Improved Algorithm and Lower Bound for Variable Time Quantum Search

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───── **Abstract** ─────

We study variable time search, a form of quantum search where queries to different items take different time. Our first result is a new quantum algorithm that performs variable time search with complexity $O(\sqrt{T}\log n)$ where $T = \sum_{i=1}^{n} t_i^2$ with t_i denoting the time to check the ith item. Our second result is a quantum lower bound of $\Omega(\sqrt{T \log T})$. Both the algorithm and the lower bound improve over previously known results by a factor of $\sqrt{\log T}$ but the algorithm is also substantially simpler than the previously known quantum algorithms.

1 Introduction

We study variable time search [2], a form of quantum search in which the time needed for a query depends on which object is being queried. Variable time search and its generalization, variable time amplitude [3] amplification, are commonly used in quantum algorithms. For example,

- Ambainis [3] used variable time amplitude amplification to improve the running time of HHL quantum algorithm for solving systems of linear equations [12] from $\widetilde{O}(\kappa^2)$ (where κ is the condition number of the system) to $\widetilde{O}(\kappa^{1+o(1)})$ in different contexts;
- Childs et al. [8] used variable time amplitude amplification to design a quantum algorithm for solving systems of linear equations with an exponentially improved dependence of the running time on the required precision;
- Le Gall [15] used variable time search to construct the best known quantum algorithm for triangle finding, with a running time $\widetilde{O}(n^{5/4})$ where n is the number of vertices;
- De Boer et al. [10] used variable time search to optimize the complexity of quantum attacks against a post-quantum cryptosystem;
- Glos et al. [11] used variable time search to develop a quantum speedup for a classical dynamic programming algorithm.
- Schrottenloher and Stevens [16] used variable time amplitude amplification to transform a classical nested search into a quantum algorithm, with applications to quantum attacks on AES.

In those applications, the oracle for the quantum search is a quantum algorithm whose running time depends on the item that is being queried. For example, we might have a graph algorithm that uses quantum search to find a vertex with a certain property and the time t_v to check the property may depend on the degree of the vertex v.

In such situations, using standard quantum search would mean that we run the checking algorithm for the maximum possible time $t_{\max} = \max_v t_v$. If most times t_v are substantially smaller, this results in suboptimal quantum algorithms.

A more efficient strategy is to use the variable time quantum search algorithm [2]. It has two variants: the "known times" variant when times t_v for checking various v are known in advance and can be used to design the algorithm and the "unknown times" variant in which t_v are only discovered when running the algorithm. In the "known times" case, VTS (variable time search) has complexity $O(\sqrt{T})$ where $T = \sum_v t_v^2$ and there is a matching lower bound [2].

For the "unknown times" case, the complexity of the variable time search increases to $O(\sqrt{T} \log^{1.5} T)$ and the quantum algorithm becomes substantially more complicated. Since almost all of the applications of VTS require the "unknown times" setting, it may be interesting to develop a simpler quantum algorithm.

In more detail, the "unknown times" search works by first running the query algorithm for a small time T_1 and then amplifying v for which the query either returns a positive result or does not finish in time T_1. This is followed by running the query algorithm for longer time $T_2, T_3, \ldots$ and each time, amplifying v for which the query either returns a positive result or does not finish in time T_i. To determine the necessary amount of amplification, quantum amplitude estimation is used. This results in a complex algorithm consisting of interleaved amplification and estimation steps. This complex structure contributes to the complexity of the algorithm, via log factors and may also lead to large constants hidden under the big-O.

In this paper, we develop a simple algorithm for variable time search that uses only amplitude amplification. Our algorithm achieves the complexity of $O(\sqrt{T} \log n)$ where T is an upper bound for $\sum_v t_v^2$ provided to the algorithm. (Unlike in the "known times" model, we do not need to provide $t_1, \ldots, t_n$ but only an estimate for T.) This also improves over the previous algorithm by a $\sqrt{\log}$ factor.

To summarize, the key difference from the earlier algorithms [2, 3] is that the earlier algorithms would use amplitude estimation (once for each amplification step) to determine the optimal schedule for amplitude amplification for this particular $t_1, \ldots, t_n$. In contrast, we use one fixed schedule for amplitude amplification (that depends only on the estimate for T and not on $t_1, \ldots, t_n$). While this schedule may be slightly suboptimal, the losses from it being suboptimal are less than savings from not performing multiple rounds of amplitude estimations. This also leads to the quantum algorithm being substantially simpler.

Our second result is a lower bound of $\Omega(\sqrt{T \log T})$, showing that a complexity of $\Theta(\sqrt{T})$ is not achievable. The lower bound is by creating a query problem which can be solved by variable time search and using the quantum adversary method to show a lower bound for this problem. In particular, this proves that "unknown times" search is more difficult than "known times" search (which has the complexity of $\Theta(\sqrt{T})$).

2 Model, definitions, and previous results

We consider the standard search problem in which the input consists of variables $x_1, \ldots, x_n \in \{0, 1\}$ and the task is to find $i : x_i = 1$ if such i exists.

Our model is a generalization of the usual quantum query model. We model a situation when the variable x_i is computed by a query algorithm Q_i which is initialized in the state $|0\rangle$ and, after t_i steps, outputs the final state $|x_i\rangle |\psi_i\rangle$ for some unknown $|\psi_i\rangle$. (For most of the paper, we restrict ourselves to the case when Q_i always outputs the correct x_i. The bounded error case is discussed briefly at the end of this section.) In the first $t_i - 1$ steps, Q_i can be in arbitrary intermediate states.

The goal is to construct an algorithm A that finds $i : x_i = 1$ (if such i exists). The algorithm A can run the query Q_i for a chosen t, with Q_i outputting x_i if $t_i \le t$ or * (an indication that the computation is not complete) if $t_i > t$. The complexity of A is the amount of time that is spent running the query algorithms Q_i. Transformations that does not involve running Q_i do not count towards the complexity.

More formally, we assume that, for any T, there is a circuit C_T which, on an input $\sum_{i=1}^{n} |i\rangle \otimes |0\rangle$ outputs

$$\sum_{i=1}^{n} |i\rangle \otimes |y_i\rangle \otimes |\psi_i\rangle ,$$

where $y_i = x_i$ if $t_i \le T$ and $y_i = *$ if $t_i > T$. The state $|\psi_i\rangle$ contains intermediate results of the computation and can be arbitrary. An algorithm A for variable time search consists of two types of transformations:

- circuits C_T for various T;
- transformations U_i that are independent of $x_1, \ldots, x_n$.

If there is no intermediate measurements, an algorithm A is of the form

$$U_k C_{T_k} U_{k-1} \ldots U_1 C_{T_1} U_0$$

and its complexity is defined as $T_1 + T_2 + \ldots + T_k$. In the general case, an algorithm is a sequence

$$U_0, C_{T_1}, U_1, \ldots, C_{T_k}, U_k$$

with intermediate measurements. Depending on the outcomes of those measurements, the algorithm may stop and output the result or continue with the next transformations. The complexity of the algorithm is defined as $p_1 T_1 + \ldots + p_k T_k$ where p_i is the probability that C_{T_i} is performed. (One could also allow U_i and T_i to vary depending on the results of previous measurements but this will not be necessary for our algorithm.)

If there exists $i : x_i = 1$, A must output one of such i with probability at least $2/3$. If $x_i = 0$, A must output "no such i" with probability at least $2/3$.

Known vs. unknown times. This model can be studied in two variants. In the "known times" variant, the times t_i for each $i \in [n]$ are known in advance and can be used to design the search algorithm. In the "unknown times" variant, the search algorithm should be independent of the times t_i, $i \in [n]$.

The complexity of the variable time search is characterized by the parameter $T = \sum_{i=1}^{n} t_i^2$. We summarize the previously known results below.

▶ **Theorem 2.1** ([2, 3]).

(a) Algorithm – known times: *For any $t_1, \ldots, t_n$, there is a variable time search algorithm $A_{t_1,\ldots,t_n}$ with the complexity $O(\sqrt{T})$.*

(b) Algorithm – unknown times: *There is a variable time search algorithm A with the complexity $O(\sqrt{T} \log^{1.5} T)$ for the case when $t_1, \ldots, t_n$ are not known in advance.*

(c) Lower bound – known times. *For any $t_1, \ldots, t_n$ and any variable time search algorithm $A_{t_1,\ldots,t_n}$, its complexity must be $\Omega(\sqrt{T})$.*

Parts (a) and (c) of the theorem are from [2]. Part (b) is from [3], specialized to the case of search.

In the recent years there have been attempts to reproduce and improve the aforementioned results by other means. In [9], the authors obtain a variant of Theorem 2.1(a) by converting the original algorithms into span programs, which then are composed and subsequently converted back to a quantum algorithm. More recently, [14] gives variable time quantum walk algorithm (which generalizes variable time quantum search) by employing a recent technique of multidimensional quantum walks. While the focus of these two papers is on developing very general frameworks, our focus is on making the variable time search algorithm simpler.

Concurrently and independently of our work, a similar algorithm for variable time amplitude amplification was presented in [16], which also relies on recursive nesting of quantum amplitude amplifications.

Variable time search with bounded error inputs. We present our results for the case when the queries Q_i are perfect (have no error) but our algorithm can be extended to the case if Q_i are bounded error algorithms, at the cost of an extra logarithmic factor.

Let k be the maximum number of calls to C_T's in an algorithm A. Then, it suffices that each C_T outputs a correct answer with a probability $1 - o(1/k^2)$. This can be achieved by repeating C_T $O(\log k)$ times and taking the majority of answers.

Possibly, this logarithmic factor can be removed using methods similar to ones for search with bounded error inputs in the standard (not variable time) setting [13].

3   Algorithm

We proceed in two steps. We first present a simple algorithm for the case when a sufficiently good bound on the number of solutions $m = |i : x_i = 1|$ are known (Section 3.2). We then present an algorithm for the general case that calls the simple algorithm multiple times, with different estimates for the parameter ℓ corresponding to m (Section 3.3).

Both algorithms require an estimate T for which $\sum_{i=1}^{n} t_i^2 \leq T$, with the complexity depending on T.

3.1   Tools and methods

Before presenting our results, we describe the necessary background about quantum amplitude amplification [6].

Amplitude amplification – basic construction. Assume that we have an algorithm A that succeeds with a small probability and it can be verified whether A has succeeded. Amplitude amplification is a procedure for increasing the success probability. Let

$$A \left|0\right\rangle = \sin \alpha \left|\psi_{\text{succ}}\right\rangle + \cos \alpha \left|\psi_{\text{fail}}\right\rangle.$$

Then, there is an algorithm $A(k)$ that involves $k+1$ applications of A and k applications of A^{-1} such that

$$A(k)|0\rangle = \sin\left((2k+1)\alpha\right)|\psi_{\text{succ}}\rangle + \cos\left((2k+1)\alpha\right)|\psi_{\text{fail}}\rangle.$$

Knowledge of α is not necessary (the way how $A(k)$ is obtained from A is independent of α).

Amplitude amplification – amplifying to success probability $1 - \delta$. If α is known then one can choose $k = \lfloor \frac{\pi}{4\alpha} \rfloor$ to amplify to a success probability close to 1 (since $(2k+1)\alpha$ will be close to $\frac{\pi}{2}$). If the success probability of A is ϵ, then $\sin\alpha \approx \sqrt{\epsilon}$ and $k \approx \frac{\pi}{4\sqrt{\epsilon}}$.

For unknown α, amplification to success probability $1 - \delta$ for any $\delta > 0$ can be still achieved, via a more complex algorithm. Namely, for any $\epsilon, \delta \in (0,1)$ and any A, one can construct an algorithm $A(\epsilon, \delta)$ such that:

- $A(\epsilon, \delta)$ invokes A and A^{-1} $O(\frac{1}{\sqrt{\epsilon}} \log \frac{1}{\delta})$ times;
- If A succeeds with probability at least ϵ, $A(\epsilon, \delta)$ succeeds with probability at least $1 - \delta$.

To achieve this, we first note that performing $A(k)$ for a randomly chosen $k \in \{1, \ldots, M\}$ for an appropriate $M = O\left(\frac{1}{\sqrt{\epsilon}}\right)$ and measuring the final state gives a success probability that is close to $1/2$ (as observed in the proof of Theorem 3 in [6]). Repeating this procedure $O\left(\log \frac{1}{\delta}\right)$ times achieves the success probability of at least $1 - \delta$.

3.2 Algorithm with a fixed number of stages

Now we present an informal overview of the algorithm when tight bounds on the number of solutions $m = |i : x_i = 1|$ is known. We will define a sequence of times $T_1, T_2, \ldots$ and procedures $A_1, A_2, \ldots$. We choose $T_1 = 3\sqrt{T/n}$ (this ensures that at most $n/9$ of indices $i \in [n]$ have $t_i \geq T_1$) and $T_2 = 3T_1$, $T_3 = 3T_2$, $\ldots$ until d for which $T_d \geq \sqrt{T}$. The procedure A_1 creates the superposition $\sum_{i=1}^{n} \frac{1}{\sqrt{n}} |i\rangle$ and runs the checking procedure C_{T_1}, obtaining state of the form $\sum_{i=1}^{n} \frac{1}{\sqrt{n}} |i, a_i\rangle$, where $a_i \in \{0, 1, *\}$, with $*$ denoting a computation that did not terminate. The subsequent procedures A_j are defined as $A_j = C_{T_j} A_{j-1}(1)$, i.e., we first amplify the parts of the state with outcomes 1 or $*$ and then run the checking procedure C_{T_j}.

We express the final state of A_{j-1} as

$$\sin\alpha_{j-1}|\psi_{\text{succ}}\rangle + \cos\alpha_{j-1}|\psi_{\text{fail}}\rangle,$$

where $|\psi_{\text{succ}}\rangle$ consists of those indices $i \in [n]$ which are either 1 or are still unresolved $*$ (and thus have the potential to turn out to be '1'). Then the amplitude amplification part triples the angle α_{j-1}, i.e., amplifies both the "good" and "unresolved" states by a factor of $\sin(3\alpha_{j-1})/\sin(\alpha_{j-1}) \approx 3$. We will show that $\ell = \lceil \log_9 \frac{n}{m} \rceil$ stages are sufficient, i.e., the procedure A_ℓ the amplitude at the "good" states (if they exist) is sufficiently large.

We note that the idea of recursive tripling via amplitude amplification has been used in other contexts. It has been used to build an algorithm for bounded-error search in [13]; more recently, the recursive tripling trick has also been used in, e.g., [7]. Furthermore, the repeated tripling of the angle α also explains the scaling factor 3 when defining the sequence $T_1, T_2, T_3 \ldots$

A formal description follows.

We assume an estimate $T \geq \sum_i t_i^2$ to be known and set

$$T_1 = 3\sqrt{T/n}, \ T_2 = 3T_1, \ \ldots, T_d = 3T_{d-1},$$

with $d \in \mathbb{N}$ s.t. $T_{d-1} < \sqrt{T} \leq T_d$ (equivalently, $9^{d-1} < n \leq 9^d$).

Let $\mathcal{M} = \{i \in [n] : x_i = 1\}$, $m = |\mathcal{M}|$. We assume that we know ℓ for which m belongs to the interval $\left[\frac{n}{9^\ell}, \frac{n}{9^{\ell-1}}\right)$ (so that $\ell = \lceil \log_9 \frac{n}{m} \rceil$).

Under those assumptions, we now describe a variable time search algorithm with parameters T, ℓ.

■ **Algorithm 1** VTS algorithm with a fixed number of stages.
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Parameters: T, n, ℓ, δ.
 1: Run the amplified algorithm $A(0.04, \delta)$ where A is the procedure defined below and we amplify the part of the state for the second register contains '1'
 2: **procedure** A
 3:     Run A_ℓ ▷ (defined below)
 4:     Run $C_{T_{\ell+1}}$ (or C_{T_d} if $\ell = d$)
 5: **end procedure**
 6: Measure the state
 7: **if** The second register is '1' **then**
 8:     Output i from the first register
 9: **else**
10:     Output `No solutions.`
11: **end if**
12: **procedure** A_j ▷ $j \in [d]$
13:     **if** $j = 1$ **then**
14:         Create the state $\sum_{i=1}^n \frac{1}{\sqrt{n}} |i\rangle$
15:         Run C_{T_1}, obtaining state of the form $\sum_{i=1}^n \frac{1}{\sqrt{n}} |i, a_i\rangle$ where $a_i \in \{0, 1, *\}$.
16:     **else**
17:         Perform the amplified algorithm $A_{j-1}(1)$, amplifying the basis states with 1 or * in the second register
18:         **if** $j < \ell$ **then**
19:             Run C_{T_j}.
20:         **end if**
21:     **end if**
22: **end procedure**
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▶ **Lemma 3.1.** *Algorithm 1 with parameter $\ell = \lceil \log_9 \frac{n}{m} \rceil$ finds an index $i \in \mathcal{M}$ with probability at least $1 - \delta$ in time $O\left(\sqrt{\frac{T}{m}} \log \frac{n}{m} \log \frac{1}{\delta}\right)$.*

Proof. By $\mathcal{S}_j$ we denote the sets of those indices whose amplitudes will be amplified after running A_j, namely, the set of indices for which the query either returns a positive result or does not finish in time T_j:

$$\mathcal{S}_j = \{i \in [n] \; : \; (T_j < t_i) \vee (t_i \leq T_j \wedge x_i = 1)\}, \quad j = 0, 1, 2, \ldots, d,$$

where $T_0 := 0$. We note that the sets $\mathcal{S}_j$ form a decreasing sequence[1], i.e.,

$$[n] = \mathcal{S}_0 \supseteq \mathcal{S}_1 \supseteq \mathcal{S}_2 \supseteq \ldots \supseteq \mathcal{S}_{d-1} \supseteq \mathcal{S}_d = \mathcal{M}.$$

We shall denote the cardinality of $\mathcal{S}_j$ by s_j; then

$$n = s_0 \geq s_1 \geq \ldots \geq s_d = m.$$

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[1] Since each i s.t. $t_i \leq T_j \wedge x_i = 1$ either satisfies $t_i \leq T_{j-1} \wedge x_i = 1$ or $t_i > T_{j-1}$; in both cases $i \in \mathcal{S}_{j-1}$.

We express the final state of A_j as

$$\sin\alpha_j \, |\psi_{\mathrm{succ},j}\rangle + \cos\alpha_j \, |\psi_{\mathrm{fail},j}\rangle$$

where $|\psi_{\mathrm{succ},j}\rangle$ consists of basis states with $|i\rangle$, $i \in \mathcal{S}_j$, in the first register and $|\psi_{\mathrm{fail},j}\rangle$ consists of basis states with $|i\rangle$, $i \notin \mathcal{S}_j$, in the first register.

We begin by describing how the cardinality of $\mathcal{S}_j$ is related to the amplitude $\sin\alpha_j$ (the proof is deferred to Appendix A).

▶ **Lemma 3.2.** *For all $j = 1, 2, \ldots, \ell$,*

$$\sin^2\alpha_j = \frac{s_j}{n} \prod_{k=1}^{j-1} \left(\frac{\sin(3\alpha_k)}{\sin\alpha_k} \right)^2 . \tag{1}$$

Moreover, for any $i \in \mathcal{S}_j$, the amplitude at $|i, 1\rangle$ (or $|i, *\rangle$, if $t_i > T_j$) equals $\frac{\sin\alpha_j}{\sqrt{s_j}}$.

Equation (1) and the trigonometric identity

$$\sin(3\alpha) = (3 - 4\sin^2\alpha)\sin\alpha$$

allows to obtain (for $j = 1, 2, \ldots, \ell$)

$$\frac{\sin(3\alpha_j)}{\sin\alpha_j} = 3 - 4\sin^2\alpha_j = 3 - \frac{4s_j \cdot 9^{j-1}}{n} \prod_{k=1}^{j-1} \left(\frac{\sin(3\alpha_k)}{3\sin\alpha_k} \right)^2 \geq 3 - \frac{4s_j}{n} \cdot 9^{j-1}, \tag{2}$$

where the inequality is justified by the observation $\left| \frac{\sin(3\alpha)}{3\sin\alpha} \right| \leq 1$. This allows to estimate

$$\sin\alpha_\ell = \sqrt{\frac{s_\ell}{n}} \prod_{j=1}^{\ell-1} \frac{\sin(3\alpha_j)}{\sin\alpha_j} \geq 3^{\ell-1} \sqrt{\frac{s_\ell}{n}} \prod_{j=1}^{\ell-1} \left(1 - \frac{4s_j}{27n} \cdot 9^j \right), \tag{3}$$

as long as each factor on the RHS is positive. We argue that it is indeed the case; moreover, the whole product is lower-bounded by a constant (the proof is deferred to Appendix A):

▶ **Lemma 3.3.** *The following claims hold:*
C-1 *Each factor on the RHS of (3) is positive: $\frac{9^j s_j}{n} \leq \frac{9}{4}$, thus*

$$\left(1 - \frac{4s_j}{27n} \cdot 9^j \right) \geq \frac{2}{3}, \quad \text{for all } j \in [\ell - 1].$$

C-2 *The product $\prod_{j=1}^{\ell-1} \left(1 - \frac{4s_j}{27n} \cdot 9^j \right)$ is lower bounded by 2/3.*
C-3 $9^\ell s_\ell \geq 9^\ell s_d \geq n$.

From (3) and Lemma 3.3 it is evident that $\sin\alpha_\ell \geq \frac{2}{9} \sqrt{\frac{9^\ell s_\ell}{n}} \geq \frac{2}{9}$.

However, after running A_ℓ, there still could be some unresolved indices i with $t_i > T_\ell$ and some of these unresolved indices may correspond to $x_i = 0$. Our next argument is that running $C_{T_{\ell+1}}$, i.e., the checking procedure for $3T_\ell$ steps, resolves sufficiently many indices in $\mathcal{M}$. This argument, however, is necessary only for $\ell < d$; for $\ell = d$, one runs C_{T_d} instead of $C_{T_{\ell+1}}$ and the same estimate (4) of the success probability applies, with $8m/9$ replaced by m. Also notice that in Algorithm 1 we skipped running C_{T_ℓ} at the end of A_ℓ and immediately proceeded with running $C_{T_{\ell+1}}$ instead. In the analysis, this detail is omitted for convenience (since it is equivalent to running C_{T_j} at the end of each procedure A_j and additionally running $C_{T_{\ell+1}}$ after A_ℓ).

By the choice of ℓ we have $\sqrt{\frac{T}{m}} \leq T_\ell = \sqrt{\frac{9^\ell T}{n}}$ and $T_{\ell+1} \geq 3\sqrt{\frac{T}{m}}$. Notice that at most $m/9$ of the indices $i \in [n]$ can satisfy $t_i^2 > T_{\ell+1}^2$ (otherwise, the sum over those indices already exceeds $\frac{m}{9} \cdot \frac{9T}{m} = T$). Consequently, after running the checking procedure $C_{T_{\ell+1}}$, at least $8m/9$ of the indices in $\mathcal{M}$ will be resolved to '1'. By Lemma 3.2, the amplitude at each of the respective states $|i, 1\rangle$ is equal to $\frac{\sin \alpha_\ell}{\sqrt{s_\ell}}$, therefore the probability to measure '1' in the second register is at least

$$\frac{8m}{9} \cdot \frac{\sin^2 \alpha_\ell}{s_\ell} \geq \frac{8m}{9} \cdot \frac{9^\ell}{n} \left(\frac{1}{3} \prod_{j=1}^{\ell-1} \left(1 - \frac{4s_j}{27n} \cdot 9^j \right) \right)^2 \geq \frac{8}{9} \cdot \left(\frac{2}{9} \right)^2 > 0.04, \tag{4}$$

where the first inequality follows from (3) and the second inequality is due to **C-2** and **C-3**.

We conclude that the procedure A finds an index $i \in \mathcal{M}$ with probability at least 0.04; its running time is easily seen to be

$$T_{\ell+1} + T_\ell + 3\left(T_{\ell-1} + 3\left(T_{\ell-2} + \ldots + 3\left(T_2 + 3T_1\right)\right)\right) = (3 + \ell)T_\ell,$$

which for our choice of ℓ is of order

$$O\left(\log \frac{n}{m} \sqrt{9^\ell \frac{T}{n}} \right) = O\left(\log \frac{n}{m} \sqrt{\frac{T}{m}} \right).$$

Use $O(\log \frac{1}{\delta})$ rounds amplitude amplification to amplify the success probability of A to $1 - \delta$, concluding the proof. ◀

3.3 Algorithm for the general case

When the cardinality of $|\mathcal{M}|$ is not known in advance, we run Algorithm 1 with increasing values of ℓ (which corresponds to exponentially decreasing guesses of m) until either $i : x_i = 1$ is found or we conclude that no such i exists. Algorithm 1 also suffers the "soufflé problem" [5] in which iterating too much (choosing ℓ in Algorithm 1 larger than its optimal value) may "overcook" the state and decrease the success probability. For this reason, before running Algorithm 1 with the next value of ℓ, we re-run it with all the previous values of ℓ to ensure that the probability of running Algorithm 1 with too large ℓ is small. This ensures that the algorithm stops in time $O\left(\sqrt{\frac{T}{m}} \log \frac{n}{m} \right)$ with high probability. Formally, we make the following claim:

▶ **Lemma 3.4.** *If $\mathcal{M}$ is nonempty, Algorithm 2 finds an index $i \in \mathcal{M}$ with probability at least* $5/6$ *with complexity* $O\left(\sqrt{\frac{T}{m}} \log \frac{n}{m} \right)$. *If $\mathcal{M}$ is empty, Algorithm 2 outputs* `No solutions.` *with complexity* $O\left(\sqrt{T} \log n \right)$.

Proof of Lemma 3.4. Let $\delta = 1/6$; let us remark that each procedure B_k runs in time $O\left(k3^k \sqrt{T/n} \right)$.

Let us consider the case when $m = |\mathcal{M}| > 0$; denote $\ell := \lceil \log_9 \frac{n}{m} \rceil$. The probability of B_k, $k \neq \ell$, finding an index $i \in \mathcal{M}$ is lower-bounded by 0; the probability of B_ℓ finding an index $i \in \mathcal{M}$ is lower-bounded by $1 - \delta$.

Hence, the total complexity of the algorithm stages $j = 1, 2, \ldots, \ell$, is of order

$$\sqrt{\frac{T}{n}} \sum_{j=1}^{\ell} \sum_{k=1}^{j} k3^k = \sqrt{\frac{T}{n}} \sum_{j=1}^{\ell} (\ell + 1 - j)j3^j \asymp \ell 3^\ell \sqrt{\frac{T}{n}}.$$

and the last step B_ℓ finds $i \in \mathcal{M}$ with probability at least $1 - \delta$.

■ **Algorithm 2** VTS algorithm for arbitrary number of solutions m.

Parameters: T, n.
Let B_k stand for Algorithm 1 with parameters T, n, k and $\delta = 1/6$.
1: **for** $j = 1, 2, \ldots, d$ **do**
2:     **for** $k = 1, 2, \ldots, j$ **do**
3:         Run B_k
4:         If B_k returned $i \in \mathcal{M}$, output this i and quit
5:     **end for**
6: **end for**
7: Output `No solutions.`

With probability at most δ, the last step fails to find $i \in \mathcal{M}$, and then Algorithm 2 proceeds with $j = \ell + 1$ and runs the sequence B_1, B_2, $\ldots$, B_ℓ, where the last step finds $i \in \mathcal{M}$ with (conditional) probability at least $1 - \delta$ (conditioned on the failure to find $i \in \mathcal{M}$ in the previous batch). The complexity of this part is of order

$$\delta\sqrt{\frac{T}{n}}\left(\sum_{j=1}^{\ell} j3^j\right) \asymp \delta\sqrt{\frac{T}{n}}\ell3^\ell,$$

where the δ factor reflects the fact the respective procedures are invoked with probability δ.

With (total) probability at most δ^2, the algorithm still has not found $i \in \mathcal{M}$. Then Algorithm 2 runs the sequence $B_{\ell+1}$, B_1, B_2, $\ldots$, B_ℓ (i.e., finishes with $j = \ell + 1$ and continues with $j = \ell + 2$), where the last step finds $i \in \mathcal{M}$ with (conditional) probability at least $1 - \delta$. The complexity of this part is of order

$$\delta^2\sqrt{\frac{T}{n}}\left((\ell+1)3^{\ell+1} + \ell3^\ell\right) \asymp \delta^2\sqrt{\frac{T}{n}}(\ell+1)3^{\ell+1}.$$

With (total) probability at most δ^3, the algorithm still has not found $i \in \mathcal{M}$. Then Algorithm 2 runs the sequence $B_{\ell+1}, B_{\ell+2}, B_1, B_2, \ldots, B_\ell$, where the last step finds $i \in \mathcal{M}$ with (conditional) probability at least $1 - \delta$. The complexity of this part is of order

$$\delta^3\sqrt{\frac{T}{n}}\left((\ell+1)3^{\ell+1} + (\ell+2)3^{\ell+2} + \ell3^\ell\right) \asymp \delta^3\sqrt{\frac{T}{n}}(\ell+2)3^{\ell+2},$$

and so on.

For $j = d$, the final batch $B_1, B_2, \ldots, B_\ell$ is invoked with probability at most $\delta^{d-\ell}$; with conditional probability at most δ we still fail to find $i \in \mathcal{M}$ and run the remaining sequence $B_{\ell+1}, \ldots, B_d$ (which can completely fail finding any $i \in \mathcal{M}$ as it has no non-trivial lower bounds on the success probability). The complexity of the latter sequence is of order

$$\delta^{d+1-\ell}\sqrt{\frac{T}{n}}\left((\ell+1)3^{\ell+1} + (\ell+2)3^{\ell+2} + \ldots + d3^d\right) \asymp \delta^{d+1-\ell}\sqrt{\frac{T}{n}}\,d\,3^d.$$

We see that Algorithm 2 fails with probability at most $\delta^{d+1-\ell}$; since $\ell \leq d$, this is upper-bounded by $\delta = 1/6$. The total complexity of the algorithm is of order

$$3^\ell \sqrt{\frac{T}{n}} \left(\ell + 3\delta^2(\ell+1) + 9\delta^3(\ell+2) + \ldots + (3\delta)^{d-\ell} \cdot d\delta \right)$$

$$< 3^\ell \sqrt{\frac{T}{n}} \left(\ell + \ell 3\delta^2 \sum_{i=0}^{\infty} (3\delta)^i + 3\delta^2 \sum_{i=1}^{\infty} i(3\delta)^{i-1} \right)$$

$$= 3^\ell \sqrt{\frac{T}{n}} \left(\ell + \ell \frac{3\delta^2}{1-3\delta} + \frac{3\delta^2}{(1-3\delta)^2} \right) \asymp \ell 3^\ell \sqrt{\frac{T}{n}},$$

since $3\delta = 1/2$. Since $3^\ell \asymp \sqrt{\frac{n}{m}}$ and $\ell \asymp \log \frac{n}{m}$, we conclude that the complexity of the algorithm is $O\left(\sqrt{\frac{T}{m}} \log \frac{n}{m} \right)$, as claimed.

Let us consider the case when $\mathcal{M}$ is empty; then with certainty each B_j fails to output any i, and Algorithm 2 correctly outputs `No solutions`. In this case, the complexity of the algorithm is of order

$$\sqrt{\frac{T}{n}} \sum_{j=1}^{d} \sum_{k=1}^{j} k 3^k = \sqrt{\frac{T}{n}} \sum_{j=1}^{d} (d+1-j) j 3^j \asymp d 3^d \sqrt{\frac{T}{n}} \asymp \sqrt{T} \log n,$$

since $3^d \asymp \sqrt{n}$. ◀

4 Lower bound

For the improved lower bound, we consider a query problem which can be solved with variable time search. Let $g : \{0,1,\star\}^m \to \{0,1\}$ be a partial function defined on the strings with exactly one non-$\star$ value, which is the value of the function. The function f we examine then is the composition of OR_n with g. We note that g is also known in the literature as pSEARCH, which has been used for quantum lower bounds in cryptographic applications [4].

For any $i \in [n]$, if the index of the non-$\star$ element in the corresponding instance of g is $j_i \in [m]$, then we can find this value in $O(\sqrt{j_i})$ queries using Grover's search. This creates an instance of the variable search problem with unknown times $t_i = \sqrt{j_i}$. By examining only inputs with fixed $T = \sum_{i=1}^{n} t_i^2 = \sum_{i=1}^{n} j_i$ and the restriction of f on these inputs f_T, we are able to prove a $\Omega(\sqrt{T \log T})$ query lower bound using the weighted quantum adversary bound [1]. Since any quantum algorithm for the variable time search also solves f_T, this gives the required lower bound.

▶ **Theorem 4.1.** *Any algorithm that solves variable time search with unknown times t_i requires time $\Omega(\sqrt{T \log T})$, where $T = \sum_{i \in [n]} t_i^2$.*

We note that the lower bound of Theorem 4.1 contains a factor of $\sqrt{\log T}$ while the upper bound of Lemma 3.4 contains a factor of $\log n$. There is no contradiction between these two results as the lower bound uses inputs with $T = \Theta(n \log n)$ and for those inputs $\log T = (1 + o(1)) \log n$.

Proof of Theorem 4.1. Consider a partial function $f : D \to \{0,1\}$, where $D \subset \{\star, 0, 1\}^{[n] \times [m]}$, defined as follows. An input $x \in D$ if for each $i \in [n]$ there is a unique $j \in [m]$ such that $x_{i,j} \neq \star$; denote this j by $j_{x,i}$. Then $f(x) = 1$ iff there exists an i such that $x_{i,j_{x,i}} = 1$.

Suppose that x is given by query access to $x_{i,j}$. For any i, we can check whether $x_{i,j_{x,i}} = 1$ in $O(\sqrt{j_{x,i}})$ queries with certainty in the following way. There is a version of Grover's search that detects a marked element out of N elements in $O(\sqrt{N})$ queries with certainty, if the

number of marked elements is either 0 or 1 [6]. By running this algorithm for the first N elements, where we iterate over $N = 1, 2, \ldots, 2^{\lceil \log_2 j_{x,i} \rceil}$, we will detect whether $x_{i,j_{x,i}} = 1$ in $O(\sqrt{j_{x,i}})$ queries with certainty.

Letting $t_i = \sqrt{j_{x,i}}$ and $T = \sum_{i \in [n]} t_i^2$, we get an instance of a variable search problem. Now fix any value of T and examine only inputs with such T. Denote f restricted on T by f_T. If the quantum query complexity of f_T is $Q(f_T)$, then any algorithm that solves variable time search must require at least $\Omega(Q(f_T))$ time. In the following, we will prove that $Q(f_T) = \Omega(\sqrt{T \log T})$.

Adversary bound

We will use the relational version of the quantum adversary bound [1]. Let $X \subseteq f_T^{-1}(0)$ and $Y \subseteq f_T^{-1}(1)$ and $R : X \times Y \to \mathbb{R}_{\geq 0}$ be a weight function. For any input $x \in X$, define $w(x) = \sum_{y \in Y} R(x, y)$ and for any $i \in [n]$, $j \in [m]$, define $w(x, i, j) = \sum_{y \in Y, x_{i,j} \neq y_{i,j}} R(x, y)$. Similarly define $w(y)$ and $w(y, i, j)$. Then

$$Q(f_T) = \Omega \left(\min_{\substack{x \in x, y \in Y \\ i \in [n], j \in [m] \\ R(x,y) > 0 \\ x_{i,j} \neq y_{i,j}}} \sqrt{\frac{w(x) w(y)}{w(x, i, j) w(y, i, j)}} \right).$$

Input sets

Here we define the subsets of inputs X and Y. First, let k be the smallest positive integer such that $T \leq 2^k k$ and k is a multiple of 4. Denote $d = 2^k$, then $k = \log_2 d$ and $T = \Theta(d \log d)$. An input z from either X or Y must then satisfy the following conditions.

- for each $p \in \left[0, \frac{k}{2}\right]$, there are exactly $\frac{d}{2^p}$ indices i such that $j_{z,i} \in [2^p, 2^{p+1})$; we will call the set of such indices the *p-th block* of z;
- moreover, for each p and each $\ell \in [0, 2^p)$, there are exactly $\frac{d}{2^{2p}}$ indices i such that $j_{z,i} = 2^p + \ell$.

Consequently, we examine inputs with $n = 2^k + 2^{k-1} + \ldots 2^{\frac{k}{2}}$ and $m = 2^{\frac{k}{2}+1} - 1$. Additionally, an input y belongs to Y only if there is a unique i such that $y_{i,j_{y,i}} = 1$. For this i, we also require $j_{y,i} \geq 2^{\frac{k}{4}+1}$: equivalently this means that i belongs to a block with $p > \frac{k}{4}$.

We verify the value of $T' = \sum_{i \in [n]} t_i^2$ for these inputs. If i belongs to the p-th block of an input z, then $j_{z,i} = \Theta(2^p)$, as $j_{z,i} \in [2^p, 2^{p+1})$. Then

$$T' = \sum_{i \in [n]} t_i^2 = \sum_{i \in [n]} j_{z,i} = \sum_{p \in \left[0, \frac{k}{2}\right]} \frac{d}{2^p} \cdot 2^p = d \left(\frac{k}{2} + 1 \right) = \Theta(T).$$

Note that since $Q(f_{T'}) \leq Q(f_T)$, a lower bound on $Q(f_{T'})$ in terms of T will also give us a lower bound on $Q(f_T)$. In the remainder of the proof, we will thus lower bound $Q(f_{T'})$.

Relation

For an index $i \in [n]$ of an input z that belongs to the p-th block, we define an *index weight* $W_{z,i} = 2^p$. Then we also define values

- $B_p = \frac{d}{2^p} \cdot 2^p = d$ is the total index weight of the p-th block;
- $J_p = \frac{d}{2^{2p}} \cdot 2^p = \frac{d}{2^p}$ is the total index weight in the p-th block for any $j_{z,i} \in [2^p, 2^{p+1})$.

Note that these values do not depend on the input.

For the relation, we will call the p-th block *light* if $p \in \left[0, \frac{k}{4}\right]$ and *heavy* if $p \in \left(\frac{k}{4}, \frac{k}{2}\right]$. Two inputs $x \in X$ and $y \in Y$ have $R(x, y) > 0$ iff:

- there are exactly two indices $i_0, i_1 \in [n]$ such that $j_{x,i_b} \neq j_{y,i_b}$;
- i_0 is from some light block p_0 and i_1 is from some heavy block p_1 of y; let $j_0 = j_{y,i_0}$ and $j_1 = j_{y,i_1}$;
- $y_{i_0,j_0} = 0$, $y_{i_1,j_1} = 1$.
- $x_{i_0,j_1} = x_{i_1,j_0} = 0$.

Then let the weight in the relation be

$$R(x,y) = W_{y,i_0} W_{y,i_1} = W_{x,i_1} W_{x,i_0} = 2^{p_0} 2^{p_1}.$$

Figure 1 illustrates the structure of the inputs and the relation.
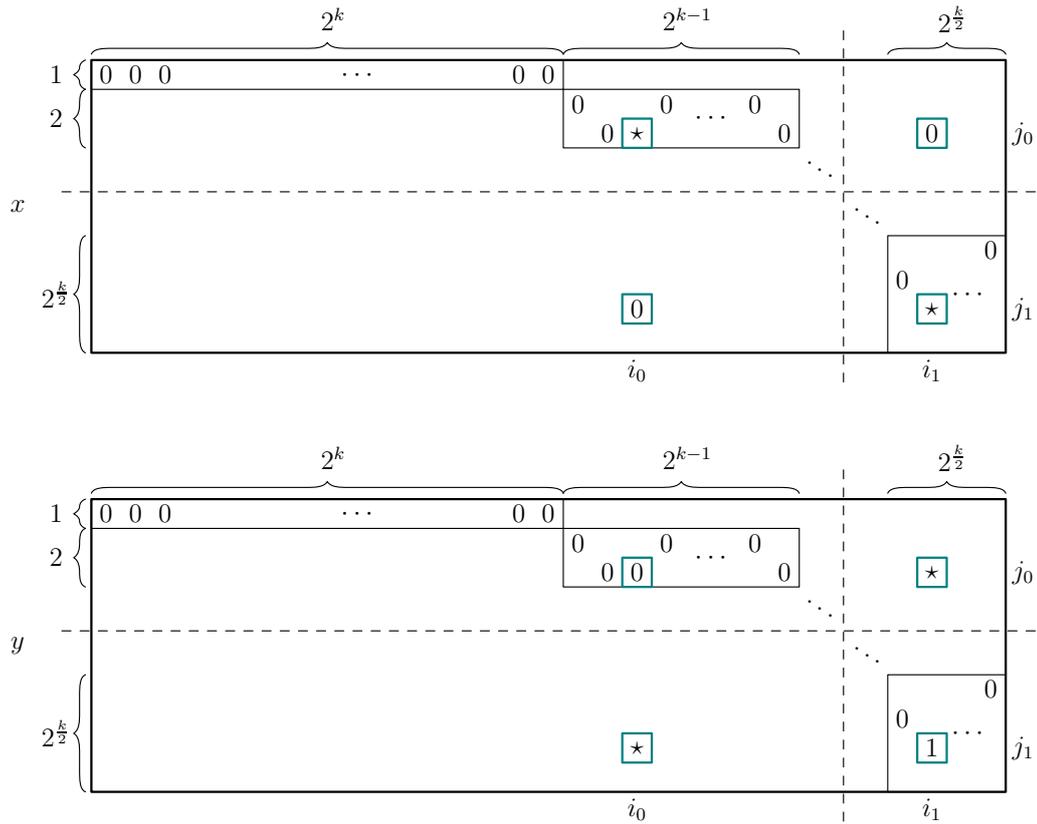
Figure 1 An example of two inputs $x \in X$ and $y \in Y$ in the relation. Inputs x and y differ only in the 4 highlighted positions. All of the empty cells contain $\star$, not shown for readability. For y, the non-$\star$ symbols of the light blocks are located in the left upper area separated by the dashed lines, while the non-$\star$ symbols of the heavy blocks are located in the lower right area. Note that for x, i_0 is in a heavy block and i_1 is in a light block.

Lower bound

Now we will calculate the values for the adversary bound. Fix two inputs $x \in X$ and $y \in Y$ with $R(x,y) > 0$. First, since for x the index i_1 can be any index from any light block and i_0 can be any index from any heavy block,

$$w(x) = \left(\sum_{p_0 \in \left[0, \frac{k}{4}\right]} B_{p_0} \right) \cdot \left(\sum_{p_1 \in \left(\frac{k}{4}, \frac{k}{2}\right]} B_{p_1} \right) = \Theta(d^2 k^2).$$

For $w(y)$, note that p_1 is uniquely determined by the position of the unique symbol 1 in y. However, the choice for i_0 is not additionally constrained, hence

$$w(y) = \left(\sum_{p_0 \in \left[0, \frac{k}{4}\right]} B_{p_0} \right) \cdot 2^{p_1} = \Theta(dk2^{p_1}).$$

Therefore, the nominator in the ratio in the adversary bound is

$$w(x)w(y) = \Theta(d^3 k^3 2^{p_1}).$$

Now note the following important property: if $x_{i,j} \neq y_{i,j}$, then one of $x_{i,j}$ and $y_{i,j}$ is $\star$, and the other is either 0 or 1. There are in total exactly 4 positions (i, j) where x and y differ. We will examine each case separately.

(a) $i = i_0$, $j = j_0$. In this case $x_{i,j} = \star$ and $y_{i,j} = 0$.

For x, i_1 is not fixed but j_0 is known and hence also p_0 is known. Therefore, the total index weight from the light blocks is J_{p_0}. On the other hand, the positions of i_0 and, therefore, also p_1 are fixed. Thus,

$$w(x, i, j) = J_{p_0} \cdot 2^{p_1} = \frac{d}{2^{p_0}} \cdot 2^{p_1}.$$

For y, both i_0 and i_1 are fixed, hence

$$w(y, i, j) = 2^{p_0} \cdot 2^{p_1} < d,$$

since $p_0 + p_1 \leq \frac{k}{4} + \frac{k}{2} < k$. Overall,

$$w(x, i, j)w(y, i, j) < \frac{d}{2^{p_0}} \cdot 2^{p_1} \cdot d = \frac{d^2 \cdot 2^{p_1}}{2^{p_0}}.$$

(b) $i = i_0$, $j = j_1$. In this case $x_{i,j} = 0$ and $y_{i,j} = \star$.

For x, now the position i_0 is fixed, but i_1 can be chosen without additional constraints. The index i_0 uniquely defines the value of p_1. Hence,

$$w(x, i, j) = \left(\sum_{p_0 \in \left[0, \frac{k}{4}\right]} B_{p_0} \right) \cdot 2^{p_1} = \Theta(dk2^{p_1}).$$

For y, similarly as in the previous case, we have i_0 and i_1 fixed, thus

$$w(y, i, j) = 2^{p_0} \cdot 2^{p_1} < d.$$

Then

$$w(x, i, j)w(y, i, j) = O(dk2^{p_1} \cdot d) = O(d^2 k2^{p_1}).$$

(c) $i = i_1$, $j = j_0$. In this case $x_{i,j} = 0$ and $y_{i,j} = \star$.

For x, i_1 is fixed, so it uniquely determines p_0. The index i_0 can be chosen without additional restrictions. Hence,

$$w(x, i, j) = 2^{p_0} \cdot \left(\sum_{p_1 \in \left(\frac{k}{4}, \frac{k}{2}\right]} B_{p_1} \right) = \Theta(2^{p_0} \cdot dk).$$

For y, i_0 is not fixed but j_0 is fixed, which also fixes p_0. Therefore, the total index weight from the light blocks is J_{p_0}. On the other hand, i_1 and p_1 are fixed for y by the position of the symbol 1, thus

$$w(y, i, j) = J_{p_0} \cdot 2^{p_1} = \frac{d}{2^{p_0}} \cdot 2^{p_1}.$$

Their product is

$$w(x, i, j)w(y, i, j) = \Theta\left(2^{p_0} \cdot dk \cdot \frac{d}{2^{p_0}} \cdot 2^{p_1}\right) = \Theta(d^2 k 2^{p_1}).$$

(d) $i = i_1$, $j = j_1$. In this case $x_{i,j} = \star$ and $y_{i,j} = 1$.

For x, i_1 is fixed, hence p_0 is also fixed; i_0 is not fixed, but $j_1 = j$ and p_1 is uniquely defined. Hence,

$$w(x, i, j) = 2^{p_0} \cdot J_{p_1} = 2^{p_0} \cdot \frac{d}{2^{p_1}}.$$

For y, the position of the symbol 1 must necessarily change, hence

$$w(y, i, j) = w(y) = \Theta(dk 2^{p_1}).$$

The product then is

$$w(x, i, j)w(y, i, j) = \Theta\left(2^{p_0} \cdot \frac{d}{2^{p_1}} \cdot dk 2^{p_1}\right) = \Theta(d^2 k 2^{p_0}) = O(d^2 k 2^{p_1}),$$

as $p_0 \le \frac{k}{4} < p_1$.

We can see that in all cases the denominator in the ratio of the adversary bound is $O(d^2 k 2^{p_1})$. Therefore,

$$\frac{w(x)w(y)}{w(x, i, j)w(y, i, j)} = \Omega\left(\frac{d^3 k^3 2^{p_1}}{d^2 k 2^{p_1}}\right) = \Omega(dk^2) = \Omega(d \log^2 d)$$

and since $\log T = \Theta(\log(d \log d)) = \Theta(\log d + \log \log d) = \Theta(\log d)$, we have

$$\mathrm{Q}(f_T) \ge \mathrm{Q}(f_{T'}) = \Omega\left(\sqrt{d \log^2 d}\right) = \Omega\left(\sqrt{T \log T}\right). \qquad \blacktriangleleft$$

5 Conclusion

In this paper, we developed a new quantum algorithm and a new quantum lower bound for variable time search. Our quantum algorithm has complexity $O(\sqrt{T} \log n)$, compared to $O(\sqrt{T} \log^{1.5} T)$ for the best previously known algorithm (quantum variable time amplitude amplification [3] instantiated to the case of search). It also has the advantage of being simpler than previous quantum algorithms for variable time search. If the recursive structure is unrolled, our algorithm consists of checking algorithms C_{T_i} for various times T_i interleaved with Grover diffusion steps. Thus, the structure is the essentially same as for regular search and the main difference is that C_{T_i} for different i are substituted at different query steps.

We note that our algorithm has a stronger assumption about T: we assume that an upper bound estimate $T \ge \sum_{i=1}^n t_i^2$ is provided as an input to the algorithm and the complexity depends on this estimate T, rather than the actual $\sum_{i=1}^n t_i^2$. Possibly, this assumption can be removed by a doubling strategy that tries values of T that keep increasing by a factor of 2 but the details remain to be worked out.

Our quantum lower bound is $\Omega(\sqrt{T \log T})$ which improves over the previously known $\Omega(\sqrt{T})$ lower bound. This shows that variable time search for the "unknown times" case (when the times $t_1, \ldots, t_n$ are not known in advance and cannot be used to design the quantum algorithm) is more difficult than for the "known times" case (which can be solved with complexity $\Theta(\sqrt{T})$).

A gap between the upper and lower bounds remains but is now just a factor of $\sqrt{\log T}$. Possibly, this is due to the lower bound using a set of inputs for which an approximate distribution of values t_i is fixed. In such a case, the problem may be easier than in the general case, as an approximately fixed distribution of t_i can be used for algorithm design.

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A Proofs of Lemmas 3.2 and 3.3

▶ **Lemma 3.2.** *For all $j = 1, 2, \ldots, \ell$,*

$$\sin^2 \alpha_j = \frac{s_j}{n} \prod_{k=1}^{j-1} \left(\frac{\sin(3\alpha_k)}{\sin \alpha_k} \right)^2 . \tag{1}$$

Moreover, for any $i \in \mathcal{S}_j$, the amplitude at $|i, 1\rangle$ (or $|i, *\rangle$, if $t_i > T_j$) equals $\frac{\sin \alpha_j}{\sqrt{s_j}}$.

Proof. For each j express the final state of A_j in the canonical basis as

$$\sum_{i=1}^{n} \beta_{ij} |i, a_{ij}\rangle ,$$

where $a_{ij} \in \{0, 1, *\}$ and $a_{ij} = 0$ iff $x_i = 0$ and $t_i \leq T_j$ (i.e., iff $i \notin \mathcal{S}_j$). Initially, $\beta_{i0} = n^{-1/2}$ for all i. Then

$$\sin^2 \alpha_j = \sum_{i \in \mathcal{S}_j} |\beta_{ij}|^2 ,$$

for all j. To see how the amplitude $\beta_{i(j+1)}$ is related to β_{ij}, consider how the state evolves under A_{j+1}:

- the final state of A_j is

$$\sum_{i \in [n] \setminus \mathcal{S}_j} \beta_{ij} |i, 0\rangle + \sum_{i \in \mathcal{S}_j} \beta_{ij} |i, a_{ij}\rangle ,$$

by the definition of β_{ij}; moreover, $a_{ij} \in \{1, *\}$ for all $i \in \mathcal{S}_j$.

- Amplitude amplification $A_j(1)$ results in the state

$$\sum_{i \in [n] \setminus \mathcal{S}_j} \frac{\cos(3\alpha_j)}{\cos \alpha_j} \beta_{ij} |i, 0\rangle + \sum_{i \in \mathcal{S}_j} \frac{\sin(3\alpha_j)}{\sin \alpha_j} \beta_{ij} |i, a_{ij}\rangle .$$

- An application of $C_{T_{j+1}}$ transforms this state to

$$\sum_{i \in [n] \setminus \mathcal{S}_j} \frac{\cos(3\alpha_j)}{\cos \alpha_j} \beta_{ij} \ket{i, 0} + \sum_{i \in \mathcal{S}_j \setminus \mathcal{S}_{j+1}} \frac{\sin(3\alpha_j)}{\sin \alpha_j} \beta_{ij} \ket{i, 0} + \sum_{i \in \mathcal{S}_{j+1}} \frac{\sin(3\alpha_j)}{\sin \alpha_j} \beta_{ij} \ket{i, a_{i(j+1)}}.$$

We conclude that

$$\beta_{i(j+1)} = \begin{cases} \beta_{ij} \frac{\sin(3\alpha_j)}{\sin \alpha_j}, & i \in \mathcal{S}_j, \\ \beta_{ij} \frac{\cos(3\alpha_j)}{\cos \alpha_j}, & i \in [n] \setminus \mathcal{S}_j. \end{cases} \tag{5}$$

In particular, for any $j \in [\ell]$ and $i \in \mathcal{S}_j$ we have

$$\beta_{ij} = \frac{1}{\sqrt{n}} \prod_{k=1}^{j-1} \frac{\sin(3\alpha_k)}{\sin \alpha_k},$$

since each such i is in $\mathcal{S}_k$, $k \leq j-1$, thus, by (5), the respective amplitude gets multiplied by $\frac{\sin(3\alpha_k)}{\sin \alpha_k}$ at each step. This establishes the second part of the lemma (that the amplitudes β_{ij} are all equal for any $i \in \mathcal{S}_j$). For the first part, we arrive at

$$\sin^2 \alpha_j = \sum_{i \in \mathcal{S}_j} |\beta_{ij}|^2 = \sum_{i \in \mathcal{S}_j} \prod_{k=1}^{j-1} \left(\frac{\sin(3\alpha_k)}{\sin \alpha_k} \right)^2 = \frac{s_j}{n} \prod_{k=1}^{j-1} \left(\frac{\sin(3\alpha_k)}{\sin \alpha_k} \right)^2. \qquad \blacktriangleleft$$

▶ **Lemma 3.3.** *The following claims hold:*

C-1 *Each factor on the RHS of* (3) *is positive:* $\frac{9^j s_j}{n} \leq \frac{9}{4}$, *thus*

$$\left(1 - \frac{4s_j}{27n} \cdot 9^j \right) \geq \frac{2}{3}, \quad \text{for all } j \in [\ell-1].$$

C-2 *The product* $\prod_{j=1}^{\ell-1} \left(1 - \frac{4s_j}{27n} \cdot 9^j \right)$ *is lower bounded by 2/3.*

C-3 $9^\ell s_\ell \geq 9^\ell s_d \geq n.$

Proof. We will prove the following inequality:

$$\sum_{j=1}^{\ell-1} s_j 9^j < \frac{9n}{4}. \tag{6}$$

Then **C-1** will immediately follow, since each term on (6) is nonnegative. Furthermore, also **C-2** follows from (6) via the generalized Bernoulli's inequality:

$$\prod_{j=1}^{\ell-1} \left(1 - \frac{4s_j}{27n} \cdot 9^j \right) \geq 1 - \frac{4}{27n} \sum_{j=1}^{\ell-1} s_j 9^j \geq 1 - \frac{4}{27n} \cdot \frac{9n}{4} = \frac{2}{3}.$$

First we observe that

$$\sum_{j=1}^{d} \sum_{i \in \mathcal{S}_{j-1} \setminus \mathcal{S}_j} t_i^2 = \sum_{i \in [n] \setminus \mathcal{M}} t_i^2 < \sum_{i \in [n]} t_i^2 \leq T.$$

Notice that each set difference $\mathcal{S}_{j-1} \setminus \mathcal{S}_j$ can be characterized as follows:

$$\mathcal{S}_{j-1} \setminus \mathcal{S}_j = \{ i \in [n] : (T_{j-1} < t_i \leq T_j) \wedge x_i = 0 \}.$$

Therefore all t_i^2 s.t. $i \in \mathcal{S}_{j-1} \setminus \mathcal{S}_j$ satisfy the bound

$$t_i^2 \geq T_{j-1}^2 = \begin{cases} \frac{9^{j-1}T}{n}, & j > 1, \\ 0, & j = 1. \end{cases}$$

Thus we obtain the following inequality:

$$\frac{T}{n} \sum_{j=2}^d 9^{j-1} |\mathcal{S}_{j-1} \setminus \mathcal{S}_j| < \sum_{j=1}^d \sum_{i \in \mathcal{S}_{j-1} \setminus \mathcal{S}_j} t_i^2 < T$$

or

$$\sum_{k=1}^{d-1} 9^k (s_k - s_{k+1}) < n. \tag{7}$$

We also expand $9^\ell s_\ell$ as follows, taking into account $s_d = m$:

$$9^\ell s_\ell = 9^\ell (s_\ell - s_{\ell+1}) + \frac{1}{9} \cdot 9^{\ell+1} (s_{\ell+1} - s_{\ell+2}) + \ldots + \frac{1}{9^{d-1-\ell}} \cdot 9^{d-1} (s_{d-1} - s_d) + 9^\ell m.$$

From this equality, taking into account $s_k - s_{k+1} \geq 0$, we can upper bound $9^\ell s_\ell$ as

$$9^\ell s_\ell \leq \sum_{k=\ell}^{d-1} 9^k (s_k - s_{k+1}) + 9^\ell m \tag{8}$$

Rewrite (7) as

$$s_1 + \frac{8}{9} \sum_{k=1}^{\ell-1} 9^k s_k - 9^{\ell-1} s_\ell + \sum_{k=\ell}^{d-1} 9^k (s_k - s_{k+1}) < n$$

and apply (8) to obtain

$$s_1 + \frac{8}{9} \sum_{k=1}^{\ell-1} 9^k s_k + \sum_{k=\ell}^{d-1} 9^k (s_k - s_{k+1}) < n + 9^{\ell-1} s_\ell \leq n + \frac{1}{9} \sum_{k=\ell}^{d-1} 9^k (s_k - s_{k+1}) + 9^{\ell-1} m$$

$$\frac{8}{9} \sum_{k=1}^{\ell-1} 9^k s_k + \frac{8}{9} \sum_{k=\ell}^{d-1} 9^k (s_k - s_{k+1}) < n - s_1 + 9^{\ell-1} m$$

$$8 \sum_{k=1}^{\ell-1} 9^k s_k < 9n - 9s_1 + 9^\ell m < 9n + 9^\ell m.$$

By the choice of ℓ we have $9^{\ell-1} \leq \frac{n}{m}$, therefore we arrive at

$$8 \sum_{k=1}^{\ell-1} 9^k s_k < 9n + 9\frac{n}{m} \cdot m = 18n,$$

which is equivalent to (6).

Finally, to show **C-3**, we recall that $s_\ell \geq s_d = m$. Again by the choice of ℓ, $9^\ell \geq \frac{n}{m}$. Consequently,

$$9^\ell s_\ell \geq \frac{n}{m} \cdot m = n,$$

as claimed.   ◀