A Hyperbolic Extension of Kadison-Singer Type Results

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Abstract

In 2013, Marcus, Spielman, and Srivastava resolved the famous Kadison-Singer conjecture. It states that for n independent random vectors $v_1, \cdots, v_n$ that have expected squared norm bounded by ϵ and are in the isotropic position in expectation, there is a positive probability that the determinant polynomial $\det(xI - \sum_{i=1}^n v_i v_i^\top)$ has roots bounded by $(1+\sqrt{\epsilon})^2$. An interpretation of the Kadison-Singer theorem is that we can always find a partition of the vectors $v_1, \cdots, v_n$ into two sets with a low discrepancy in terms of the spectral norm (in other words, rely on the determinant polynomial).

In this paper, we provide two results for a broader class of polynomials, the hyperbolic polynomials. Furthermore, our results are in two generalized settings:

- The first one shows that the Kadison-Singer result requires a weaker assumption that the vectors have a bounded sum of hyperbolic norms.
- The second one relaxes the Kadison-Singer result's distribution assumption to the Strongly Rayleigh distribution.

To the best of our knowledge, the previous results only support determinant polynomials [Anari and Oveis Gharan'14, Kyng, Luh and Song'20]. It is unclear whether they can be generalized to a broader class of polynomials. In addition, we also provide a sub-exponential time algorithm for constructing our results.

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1 Introduction

Introduced by [30], the Kadison-Singer problem was a long-standing open problem in mathematics. It was resolved by Marcus, Spielman, and Srivastrava in their seminal work [43]: For any set of independent random vectors $u_1, \cdots, u_n$ such that each u_i has finite support, and $u_1, \cdots, u_n$ are in isotropic positions in expectation, there is positive probability that $\sum_{i=1}^n u_i u_i^*$ has spectral norm bounded by $1 + O(\max_{i \in [n]} \|u_i\|)$. The main result of [43] is as follows:

▶ **Theorem 1** (Main result of [43]). *Let $\epsilon > 0$ and let $v_1, \cdots, v_n \in \mathbb{C}^m$ be n independent random vectors with finite support, such that $\mathbb{E}[\sum_{i=1}^n v_i v_i^*] = I$, and $\mathbb{E}[\|v_i\|^2] \le \epsilon$, $\forall i \in [n]$. Then*

$$\Pr\left[\Big\| \sum_{i\in[n]} v_i v_i^* \Big\| \le (1+\sqrt{\epsilon})^2\right] > 0.$$

The Kadison-Singer problem is closely related to discrepancy theory, which is an essential area in mathematics and theoretical computer science. A classical discrepancy problem is as follows: given n sets over n elements, can we color each element in red or blue such that each set has roughly the same number of elements in each color? More formally, for vectors $a_1, \ldots, a_n \in \mathbb{R}^n$ with $\|a_i\|_\infty \le 1$ and a coloring $s \in \{\pm 1\}^n$, the discrepancy is defined by $\mathrm{Disc}(a_1, \ldots, a_n; s) := \|\sum_{i\in[n]} s_i a_i\|_\infty$. The famous Spencer's Six Standard Deviations Suffice Theorem [57] shows that there exists a coloring with discrepancy at most $6\sqrt{n}$, which beats the standard Chernoff bound showing that a random coloring has discrepancy $\sqrt{n\log n}$. More generally, we can consider the "matrix version" of discrepancy: for matrices $A_1, \ldots, A_n \in \mathbb{R}^{d\times d}$ and a coloring $s \in \{\pm 1\}^n$,

$$\mathrm{Disc}(A_1, \ldots, A_n; s) := \Big\| \sum_{i\in[n]} s_i A_i \Big\|.$$

Theorem 1 is equivalent to the following discrepancy result for rank-1 matrices:

▶ **Theorem 2** ([43]). *Let $u_1, \ldots, u_n \in \mathbb{C}^m$ and suppose $\max_{i\in[n]} \|u_i u_i^*\| \le \epsilon$ and $\sum_{i=1}^n u_i u_i^* = I$. Then,*

$$\min_{s\in\{\pm 1\}^n} \mathrm{Disc}(u_1 u_1^*, \ldots, u_n u_n^*; s) \le O(\sqrt{\epsilon}).$$

In other words, the minimum discrepancy of rank-1 isotropic matrices is bounded by $O(\sqrt{\epsilon})$, where ϵ is the maximum spectral norm. This result also beats the matrix Chernoff bound [60], which shows that a random coloring for matrices has discrepancy $O(\sqrt{\epsilon \log d})$. The main techniques in [43] are the method of interlacing polynomials and the barrier methods developed in [42].

Several generalizations of the Kadison-Singer-type results, which have interesting applications in theoretical computer science, have been established using the same technical framework as described in [43]. In particular, Kyng, Luh, and Song [36] provided a "four derivations suffice" version of Kadison-Singer conjecture: Instead of assuming every independent random vector has a bounded norm, the main result in [36] only requires that the sum of the squared spectral norm is bounded by σ^2, and showed a discrepancy bound of 4σ:

▶ **Theorem 3** ([36]). *Let $u_1, \ldots, u_n \in \mathbb{C}^m$ and $\sigma^2 = \|\sum_{i=1}^n (u_i u_i^*)^2\|$. Then, we have*

$$\Pr_{\xi\sim\{\pm 1\}^n}\left[\Big\| \sum_{i=1}^n \xi_i u_i u_i^* \Big\| \le 4\sigma\right] > 0.$$

This result was recently applied by [38] to approximate solutions of generalized network design problems.

Moreover, Anari and Oveis-Gharan [6] generalized the Kadison-Singer conjecture into the setting of real-stable polynomials. Instead of assuming the random vectors are independent, [6] assumes that the vectors are sampled from any homogeneous strongly Rayleigh distribution with bounded marginal probability, have bounded norm, and are in an isotropic position:

▶ **Theorem 4** ([6]). *Let μ be a homogeneous strongly Rayleigh probability distribution on $[n]$ such that the marginal probability of each element is at most ϵ_1, and let $u_1, \cdots, u_n \in \mathbb{R}^m$ be vectors in an isotropic position, $\sum_{i=1}^n u_i u_i^* = I$, such that $\max_{i\in[n]} \|u_i\|^2 \le \epsilon_2$. Then*

$$\Pr_{S\sim\mu}\left[\left\|\sum_{i\in S} u_i u_i^*\right\| \le 4(\epsilon_1+\epsilon_2)+2(\epsilon_1+\epsilon_2)^2\right] > 0.$$

Theorem 4 has a direct analog in spectral graph theory: Given any (weighted) connected graph $G = (V, E)$ with Laplacian L_G. For any edge $e = (u, v) \in E$, define the vector corresponding to e as $v_e = L_G^{\dagger/2}(\mathbf{1}_u - \mathbf{1}_v)$ (here $L_G^\dagger$ is the Moore-Penrose inverse). Then the set of $\{v_e : e \in E\}$ are in isotropic position, and $\|v_e\|^2$ equals to the graph effective resistance with respect to e. Also, any spanning tree distribution of the edges in E is homogeneous strongly Rayleigh. It follows from Theorem 4 that any graph with bounded maximum effective resistance has a spectrally-thin spanning tree [6]. Moreover, [7] provided an exciting application to the asymmetric traveling salesman problem and obtained an $O(\log\log n)$-approximation.

Another perspective of generalizing the Kadison-Singer theorem is to study the discrepancy with respect to a more general norm than the spectral norm, which is the largest root of a determinant polynomial. A recent work by Bränden [19] proved a high-rank version of Theorem 2 for *hyperbolic* polynomial, which is a larger class of polynomials including the determinant polynomial. Moreover, the hyperbolic norm on vectors is a natural generalization of the matrix spectral norm. We will introduce hyperbolic polynomials in the full version of our paper. From this perspective, it is very natural to ask:

Can we also extend Theorem 3 and Theorem 4 to a more general class of polynomials, e.g., hyperbolic polynomials?

1.1 Our results

In this work, we provide an affirmative answer by generalizing both Theorem 3 and Theorem 4 into the setting of hyperbolic polynomials. Before stating our main results, we first introduce some basic notation of hyperbolic polynomials below.

Hyperbolic polynomials form a broader class of polynomials that encompasses determinant polynomials and homogeneous real-stable polynomials. An m-variate, degree-d homogeneous polynomial $h \in \mathbb{R}[x_1, \cdots, x_m]$ is *hyperbolic* with respect to a direction $e \in \mathbb{R}^m$ if the univariate polynomial $t \mapsto h(te - x)$ has only real roots for all $x \in \mathbb{R}^m$. The set of $x \in \mathbb{R}^m$ such that all roots of $h(te - x)$ are non-negative (or strictly positive) is referred to as the hyperbolicity cone $\Gamma_+^h(e)$ (or $\Gamma_{++}^h(e)$). It is a widely recognized result [16] that any vector x in the open hyperbolicity cone $\Gamma_{++}^h(e)$ is itself hyperbolic with respect to the polynomial h and have the same hyperbolicity cone as e, meaning that $\Gamma_{++}^h(e) = \Gamma_{++}^h(x)$. Therefore, the unique hyperbolicity cone of h can simply be expressed as Γ_+^h.

The hyperbolic polynomials have similarities to determinant polynomials of matrices, as they both can be used to define trace, norm, and eigenvalues. Given a hyperbolic polynomial $h \in \mathbb{R}[x_1, \cdots, x_m]$ and any vector $e \in \Gamma_{++}^h$, we can define a norm with respect to $h(x)$ and e as follows: for any $x \in \mathbb{R}^m$, its *hyperbolic norm* $\|x\|_h$ is equal to the largest root (in absolute value) of the linear restriction polynomial $h(te - x) \in \mathbb{R}[t]$. Similar to the eigenvalues of matrices, we define the *hyperbolic eigenvalues* of x to be the d roots of $h(te - x)$, denoted by $\lambda_1(x) \ge \cdots \ge \lambda_d(x)$. We can also define the *hyperbolic trace* and the *hyperbolic rank*:

$$\mathrm{tr}_h[x] := \sum_{i=1}^d \lambda_i(x), \quad \text{and} \quad \mathrm{rank}(x)_h := |\{i \in [d] : \lambda_i(x) \ne 0\}|.$$

Recall that both Theorem 3 and Theorem 4 upper-bound the spectral norm of the sum $\| \sum_{i=1}^n \xi_i v_i v_i^\top \|$. In the setting of hyperbolic polynomials, we should upper bound the hyperbolic norm $\| \sum_{i=1}^n \xi_i v_i \|_h$ for vectors $v_1, \ldots, v_n$ in the hyperbolicity cone, which is the set of vectors with all non-negative hyperbolic eigenvalues.

Our main results are as follows:

▶ **Theorem 5** (Main Result I, informal hyperbolic version of Theorem 1.4, [36]). *Let $h \in \mathbb{R}[x_1, \ldots, x_m]$ denote a hyperbolic polynomial in direction $e \in \mathbb{R}^m$. Let $v_1, \ldots, v_n \in \Gamma_+^h$ be n vectors in the closed hyperbolicity cone. Let $\xi_1, \ldots, \xi_n$ be n independent random variables with finite supports and $\mathbb{E}[\xi_i] = \mu_i$ and $\mathbf{Var}[\xi_i] = \tau_i^2$. Suppose $\sigma := \| \sum_{i=1}^n \tau_i^2 \operatorname{tr}_h[v_i] v_i \|_h$. Then there exists an assignment $(s_1, \ldots, s_n)$ with s_i in the support of ξ_i for all $i \in [n]$, such that*

$$\Big\| \sum_{i=1}^n (s_i - \mu_i) v_i \Big\|_h \leq 4\sigma.$$

We remark that Theorem 5 does not require the isotropic position condition of $v_1, \cdots, v_n$ as in [19]. In addition, we only need the sum of $\operatorname{tr}_h[v_i]v_i$'s hyperbolic norm to be bounded, while [19]'s result requires each vector's trace to be bounded individually.

We would also like to note that the class of hyperbolic polynomials is much broader than that of determinant polynomials, which were used in the original Kadison-Singer-type theorems. Lax conjectured in [39] that every 3-variate hyperbolic/real-stable polynomial could be represented as a determinant polynomial, this was later resolved in [28, 40]. However, the Lax conjecture is false when the number of variables exceeds 3, as demonstrated in [17, 20] with counterexamples of hyperbolic/real-stable polynomials $h(x)$ for which even $(h(x))^k$ cannot be represented by determinant polynomials for any $k > 0$.

Our second main result considers the setting where the random vectors are not independent, but instead, sampled from a strongly Rayleigh distribution. We say a distribution μ over the subsets of $[n]$ is *strongly Rayleigh* if its generating polynomial $g_\mu(z) := \sum_{S \subseteq [n]} \mu(S) z^S \in \mathbb{R}[z_1, \ldots, z_n]$ is a *real-stable polynomial*, which means $g_\mu(z)$ does not have any root in the upper-half of the complex plane, i.e., $g_\mu(z) \neq 0$ for any $z \in \mathbb{C}^n$ with $\Re(z) \succ 0$.

▶ **Theorem 6** (Main Result II, informal hyperbolic version of Theorem 1.2, [6]). *Let $h \in \mathbb{R}[x_1, \ldots, x_m]$ denote hyperbolic polynomial in direction $e \in \mathbb{R}^m$. Let μ be a homogeneous strongly Rayleigh probability distribution on $[n]$ such that the marginal probability of each element is at most ϵ_1.*

Suppose $v_1, \cdots, v_n \in \Gamma_+^h$ are in the hyperbolicity cone of h such that $\sum_{i=1}^n v_i = e$, and for all $i \in [n]$, $\|v_i\|_h \leq \epsilon_2$. Then there exists $S \subseteq [n]$ in the support of μ, such that

$$\Big\| \sum_{i \in S} v_i \Big\|_h \leq 4(\epsilon_1 + \epsilon_2) + 2(\epsilon_1 + \epsilon_2)^2.$$

It is worth mentioning that the previous paper [36, 6] focused on the determinant polynomial, leaving the question of whether their techniques could be extended to the hyperbolic/real-stable setting unresolved. In our paper, we address this gap by developing new techniques specifically tailored to hyperbolic polynomials.

In addition, we follow the results from [11] and give an algorithm that can find the approximate solutions of both Theorem 5 and Theorem 6 in time sub-exponential to m:

▶ **Proposition 7** (Sub-exponential algorithm for Theorem 5, informal). *Let $h \in \mathbb{R}[x_1, \ldots, x_m]$ denote a hyperbolic polynomial with direction $e \in \mathbb{R}^m$. Let $v_1, \ldots, v_n \in \Gamma_+^h$ be n vectors in the hyperbolicity cone Γ_+^h of h. Suppose $\sigma = \| \sum_{i=1}^n \operatorname{tr}_h[v_i] v_i \|_h$.*

Let $\mathcal{P}$ be the interlacing family used in the proof of Theorem 6. Then there exists an sub-exponential time algorithm KadisonSinger$(\delta, \mathcal{P})$*, such that for any $\delta > 0$, it returns a sign assignment $(s_1, \cdots, s_n) \in \{\pm 1\}^n$ satisfying*

$$\Big\| \sum_{i=1}^{n} s_i u_i \Big\|_h \leq 4(1+\delta)\sigma.$$

▶ **Proposition 8** (Sub-Exponential algorithm for Theorem 6, informal)**.** *Let $h \in \mathbb{R}[x_1, \ldots, x_m]$ denote a hyperbolic polynomial in direction $e \in \mathbb{R}^m$. Let μ be a homogeneous strongly Rayleigh probability distribution on $[n]$ such that the marginal probability of each element is at most ϵ_1, and let $v_1, \cdots, v_n \in \Gamma_+^h$ be n vectors such that $\sum_{i=1}^n v_i = e$, and for all $i \in [n]$, $\|v_i\|_h \leq \epsilon_2$.*

Let $\mathcal{Q}$ be the interlacing family used in the proof of Theorem 6. Then there exists an sub-exponential time algorithm KadisonSinger$(\delta, \mathcal{Q})$*, such that for any $\delta > 0$, it returns a set S in the support of μ satisfying*

$$\Big\| \sum_{i \in S} u_i \Big\|_h \leq (1+\delta) \cdot \left(4(\epsilon_1 + \epsilon_2) + 2(\epsilon_1 + \epsilon_2)^2\right).$$

2 Related work

Real-Stable Polynomials

Real-stability is an important property for multivariate polynomials. In [13], the authors used the real-stability to give a unified framework for Lee-Yang type problems in statistical mechanics and combinatorics. Real-stable polynomials are also related to the permanent. Gurvits [25] proved the Van der Waerden conjecture, which conjectures that the permanent of n-by-n doubly stochastic matrices are lower-bounded by $n!/n^n$, via the capacity of real-stable polynomials. Recently, [26] improved the capacity lower bound for real-stable polynomials, which has applications in matrix scaling and metric TSP. In addition, real-stable polynomials are an important tool in solving many counting and sampling problems [46, 9, 8, 58, 10, 5, 12, 3, 4].

Hyperbolic Polynomials

Hyperbolic polynomial was originally defined to study the stability of partial differential equations [23, 29, 34]. In theoretical computer science, Güler [24] first introduced hyperbolic polynomial for optimization (hyperbolic programming), which is a generalization of LP and SDP. Later, a few algorithms [50, 44, 53, 51, 45, 52] were designed for hyperbolic programming. On the other hand, a significant effort has been put into the equivalence between hyperbolic programming and SDP, which is closely related to the “Generalized Lax Conjecture” (which conjectures that every hyperbolicity cone is spectrahedral) and its variants [28, 40, 18, 35, 54, 2, 48].

Strongly Rayleigh Distribution

The strongly Rayleigh distribution was introduced by [14]. The authors also proved numerous basic properties of strongly Rayleigh distributions, including negative association, and closure property under operations such as conditioning, product, and restriction to a subset. [47] proved a concentration result for Lipschitz functions of strongly Rayleigh variables. [37] showed a matrix concentration for strongly Rayleigh random variables, which implies that adding a small number of uniformly random spanning trees gives a graph spectral sparsifier.

Strongly Rayleigh distribution also has many algorithmic applications. [9] exploited the negative dependence property of homogeneous strongly Rayleigh distributions, and designed efficient algorithms for generating approximate samples from Determinantal Point Process using Monte Carlo Markov Chain. The strongly Rayleigh property of spanning tree distribution is a key component for improving the approximation ratios of TSP [31, 32] and k-edge connected graph problem [33].

Other generalizations of the Kadison-Singer-type results

The upper bound of the rank-one Kadison-Singer theorem was improved by [15, 49]. [1] further extended [49]'s result to prove a real-stable version of Anderson's paving conjecture. However, they used a different norm for real-stable polynomials, and hence their results and ours are incomparable. In the high-rank case, [21] also proved a Kadison-Singer result for high-rank matrices. [56] relaxed [19]'s result to the vectors in sub-isotropic position. In addition, they proved a hyperbolic Spencer theorem for constant-rank vectors.

Another direction of generalizing the Kadison-Singer-type result is to relax the $\{+1,-1\}$-coloring to $\{0,1\}$-coloring, which is called the one-sided version of Kadison-Singer problem in [61]. More specifically, given n isotropic vectors $v_1,\dots,v_n \in \mathbb{R}^m$ with norm $\frac{1}{\sqrt{N}}$, the goal is to find a subset $S \subset [n]$ of size k such that $\|\sum_{i\in S} v_i v_i^\top\| \leq \frac{k}{n} + O(1/\sqrt{N})$. Unlike the original Kadison-Singer problem, Weaver [61] showed that this problem can be solved in polynomial time. Very recently, Song, Xu and Zhang [55] improved the time complexity of the algorithm via an efficient inner product search data structure.

Applications of Kadison-Singer Problem

There are many interesting results developed from the Kadison-Singer theorem. In spectral graph theory, [27] exploited the same proof technique of interlacing families to show a sufficient condition of the spectrally thin tree conjecture. [6] used the strongly-Rayleigh extension of Kadison-Singer theorem to show a weaker sufficient condition. Based on this result, [7] showed that any k-edge-connected graph has an $O(\frac{\log\log(n)}{k})$-thin tree, and gave a $\mathrm{poly}(\log\log(n))$-integrality gap of the asymmetric TSP. [41, 22] used the Kadison-Singer theorem to construct bipartite Ramanujan graphs of all sizes and degrees. In the network design problem, [38] exploited the result in [36], and built a spectral rounding algorithm for the general network design convex program, which has applications in weighted experimental design, spectral network design, and additive spectral sparsifier.

3 Proof Overview

3.1 Hyperbolic Deviations

In this section, we will sketch the proof of our hyperbolic generalization of the Kadison-Singer theorem (Theorem 5). Details of the proof are deferred to the full version of the paper. We will use the same strategy as the original Kadison-Singer theorem (Theorem 1) in [42, 43], following three main technical steps.

For simplicity, we assume that the random variables $\xi_1,\dots,\xi_n \in \{\pm 1\}$ are independent Rademacher random variables, i.e., $\Pr[\xi_i = 1] = \frac{1}{2}$ and $\Pr[\xi_i = -1] = \frac{1}{2}$ for all $i \in [n]$.

To generalize the Kadison-Singer statement into the hyperbolic norm, one main obstacle is to define the variance of the hyperbolic norm of the sum of random vectors $\sum_{i=1}^n \xi_i v_i$. In the determinant polynomial case, each v_i corresponds to a rank-1 matrix $u_i u_i^*$, and it is easy to see that the variance of the spectral norm is $\| \sum_{i=1}^n (u_i u_i^*)^2 \|$. However, there is no analog of "matrix square" in the setting of hyperbolic/real-stable polynomials. Instead, we define the *hyperbolic variance*:

$$\left\| \sum_{i=1}^n \mathrm{tr}_h[v_i] v_i \right\|_h$$

in terms of the hyperbolic trace, and show that *four hyperbolic deviations suffice.*

Defining interlacing family of characteristic polynomials

In the first step, we construct a family of *characteristic polynomials* $\{p_s : s \in \{\pm 1\}^t, t \in \{0, \cdots, n\}\}$ as follows: For each $\mathbf{s} \in \{\pm 1\}^n$, define the leaf-node-polynomial:

$$p_{\mathbf{s}}(x) := \left(\prod_{i=1}^n p_{i,s_i} \right) \cdot h\left(xe + \sum_{i=1}^n s_i v_i \right) \cdot h\left(xe - \sum_{i=1}^n s_i v_i \right),$$

and for all $\ell \in \{0, \ldots, n-1\}$, $\mathbf{s}' \in \{\pm 1\}^\ell$, we construct an inner node with a polynomial that corresponds to the bit-string $\mathbf{s}'$:

$$p_{\mathbf{s}'}(x) := \sum_{\mathbf{t} \in \{\pm 1\}^{n-\ell}} p_{(\mathbf{s}', \mathbf{t})}(x).$$

where $(\mathbf{s}', \mathbf{t}) \in \{\pm 1\}^n$ is the bit-string concatenated by $\mathbf{s}'$ and $\mathbf{t}$.

We will then show that the above family of characteristic polynomials forms an *interlacing family.* By basic properties of interlacing family, we can always find a leaf-root-polynomial p_s (where $s \in \{\pm 1\}^n$) whose largest root is upper bounded by the largest root of the top-most polynomial.

$$p_\emptyset(x) = \mathop{\mathbb{E}}_{\xi_1, \cdots, \xi_n} \left[h\Big(xe + \sum_{i=1}^n \xi_i v_i \Big) \cdot h\Big(xe - \sum_{i=1}^n \xi_i v_i \Big) \right].$$

(we call $p_\emptyset$ to be the *mixed characteristic polynomial*). Notice that by rewriting the largest root of p_s to be the expected hyperbolic norm of $\sum_{i=1}^n s_i v_i$, we get that

$$\lambda_{\max}(p_\emptyset) = \left\| \sum_{i=1}^n s_i v_i \right\|_h. \tag{1}$$

Also, we will take $s \in \{\pm 1\}^n$ as the corresponding sign assignment in the main theorem (Theorem 5) It then suffices to upper-bound the largest root of the mixed characteristic polynomial.

From mixed characteristic polynomial to multivariate polynomial

In the second step, we will show that the mixed characteristic polynomial that takes the average on n random variables

$$p_\emptyset(x) = \mathop{\mathbb{E}}_{\xi_1, \ldots, \xi_n} \left[h\Big(xe + \sum_{i=1}^n \xi_i v_i \Big) \cdot h\Big(xe - \sum_{i=1}^n \xi_i v_i \Big) \right]$$

is equivalent to a polynomial with n extra variables $z_1, \cdots, z_n$:

$$\prod_{i=1}^n \Big(1 - \frac{1}{2} \frac{\partial^2}{\partial z_i^2} \Big) \bigg|_{z=0} \left(h\Big(xe + \sum_{i=1}^n z_i v_i \Big) \right)^2. \tag{2}$$

Thus, we can reduce the upper bound of $\chi_{\max}(p_\emptyset)$ to an upper bound of the largest root in (2). The latter turns out to be easier to estimate with the help of a barrier argument [43].

To show such equivalence holds, we use induction on the random variables $\xi_1, \ldots, \xi_n$. More specifically, we start from ξ_1 and are conditioned on any fixed choice of $\xi_2, \ldots, \xi_n$. We prove that taking expectation over ξ_1 is equivalent to applying the operator $(1 - \frac{\partial^2}{\partial z_1^2})$ to the polynomial

$$\left(h(xe + z_1 v_1 + \sum_{i=2}^{n} \xi_i v_i)\right)^2$$

and setting $z_1 = 0$. Here we use the relation between expectation and the second derivatives: for any Rademacher random variable ξ,

$$\mathop{\mathbb{E}}_{\xi}[h(x_1 - \xi v) \cdot h(x_2 + \xi v)] = \left(1 - \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right)\Big|_{t=0} h(x_1 + tv)h(x_2 + tv).$$

Repeating this process and removing one random variable at a time. After n iterations, we obtain the desired multivariate polynomial.

We also need to prove the real-rootedness of the multivariate polynomial (Eqn. (2)). We first consider an easy case where h itself is a real-stable polynomial, as in the determinant polynomial case. Then the real-rootedness easily follows from the closure properties of the real-stable polynomial. More specifically, we can show that $(h(xe + \sum_{i=1}^n z_i v_i))^2$ is also a real-stable polynomial. Furthermore, applying the operators $(1 - \frac{1}{2}\frac{\partial^2}{\partial z_i^2})$ and restricting $z = 0$ preserve the real-stability. Therefore, the multivariate polynomial is a univariate real-stable polynomial, which is equivalent to being real-rooted.

Next, we show that when h is a hyperbolic polynomial, the multivariate polynomial (Eqn. (2)) is also real-rooted. our approach is to show that the linear restriction of h: $h(xe + \sum_{i=1}^n z_i v_i)$ is a real-stable polynomial in $\mathbb{R}[x, z_1, \ldots, z_n]$. A well-known test for real-stability is that if for any $a \in \mathbb{R}_{>0}^{n+1}, b \in \mathbb{R}^{n+1}$, the one-dimensional restriction $p(at+b) \in \mathbb{R}[t]$ is non-zero and real-rooted, then $p(x)$ is real-stable. We test $h(xe + \sum_{i=1}^n z_i v_i)$ by restricting to $at + b$, and get the following polynomial:

$$h\Big((a_1 e + \sum_{i=1}^{n} a_{i+1} v_i)t + y\Big) \in \mathbb{R}[t],$$

where y is a fixed vector depending on b. Since $a_i > 0$ for all $i \in [n+1]$ and $e, v_1, \ldots, v_n$ are vectors in the hyperbolicity cone, it implies that the vector $a_1 e + \sum_{i=1}^n a_{i+1} v_i$ is also in the hyperbolicity cone. Then, by the definition of hyperbolic polynomial, we immediately see that $h((a_1 e + \sum_{i=1}^n a_{i+1} v_i)t + y)$ is real-rooted for any $a \in \mathbb{R}_{>0}^{n+1}$ and $b \in \mathbb{R}^{n+1}$. Hence, we can conclude that the restricted hyperbolic polynomial $h(xe + \sum_{i=1}^n z_i v_i)$ is real-stable and the remaining proof is the same as the real-stable case.

Applying barrier argument

Finally, we use *barrier argument* to find an "upper barrier vector" whose components lie above any roots of multivariate polynomial can take. In particular, we consider the multivariate polynomial $P(x, z) = (h(xe + \sum_{i=1}^n z_i v_i))^2$. Define the *barrier function* of any variable $i \in [n]$ as the following:

$$\Phi_P^i(\alpha(t), -\delta) = \frac{\partial_{z_i} P(x, z)}{P(x, z)}\Big|_{x=\alpha(t), z=-\delta},$$

where $\delta \in \mathbb{R}^n$ where $\delta_i = t\,\mathrm{tr}_h[v_i]$ for $i \in [n]$ and $\alpha(t) > t$ is a parameter that depends on t.

As a warm-up, consider the case when $\sigma = 1$ and assuming $\| \sum_{i=1}^n \mathrm{tr}_h[v_i]v_i\|_h \le 1$. It is easy to show that $(\alpha(t), -\delta)$ is an upper barrier of P, from the linearity of the hyperbolic eigenvalues and the assumption. Next, we upper-bound the barrier function's value at $(\alpha(t), -\delta)$. When h is a determinant polynomial, this step is easy because the derivative of $\log\det$ is the trace of the matrix. For a general hyperbolic polynomial, we will rewrite the partial derivative ∂_{z_i} as a directional derivative D_{v_i} and get

$$\Phi_P^i(\alpha(t), -\delta) = 2 \cdot \frac{(D_{v_i}h)(\alpha e - te + t(e - \sum_{j=1}^n \mathrm{tr}_h[v_j]v_j))}{h(\alpha e - te + t(e - \sum_{j=1}^n \mathrm{tr}_h[v_j]v_j))}.$$

We observe that our assumption $\| \sum_{i=1}^n \mathrm{tr}_h[v_i]v_i\|_h \le 1$ implies that $e - \sum_{j=1}^n \mathrm{tr}_h[v_j]v_j \in \Gamma_+^h$. By the concavity of the function $\frac{h(x)}{D_{v_i}h(x)}$ in the hyperbolicity cone, we can prove that

$$\Phi_P^i(\alpha(t), -\delta) \le \frac{2\,\mathrm{tr}_h[v_i]}{\alpha(t) - t}.$$

Now, we can apply the barrier update lemma in [36] with $\alpha(t) = 2t = 4$ to show that

$$\Phi^j_{(1-\frac{1}{2}\partial_{z_i}^2)P}(4, -\delta + \delta_i \mathbf{1}_i) \le \Phi_P^j(4, -\delta).$$

In other words, the partial differential operator $(1 - \frac{1}{2}\partial_{z_i}^2)$ shifts the upper-barrier by $(0, \cdots, 0, \delta_i, 0, \cdots, 0)$. Using induction for the variables $\delta_1, \cdots, \delta_n$, we can finally finally get an upper-barrier of

$$(4, -\delta + \sum_{i=1}^n \delta_i \mathbf{1}_i) = (4, 0, \ldots, 0),$$

which implies that $(4, 0, \ldots, 0)$ is above the roots of

$$\prod_{i=1}^n \Big(1 - \frac{1}{2}\frac{\partial^2}{\partial z_i^2}\Big) \left(h\Big(xe + \sum_{i=1}^n z_i \tau_i v_i\Big)\right)^2 \tag{3}$$

A challenge in this process is ensuring that the barrier function remains nonnegative. To achieve this, we use the multidimensional convexity of the hyperbolic barrier function as established in [59]. For cases where $\sigma \ne 1$, this requirement is satisfied through a simple scaling argument.

Combining the above three steps together, we can prove that $\Pr_{\xi_1, \cdots, \xi_n}[\| \sum_{i=1}^n \xi_i v_i\|_h \le 4\sigma] > 0$ for vectors $v_1, \ldots, v_n$ in the hyperbolicity cone with $\| \sum_{i=1}^n \mathrm{tr}_h[v_i]v_i\|_h = \sigma^2$.

3.2 Generalization to Strongly Rayleigh Distributions

Our main technical contribution to Theorem 6 is a more universal and structured method to characterize the mixed characteristic polynomial. Define the mixed characteristic polynomial as

$$q_S(x) = \mu(S) \cdot h\left(xe - \sum_{i \in S} v_i\right). \tag{4}$$

we want to show that it is equivalent to the restricted multivariate polynomial:

$$\prod_{i=1}^n (1 - \frac{1}{2}\frac{\partial^2}{\partial z_i^2})\Big(h(xe + \sum_{i=1}^n z_i v_i) g_\mu(x\mathbf{1} + z)\Big)\Big|_{z=0} \in \mathbb{R}[x, z_1, \cdots, z_n]. \tag{5}$$

Although Eqn. (4) and Eqn. (5) are the hyperbolic generalization of [6], we are unable to apply the previous techniques. This is because [6] computes the mixed characteristic polynomial explicitly, which heavily relies on the fact that the characteristic polynomial is a determinant. It is unclear how to generalize this method to hyperbolic/real-stable characteristic polynomials.

The key step in [6] is to show the following equality between mixed characteristic polynomial and multivariate polynomial:

$$x^{d_\mu - d} \cdot \mathop{\mathbb{E}}_{S\sim\mu}\left[\det\left(x^2 I - \sum_{i\in S} 2 v_i v_i^\top\right)\right]$$
$$= \prod_{i=1}^n (1-\partial_{z_i}^2)\left(g_\mu(x\mathbf{1}+z)\cdot\det(xI + \sum_{i=1}^n z_i v_i v_i^\top)\right)\Big|_{z=\mathbf{0}}$$

where d_μ is the degree of the homogeneous strongly-Rayleigh distribution μ (i.e. the degree of g_μ), and m is the dimension of v_i.

Then they expand the right-hand side to get:

$$\begin{aligned}\text{RHS} = &\sum_{k=0}^m (-1)^k x^{d_\mu+m-2k} \sum_{S\in\binom{[n]}{k}} \Pr_{T\sim\mu}[S\subseteq T]\cdot\sigma_k(\sum_{i\in S} 2v_i v_i^\top)\\ &= x^{d_\mu - m}\cdot \mathop{\mathbb{E}}_{S\sim\mu}\left[\det\left(x^2 I - \sum_{i\in S} 2 v_i v_i^\top\right)\right] = \text{LHS},\end{aligned}$$

where $\sigma_k(M)$ equals to the sum of all $k\times k$ principal minors of $M\in\mathbb{R}^{m\times m}$. The first step comes from expanding the product $\prod_{i=1}^n(1-\partial^2 z_i)$, and the second step comes from that

$$\det(x^2 I - \sum_{i=1}^n v_i v_i^\top) = \sum_{k=0}^m (-1)^{2k} x^{2m-2k} \sum_{S\in\binom{[n]}{k}} \sigma_k(\sum_{i\in S} v_i v_i^\top).$$

The naive generalization of a technique to hyperbolic/real-stable polynomial h faces challenges. One such challenge is the absence of an explicit form for h, unlike in the case of $h=\det$ where the determinant can be expressed as a combination of minors. This lack of a well-defined minor presents difficulty in rewriting the hyperbolic/real-stable polynomial. To tackle this issue, we devised a new and structured proof that relies on induction, offering a novel solution to this problem.

Inductive step

We first rewrite the expectation over the Strongly-Rayleigh distribution $T\sim\mu$ as follows:

$$\begin{aligned}x^{d_\mu}\cdot 2^{-n}\cdot \mathop{\mathbb{E}}_{T\sim\mu}[h(xe - \sum_{i\in T} v_i)] = \frac{1}{2}\mathop{\mathbb{E}}_{\xi_2,\cdots,\xi_n\sim\{0,1\}^{n-1}}\Big[&(1-\partial_{z_1})h(x_2+z_1v_1)x\partial_{z_1}g_2(x+z_1)\\ &+ h(x_2)(1-x\partial_{z_1})g_2(x+z_1)\Big|_{z_1=0}\Big]\end{aligned}$$

where g_2 is defined as

$$\begin{aligned}g_2(t) := x^{\sum_{i=2}^n \xi_i}\cdot &\\ &\prod_{i=2}^n\Big(\xi_i\partial_{z_i} + (1-\xi_i)(1-x\partial_{z_i})\Big)g_\mu(t, x+z_2, x+z_3,\cdots, x+z_n)\Big|_{z_2,\ldots,z_n=0}\end{aligned}$$

and $x_2 = x^2 e - \sum_{i=2}^n \xi_i v_i$. The main observation is that the marginals of a homogeneous Strongly-Rayleigh distribution can be computed from the derivatives of its generating polynomial.

Then, we can expand the term inside the expectation as

$$(1 - \frac{x}{2}\partial_{z_1}^2)\Big(h(x_2 + z_1 v_1) g_2(x + z_1)\Big)\Big|_{z_1=0},$$

using the fact that $\mathrm{rank}(v_1)_h \leq 1$ and the degree of $g_2(t)$ is at most 1.

Hence, we obtain our inductive step as

$$\begin{aligned} & x^{d_\mu} \cdot 2^{-n} \cdot \mathop{\mathbb{E}}_{\xi\sim\mu}\left[h(xe - \sum_{i=1}^n \xi_i v_i)\right] \\ = & \frac{1}{2}(1 - \frac{x}{2}\partial_{z_1}^2)\left(\mathop{\mathbb{E}}_{\xi_2,\cdots,\xi_n}\left[h(xe - \sum_{i=2}^n \xi_i v_i + z_1 v_1)\cdot g_2(x+z_1)\right]\right)\Bigg|_{z_1=0}. \end{aligned}$$

Applying the step inductively

Repeating the above process for n times, we finally get

$$x^{d_\mu} \cdot \mathop{\mathbb{E}}_{\xi\sim\mu}\left[h(x^2 e - (\sum_{i=1}^n \xi_i v_i))\right] = \sum_{T\subseteq[n]} (-\frac{x}{2})^{|T|}\partial_{z^T}^2\Big(h(x^2 e + \sum_{i=1}^n z_i v_i) g_\mu(x\mathbf{1} + z)\Big)\Bigg|_{z=0}.$$

Then, we rewrite the partial derivatives as directional derivatives. For any subset $T \subseteq [n]$ of size k, we have

$$\begin{aligned} & (-\frac{x}{2})^k \partial_{z^T}^2\Big(h(x^2 e + \sum_{i=1}^n z_i v_i) g_\mu(x\mathbf{1}+z)\Big)\Bigg|_{z=0} \\ = & (-\frac{x}{2})^k \cdot 2^k \cdot \left(\prod_{i\in T} D_{v_i}\right) h(x^2 e)\cdot g_\mu^{(T)}(x\mathbf{1}), \end{aligned}$$

where $g_\mu^{(T)}(x\mathbf{1}) = \prod_{i\in T}\partial_{z_i} g_\mu(x\mathbf{1}+z)\Big|_{z=0}$. And by the homogeneity of h, it further equals to

$$x^d \cdot (-\frac{1}{2})^k \partial_{z^T}^2\left(h(xe + \sum_{i=1}^n z_i v_i) g_\mu(x\mathbf{1}+z)\right)\Bigg|_{z=0}.$$

Therefore, we prove the following formula that relates the characteristic polynomial under SR distribution to the multivariate polynomial:

$$x^{d_\mu} \cdot \mathop{\mathbb{E}}_{\xi\sim\mu}\left[h(x^2 e - (\sum_{i=1}^n \xi_i v_i))\right] = x^d \cdot \prod_{i=1}^n (1 - \frac{1}{2}\partial_{z_i}^2)\Big(h(xe + \sum_{i=1}^n z_i v_i) g_\mu(x\mathbf{1}+z)\Big)\Bigg|_{z=0}.$$

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