcal{F} \rightarrow \mathcal{G}} = (\mathbf{E}^{1,0} \circ \mathbf{E}^{0,1})$, so we may assume \mathcal{F} and \mathcal{G} both contain $M(\mathbf{E}^{1,0} \circ \mathbf{E}^{0,1}) = J$, the all-1s matrix. J is a connected constraint function, so we may immediately apply Lemma 28. Moreover, since J is already binary, we don't have to worry about reducing arity in the proof of Lemma 28.