Local Hamiltonians with No Low-Energy Stabilizer States

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Abstract

The recently-defined No Low-energy Sampleable States (NLSS) conjecture of Gharibian and Le Gall [16] posits the existence of a family of local Hamiltonians where all states of low-enough constant energy do not have succinct representations allowing perfect sampling access. States that can be prepared using only Clifford gates (i.e. stabilizer states) are an example of sampleable states, so the NLSS conjecture implies the existence of local Hamiltonians whose low-energy space contains no stabilizer states. We describe families that exhibit this requisite property via a simple alteration to local Hamiltonians corresponding to CSS codes. Our method can also be applied to the recent NLTS Hamiltonians of Anshu, Breuckmann, and Nirkhe [4], resulting in a family of local Hamiltonians whose low-energy space contains neither stabilizer states nor trivial states. We hope that our techniques will eventually be helpful for constructing Hamiltonians which simultaneously satisfy NLSS and NLTS.

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1 Introduction

Local Hamiltonians are ubiquitous in quantum physics and quantum computation. From the physical perspective, Hamiltonians describe the dynamics and energy spectra of closed quantum systems, with "local" Hamiltonians corresponding to models where only a small number of particles can directly interact with each other. From the computational perspective, local Hamiltonians naturally generalize well-studied constraint satisfaction problems through the "local Hamiltonian problem", which asks about the complexity of approximating the ground-state energy of local Hamiltonians.

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▶ **Definition** (LH-$\delta(n)$). *A k-local Hamiltonian, $\mathcal{H} = \frac{1}{m}\sum_{i=1}^{m}\mathcal{H}_i$, is a sum of $m = \mathsf{poly}(n)$ Hermitian matrices, $\mathcal{H}_i \in \mathbb{C}^{2^n \times 2^n}$, where each $\mathcal{H}_i$ acts non-trivially on at most $k = \mathcal{O}(1)$ qubits*[1] *and has bounded spectral norm, $\|\mathcal{H}_i\| \leq 1$.*

*Given a local Hamiltonian, $\mathcal{H}$, and two real numbers $a < b$ with $b - a > \delta(n)$, the **local Hamiltonian problem with promise gap $\boldsymbol{\delta(n)}$** is to decide if (1) there is a state with energy $\langle\psi_0|\,\mathcal{H}\,|\psi_0\rangle \leq a$ or (2) all states have energy $\langle\psi|\,\mathcal{H}\,|\psi\rangle \geq b$, given that one of these cases is true.*[2] *The value $\delta(n)$ is called the promise gap of the problem.*

LH is a natural quantum analogue of the NP-complete constraint satisfaction problem (CSP):[3] the local terms serve as *quantum* constraints on an n-qubit state, and the energy of a local term corresponds to how well the state satisfies that local constraint. The lowest energy state – or ground-state – of $\mathcal{H}$ is the state that optimally satisfies all of the local constraints.

It is straightforward to show that CSP is NP-complete for a promise gap $\delta(n) = 1/\mathsf{poly}(n)$, and the celebrated classical PCP Theorem [7, 8] shows that [surprisingly] CSP is still NP-complete when $\delta(n) = \Omega(1)$, a constant. Since LH is the quantum generalization of a CSP we can similarly ask whether it is complete for the class QMA, the quantum version of NP. Kitaev showed that LH is QMA-complete for $\delta(n) = 1/\mathsf{poly}(n)$ when he originally defined the class of QMA problems [19]. Perhaps the most important open question in quantum complexity theory is whether or not a quantum version of the PCP theorem holds. The "quantum PCP conjecture" [3, 1] states that LH with a constant promise gap is QMA-hard; the conjecture has thus far eluded proof.

As a possible step towards proving quantum PCP, Freedman and Hastings suggested the No Low-energy Trivial States (NLTS) conjecture which is implied by the quantum PCP conjecture (assuming NP $\neq$ QMA). A local Hamiltonian has the NLTS property if there is a constant strictly larger than the ground-state energy which lower bounds the energy of any state preparable in constant-depth ("trivial states"). The NLTS conjecture posits the existence of an NLTS Hamiltonian. This seemingly simpler problem remained open for nearly a decade until Anshu and Breuckmann solved the combinatorial version [5], followed shortly after by a complete proof by Anshu, Breuckmann, and Nirkhe [4]. They explicitly constructed an NLTS Hamiltonian using recently developed asymptotically-good quantum LDPC codes [20].

While the NLTS Theorem makes significant progress, there are still many other properties that a candidate Hamiltonian must satisfy in order to be QMA-hard with a constant promise gap. For instance, Gharibian and Le Gall defined the No Low-energy Sampleable States (NLSS) conjecture [16]. A state, $|\psi\rangle$ is "sampleable" if a classical computer can efficiently draw an $x \in \{0,1\}^n$ from the distribution defined by $p(x) = |\langle x|\psi\rangle|^2$ and can calculate all of the amplitudes, $\langle x|\psi\rangle$.[4] A local Hamiltonian has the NLSS property if there is a constant which lower-bounds the energy of every sampleable state. The NLSS conjecture posits the existence of an NLSS Hamiltonian, and Gharibian and Le Gall showed that unless MA = QMA the quantum PCP conjecture implies the NLSS conjecture.

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[1] $\mathcal{H}_i = h_i \otimes \mathbb{I}_{2^{n-k}}$ where h_i is a $2^k \times 2^k$ Hermitian matrix and $\mathbb{I}_{2^{n-k}}$ is the $2^{n-k} \times 2^{n-k}$ identity matrix

[2] This is equivalent to deciding if $\mathcal{H}$ has an eigenvalue less than a or if all of the eigenvalues of $\mathcal{H}$ are larger than b, which is the more typical formulation of the problem.

[3] Technically LH is a generalization of the decision problem MAX-k-CSP.

[4] The more proper terminology, as in [16], would be that $|\psi\rangle$ has a succinct representation allowing perfect sampling access. We will not be directly addressing the NLSS conjecture, so we will use the term "sampleable" for brevity.

In this paper we examine a simplified version of the NLSS conjecture, where instead of sampleable states we consider stabilizer states. A stabilizer state is the unique state stabilized by a commuting subgroup of the Pauli group with size 2^n. Equivalently, stabilizer states are those states that can be prepared using only Clifford gates, i.e. Hadamard, Phase, and CNOT gates. We say that a local Hamiltonian has the No Low-energy Stabilizer States (NLCS)[5] property if there is a constant which lower-bounds the energy of any stabilizer state.[6] The Gottesman-Knill Theorem [18] shows that any stabilizer state can be efficiently sampled, so any NLSS Hamiltonian must also be an NLCS Hamiltonian. We show that a generic construction can be used to produce many NLCS Hamiltonians.

To prove the NLCS property for a particular local Hamiltonian one must show an explicit lower bound on the energy of all stabilizer states. Let $\mathcal{H} = \frac{1}{m}\sum \mathcal{H}_i$ be a local Hamiltonian and let $|\psi\rangle$ be an n-qubit state. The energy of any particular Hamiltonian term can be expressed as $\langle\psi| \mathcal{H}_i |\psi\rangle = \mathrm{Tr}\left[\psi_{A_i} h_i\right]$, where A_i is the set of qubits where $\mathcal{H}_i$ acts non-trivially, ψ_{A_i} is the reduced state of $|\psi\rangle$ on A_i, and h_i is the non-trivial part of $\mathcal{H}_i$. Suppose for simplicity that $|A_i| = k$ for all i. One particularly strong way to lower-bound the energy of $|\psi\rangle$ would be to "locally" bound each energy term. That is, prove that each $\mathrm{Tr}\left[\psi_{A_i} h_i\right]$ is lower-bounded by a constant. In general this is not an easy task. However, stabilizer states have a rather convenient property: we show in Claim 3 that if $|\psi\rangle$ is a stabilizer state, then every ψ_{A_i} is a convex combination of stabilizer states on k qubits. Thus, to lower-bound $\mathrm{Tr}\left[\psi_{A_i} h_i\right]$ for every n-qubit stabilizer state, $|\psi\rangle$, it is sufficient to lower-bound the quantity $\langle\zeta| h_i |\zeta\rangle$ for every *k-qubit* stabilizer state $|\zeta\rangle$.

This observation leads to a rather simple NLCS Hamiltonian. First, consider the Hamiltonian $\mathcal{H}_0 = \frac{1}{n}\sum |1\rangle\!\langle 1|_i$ where $|1\rangle\!\langle 1|_i$ is the projector to $|1\rangle$ on the i-th qubit and identity elsewhere. All of the local terms are the single-qubit projector $|1\rangle\!\langle 1|$. Clearly, we cannot lower-bound the energy of stabilizer states since $|0\rangle$ has energy 0. We can fix this, however, by instead considering a "conjugated" version of $\mathcal{H}_0$:

$$\tilde{\mathcal{H}}_0 \equiv \frac{1}{n}\sum_{i=1}^{n}\left(e^{i\frac{\pi}{8}Y}\,|1\rangle\!\langle 1|\,e^{-i\frac{\pi}{8}Y}\right)|_i,$$

which can alternatively be expressed as $\tilde{\mathcal{H}}_0 = (e^{i\frac{\pi}{8}Y})^{\otimes n}\mathcal{H}_0(e^{-i\frac{\pi}{8}Y})^{\otimes n}$. Each local term is the single-qubit projector $e^{i\frac{\pi}{8}Y}\,|1\rangle\!\langle 1|\,e^{-i\frac{\pi}{8}Y}$, and it is straightforward to calculate that every single-qubit stabilizer state has high energy under this local term. We give a self-contained proof that $\tilde{\mathcal{H}}_0$ is NLCS in Appendix B.

The quantum PCP conjecture not only implies the existence of NLTS/NLCS/NLSS Hamiltonians, but also the existence of *simultaneous* NLTS/NLCS/NLSS Hamiltonians. The process of conjugating a local Hamiltonian by a low-depth circuit conveniently preserves the NLTS property. That is, if $\mathcal{H}$ is NLTS and C is a constant-depth circuit, then $C^\dagger \mathcal{H} C$ is also NLTS (see Lemma 4).

We note that since $|1\rangle\!\langle 1| = \frac{1}{2}(\mathbb{I} - Z)$ the Hamiltonian $\mathcal{H}_0$ is an example of a *CSS Hamiltonian*, i.e. the local Hamiltonian terms are of the form $\frac{1}{2}(\mathbb{I} - P_i)$ where the P_i's are commuting X and Z type Pauli operators. As the Hamiltonian $\tilde{\mathcal{H}}_0$ is simply $\mathcal{H}_0$ conjugated

[5] The "C" in NLCS stands for Clifford, since states prepared by Clifford circuits and stabilizer states are equivalent.

[6] The existence of NLCS Hamiltonians has been suggested before as a direct consequence of the quantum PCP conjecture, for instance in [6]. We discuss the relationship of NLCS and more to the quantum PCP conjecture in Section 1.1.

by a depth-1 circuit $(e^{-i\frac{\pi}{8}Y})^{\otimes n}$ it may be natural to ask whether the same procedure can be done to the NLTS Hamiltonians from [4] as they are also CSS Hamiltonians. The main result of our paper is the following:

▶ **Theorem 1** (Informal version of Theorem 12). *Let $\mathcal{H}_{NLTS}$ be the NLTS local Hamiltonian from [4]. The local Hamiltonian given by $\tilde{\mathcal{H}}_{NLTS} \equiv (e^{i\frac{\pi}{8}Y})^{\otimes n}\mathcal{H}_{NLTS}(e^{-i\frac{\pi}{8}Y})^{\otimes n}$ satisfies both NLTS and NLCS.*

We prove Theorem 12 by exhibiting local lower bounds on the individual Hamiltonian terms. In particular, we show that if $h = \frac{1}{2}(\mathbb{I} - P^{\otimes k})$ is a k-local term where $P \in \{X, Z\}$, then

$$\langle\zeta| (e^{i\frac{\pi}{8}Y})^{\otimes k} h (e^{-i\frac{\pi}{8}Y})^{\otimes k} |\zeta\rangle \geq \sin^2(\pi/8)$$

for *every* k-qubit stabilizer state $|\zeta\rangle$, as long as k is odd. Combining this lower bound with the fact that the reduced state of a stabilizer state is a convex combination of stabilizer states, we have that conjugating a CSS Hamiltonian by $(e^{-i\frac{\pi}{8}Y})^{\otimes n}$ results in an NLCS Hamiltonian, at least in the case that many of the Hamiltonian terms act on an odd number of qubits.

The condition of odd weight is unfortunately a necessary condition of our local techniques: if k is even then there is always a k-qubit stabilizer state with $\langle\zeta_0| (e^{i\frac{\pi}{8}Y})^{\otimes k} h (e^{-i\frac{\pi}{8}Y})^{\otimes k} |\zeta_0\rangle = 0$. Nonetheless, we prove in Section 4 of the Full Version that there is an explicit NLTS Hamiltonian from [4] where every local term acts on an odd number of qubits. Since conjugating by a constant-depth circuit preserves NLTS, we ultimately have that $\tilde{\mathcal{H}}_{NLTS}$ satisfies both NLTS and NLCS.

1.1 Implications of the quantum PCP conjecture

We turn now to the question of what Hamiltonians are guaranteed to exist by the quantum PCP conjecture. The quantum PCP conjecture has two main formulations; we focus here on the gap amplification version. See [2] for a great survey on the conjecture.

▶ **Conjecture** (Conjecture 1.3 of [2]). *Let $\epsilon > 0$ be a constant.* LH-ϵ *is* QMA-hard *under quantum polynomial-time reductions.*

In other words, the conjecture says there is a worst-case local Hamiltonian whose ground state energy is QMA-hard to approximate within a constant. Approximating ground-state energies and finding ground states of local-Hamiltonians are of central importance to condensed matter theory and quantum simulation algorithms. If true, the quantum PCP conjecture says that there are some Hamiltonians whose ground-state energies we could never hope to approximate, let alone find their ground states.[7]

The key insight of [14] when they defined the NLTS conjecture was that some states have properties which allow their ground state energies to be calculated in a smaller complexity class than QMA. For a constant, k, we say that an n-qubit state, ρ, is ***k*-locally-approximable** if it has a polynomial-sized classical description from which every k-local reduced state, $\rho_A \equiv \mathrm{Tr}_{-A}[\rho]$ where $|A| \leq k$, can be approximated to inverse-polynomial precision in polynomial-time. Consider the following simple result:

▶ **Fact 2.** *Suppose $\mathcal{H} = \frac{1}{m}\sum_{i=1}^{m} \mathcal{H}_i$ is a k-local Hamiltonian and ρ is a k-locally approximable state. The energy of ρ under $\mathcal{H}$ can be approximated to inverse-polynomial precision in* NP.

[7] Unless, of course, one believes QMA $\subseteq$ P or some other weakening of QMA.

Proof. Each $\mathcal{H}_i$ acts non-trivially on at most k qubits, $A_i \subset [n]$, so the energy of ρ for $\mathcal{H}_i$ is $\mathrm{Tr}[\rho\mathcal{H}_i] = \mathrm{Tr}\left[\rho_{A_i} h_i\right]$, where $h_i \equiv \mathrm{Tr}_{-A_i}[\mathcal{H}_i]$ is the non-trivial part of $\mathcal{H}_i$. Since $h_i \in \mathbb{C}^{2^k \times 2^k}$ and by assumption we can efficiently compute ρ_{A_i} to inverse-polynomial precision from the classical description of ρ, each $\mathrm{Tr}[\rho\mathcal{H}_i]$ can be brute-force approximated in polynomial-time. ◀

Trivial states are locally approximable. If $|\psi\rangle$ is a trivial state then there is a constant-depth circuit such that $|\psi\rangle = C\,|0\rangle^{\otimes n}$. For a set of k qubits, A, the only gates that contribute to ψ_A are those in the reverse-lightcone[8] of A. As the reverse-lightcone has size at most $k2^d$, a constant, only a constant number of gates from C are needed to brute-force approximate ψ_A. Thus, we can approximate local reduced states of $|\psi\rangle\langle\psi|$ from the classical description of C.

The assumption of being able to compute local reduced states also holds for stabilizer states. Suppose $|\psi\rangle$ is an n-qubit stabilizer state. Since $|\psi\rangle$ is a stabilizer state there are n independent and commuting Pauli operators $\{P_1, \ldots, P_n\}$ that stabilize $|\psi\rangle$. The list of these Pauli operators will serve as the classical description of $|\psi\rangle\langle\psi|$ from which local reduced states can be computed. The reduced state ψ_A can be written as

$$\psi_A = \frac{1}{2^k} \sum_{P \in G_A} P, \tag{1}$$

where G_A is the subgroup of the stabilizers of $|\psi\rangle$ which act non-trivially only on qubits in A. There are 4^k such Pauli group elements (ignoring phases) which we denote by $\mathcal{P}_A$. For $P \in \mathcal{P}_A$, one of $\pm P$ is in the stabilizer group of $|\psi\rangle$ if and only if P commutes with every stabilizer generator. So, we can determine the elements of G_A by brute-force checking which elements of $\mathcal{P}_A$ commute with every generator.[9] This computation can be done in polynomial-time since there are only a constant number of Pauli operators to check, so using Equation (1) we can compute ψ_A efficiently.

Thus, in addition to being an implication of NLSS, NLCS Hamiltonians are also implied by the quantum PCP conjecture assuming $\mathsf{NP} \neq \mathsf{QMA}$: if every local Hamiltonian has a low-energy stabilizer state then the ground state energy could be computed in NP via Fact 2.

2 Preliminaries

For a natural number, n, we denote $[n] \equiv \{1, \ldots, n\}$. For a subset, $A \subseteq [n]$, we denote the set complement by $-A \equiv [n] \setminus A$ and the partial trace over the qubits in A by Tr_A. In particular, $\mathrm{Tr}_{-A}[|\psi\rangle\langle\psi|]$ denotes the local density matrix of $|\psi\rangle$ on the qubits in A.

2.1 States

Let $C = \{C_n\}$ be a countable family of quantum circuits consisting of one and two-qubits gates where each C_n acts on n qubits. If the depth of C_n is upper bounded by a function $d(n)$ for all n, then we say C is a **depth-$d(n)$** family of quantum circuits. If $d(n) = \mathcal{O}(1)$ then we say C is a depth-$\mathcal{O}(1)$ (or constant-depth) family of quantum circuits. Similarly, if $d(n) = \mathsf{poly}(n)$ then we say C is a depth-$\mathsf{poly}(n)$ (or polynomial-sized) family of quantum circuits.

[8] See Figure 1(a).

[9] It remains to determine whether $+P$ or $-P$ is in the stabilizer group. Although slightly more complicated, this can be done in polynomial-time *independent* of the weight of P.

The single-qubit **Pauli group** is the set $\mathcal{P}_1 \equiv \{i^\ell P \mid P \in \{\mathbb{I}, X, Y, Z\}, \ell \in \{0,1,2,3\}\}$, and the n-qubit Pauli group is its n-fold tensor-power, $\mathcal{P}_n = \bigotimes_{i \in [n]} \mathcal{P}_1$. For an element $S = P_1 \otimes \cdots \otimes P_n \in \mathcal{P}_n$, the **weight** of S is defined to be the number of qubits where P_i is not identity, i.e. $\mathrm{wt}(S) = \left|\{P_i \mid P_i \neq i^\ell\, \mathbb{I}\}\right|$. We denote the set of these qubits where S acts non-trivially by $\mathrm{N}(S) \subseteq [n]$.

The n-qubit **Clifford group**, $\mathcal{C}_n$, is the set of unitary operators which stabilize the Pauli group.It is well-known that $\mathcal{C}_n$ is generated by the set $\{\mathrm{H}, \mathrm{P}, \mathrm{CNOT}\}$, where H is the single-qubit Hadamard gate, P is the single-qubit phase gate, and CNOT is the two-qubit controlled-NOT gate. A **Clifford circuit** is defined to be any element of the Clifford group.

Let ψ be a [possibly mixed] state on n qubits and let $N \geq n$. If there is a quantum circuit, C, acting on N qubits such that $\psi = \mathrm{Tr}_N[C \left|0^N\middle\rangle\!\middle\langle 0^N\right| C^\dagger]$ then we say that C **prepares** ψ. ψ is said to be: a **trivial state** if there is a constant-depth quantum circuit preparing it, an [efficiently] **preparable state** if there is a polynomial-sized circuit preparing it, a **Clifford state** if there is a polynomial-sized Clifford circuit preparing it, and an **almost Clifford state** if there is a polynomial-sized quantum circuit containing Clifford + $\mathcal{O}(\log(n))$ T-gates preparing it. A pure state, $|\psi\rangle$ is said to be a **sampleable state** if (1) there is a classical algorithm exactly computing $\langle x|\psi\rangle$ for every $x \in \{0,1\}^n$ and (2) there is a classical algorithm that exactly samples $x \in \{0,1\}^n$ from the distribution $p(x) = |\langle x|\psi\rangle|^2$.

A **stabilizer group** is an abelian subgroup, G, of $\mathcal{P}_n$ not containing $-\mathbb{I}$. As a finite group, we can always find a list of mutually independent and commuting generators, $\mathcal{S} = \{S_1, \ldots, S_k\}$, of G. We will refer simply to the subgroup $\langle \mathcal{S} \rangle = G$ when this generating set is clear. Note that given a stabilizer group, there is a well-defined **stabilizer code** [17, 12, 13], $\mathcal{C}_\mathcal{S}$, which is the common $+1$ eigenspace of the operators in $\langle \mathcal{S} \rangle$.

If a given stabilizer group has a generating set, $\mathcal{S}$, consisting of tensor products of only Pauli X and $\mathbb{I}$ or only Pauli Z and $\mathbb{I}$, then we say $\mathcal{C}_\mathcal{S}$ is a **CSS code** and that $\mathcal{S}$ generates a CSS code.

The **stabilizer group** of a pure state, $|\psi\rangle$, is the subgroup of the Pauli group defined by $\mathrm{Stab}(|\psi\rangle) \equiv \{P \in \mathcal{P}_n \mid P|\psi\rangle = |\psi\rangle\}$. We say that a $P \in \mathrm{Stab}(|\psi\rangle)$ **stabilizes** $|\psi\rangle$. Note that $\mathrm{Stab}(|\psi\rangle)$ is an abelian subgroup of the Pauli group not containing $-\mathbb{I}$, and so it is a valid stabilizer group as before.

A pure state, $|\psi\rangle$, is said to be a **stabilizer state** if $|\mathrm{Stab}(|\psi\rangle)| = 2^n$, or equivalently, if there are n independent Pauli operators that stabilize $|\psi\rangle$. We note that $|\psi\rangle\!\langle\psi| = \frac{1}{2^n}\sum_{g \in G} g$ where $G = \mathrm{Stab}(|\psi\rangle)$.

A mixed state, ψ, is said to be a stabilizer state if ψ is a convex combination of pure stabilizer states, i.e. $\psi = \sum_j p_j |\varphi_j\rangle\!\langle\varphi_j|$ where each $|\varphi_j\rangle$ is a pure stabilizer state on n qubits, $\sum_j p_j = 1$, and $p_j \geq 0$.

All of the states defined here are related to one another via the following:

$$\begin{tikzcd}
\text{trivial} \arrow[dd, Rightarrow, "\text{increase depth}"] & \text{Clifford/Stabilizer} \arrow[d, Rightarrow, "\text{some T gates}"] & \\
& \text{almost Clifford} \arrow[dl, Rightarrow, "\text{arbitrary T gates}"] \arrow[dr, Rightarrow, "\text{[11]}"] & \\
\text{preparable} & & \text{sampleable}
\end{tikzcd} \tag{2}$$

By definition of the Clifford group, stabilizer states and Clifford states are equivalent for pure states. We will interchangeably use the terms "stabilizer state" and "Clifford state" even for mixed states, which is motivated by the following result:

▷ Claim 3. If ψ is a Clifford state, then it is also a stabilizer state.

A proof can be found in Appendix A.1. Claim 3 says that the reduced state of a pure stabilizer state is a convex combination of pure stabilizer states on the subsystem. This is essential in our energy lower bound arguments: To lower-bound the energy of all n-qubit stabilizer states for a k-local term of the Hamiltonian, $\mathcal{H}_i$, it is sufficient to lower-bound the energy of all k-qubit stabilizer states for the non-trivial part of $\mathcal{H}_i$.

2.2 Hamiltonians

A ***k*-local Hamiltonian**, $\mathcal{H}^{(n)}$, is a Hermitian operator on the space of n qubits, $(\mathbb{C}^2)^{\otimes n}$, which can be written as a sum $\mathcal{H}^{(n)} = \frac{1}{m}\sum_{i=1}^{m}\mathcal{H}_i$, where each $\mathcal{H}_i$ is a Hermitian matrix acting non-trivially on only k qubits and with spectral norm $\|\mathcal{H}_i\| \leq 1$. A **family of *k*-local Hamiltonians**, $\{\mathcal{H}^{(n)}\}$, is a countable set of k-local Hamiltonians indexed by system size, n, where $k = \mathcal{O}(1)$ and $m = \mathsf{poly}(n)$. We will often use the term "local Hamiltonian" to mean a family of k-local Hamiltonians.

The **ground-state energy** of $\mathcal{H}$ is $E_0 \equiv \min_\rho \mathrm{Tr}[\rho\mathcal{H}]$, where the minimization is taken over all n-qubit mixed states. $\mathcal{H}$ is said to be **frustration-free** if $E_0 = 0$. A state, ψ, is said to be a **ground state** of $\mathcal{H}$ if $\mathrm{Tr}[\psi\mathcal{H}] = E_0$. A state, ψ, is said to be an **ϵ-low-energy state** of $\mathcal{H}$ if $\mathrm{Tr}[\psi\mathcal{H}] < E_0 + \epsilon$. If $\psi = |\psi\rangle\langle\psi|$ is a pure state, this condition simplifies to $\langle\psi|\,\mathcal{H}\,|\psi\rangle < \lambda_{\min}(\mathcal{H}) + \epsilon$, where $\lambda_{\min}(\mathcal{H})$ is the smallest eigenvalue of $\mathcal{H}$. For frustration-free Hamiltonians this is equivalent to $\langle\psi|\,\mathcal{H}\,|\psi\rangle < \epsilon$. All of the Hamiltonians we consider will be frustration-free.

For $S \in \mathcal{P}_n$, we denote the orthogonal projector to the $+1$ eigenspace of S by Π_S, i.e. $\Pi_S \equiv \frac{\mathbb{I}-S}{2}$. Since Π_S acts non-trivially on only $\mathrm{wt}(S)$ qubits, we can write $\Pi_S = \Pi_S\,|_{\mathrm{N}(S)} \otimes \mathbb{I}_{[n]\setminus\mathrm{N}(S)}$.

Given a stabilizer group, $\langle\mathcal{S}\rangle$, with generating set $\mathcal{S}$, the **stabilizer Hamiltonian** associated to $\mathcal{S}$ is $\mathcal{H}_{\mathcal{S}} \equiv \frac{1}{|\mathcal{S}|}\sum_{S\in\mathcal{S}}\Pi_S$. If each qubit is acted on non-trivially by at most $\mathrm{wt}(\mathcal{S})$ elements of $\mathcal{S}$, then $\mathcal{H}_{\mathcal{S}}$ is a $\mathrm{wt}(\mathcal{S})$-local Hamiltonian. If $\mathcal{C}$ is the Stabilizer code associated with $\mathcal{S}$, then every $|\psi\rangle \in \mathcal{C}$ is a zero-energy state of $\mathcal{H}_{\mathcal{S}}$. In particular, $\mathcal{H}_{\mathcal{S}}$ is frustration-free with ground-state space $\mathcal{C}$. If $\mathcal{S}$ generates a CSS code then we say $\mathcal{H}_{\mathcal{S}}$ is a **CSS Hamiltonian**.

If $\{\langle\mathcal{S}_n\rangle \mid \langle\mathcal{S}_n\rangle \leq \mathcal{P}_n\}$ is a countable family of stabilizer groups then the **family of stabilizer (or CSS) Hamiltonians** associated with $\{\mathcal{S}_n\}$ is $\{\mathcal{H}_{\mathcal{S}_n}\}$. This will be a family of local Hamiltonians when: (1) each qubit is acted on non-trivially by at most $\mathrm{wt}(\mathcal{S}_n)$ elements of $\mathcal{S}_n$, (2) $\mathrm{wt}(\mathcal{S}_n) = \mathcal{O}(1)$, and (3) $|\mathcal{S}_n| = \Theta(n)$. Such families, $\{\langle\mathcal{S}_n\rangle\}$, of stabilizer groups correspond to quantum LDPC code families.

For each of the states in the previous section we can consider an analogue of NLTS.

▶ **Definition.** *A family of k-local Hamiltonians, $\{\mathcal{H}^{(n)}\}$, is said to have the **ϵ-NLXS** property if for all sufficiently large n, $\mathcal{H}^{(n)}$ has no ϵ-low-energy states of type X. The family, $\{\mathcal{H}^{(n)}\}$, is said to have the **NLXS** property if it is ϵ-NLXS for some constant ϵ.*

The following implications between the NLXS theorems/conjectures and quantum PCP conjecture hold. A complexity inequality next to an arrow denotes an implication that holds if the separation is true, e.g. if the quantum PCP conjecture is true and $\mathsf{MA} \neq \mathsf{QMA}$, then NLSS is true.

$$\begin{tikzcd}
 & \text{qPCP conjecture} \arrow[dl, Rightarrow, "\mathsf{QCMA}\neq\mathsf{QMA}"'] \arrow[dr, Rightarrow, "\mathsf{MA}\neq\mathsf{QMA}[16]"] \arrow[dd, Rightarrow] \arrow[dddl, Rightarrow, "\mathsf{NP}\neq\mathsf{QMA}"] & \\
\text{NLPS} \arrow[dd, Rightarrow] \arrow[dr, Rightarrow] & & \text{NLSS} \arrow[dl, Rightarrow] \\
 & \text{NLACS} \arrow[d, Rightarrow] & \\
\text{NLTS} & \text{NLCS} &
\end{tikzcd} \tag{3}$$

The relationships between each of the NLXS results are implicitly given by Diagram 2. Trivial states, stabilizer states, and almost Clifford states are all examples of locally-approximable states, so they following from the quantum PCP conjecture via Fact 2. The implication of NLSS was given by Gharibian and Le Gall when they originally defined NLSS [16]. The implication of NLPS is well-known: if every local Hamiltonian has a low-energy preparable state, $C\,|0\rangle^{\otimes n}$, then given the classical description of C a quantum prover could simply prepare the state and measure its energy. This would put LH-$\epsilon \in \mathsf{QCMA}$, implying $\mathsf{QMA} = \mathsf{QCMA}$ if the quantum PCP conjecture is true.

For a family of k-local Hamiltonians, $\{\mathcal{H}^{(n)}\}$, and a family, $C = \{C_n\}$, of depth-$\mathcal{O}(1)$ quantum circuits, we define the ***C*-rotated version** of $\{\mathcal{H}^{(n)}\}$ as $\{\mathcal{H}^{(n)}\}^C \equiv \{{C_n}^\dagger \mathcal{H}^{(n)} C_n\}$. This is still a family of local Hamiltonians, albeit with a possibly different k than the original Hamiltonian. This is because the only qubits that interact non-trivially with a single Hamiltonian term, $C^\dagger \mathcal{H}_i C$, are those qubits in the reverse-lightcone of the qubits acted on by $\mathcal{H}_i$. The number of qubits in the reverse-lightcone of a single qubit grows exponentially in the depth of a circuit, which is still constant since C is constant-depth. See Figure 1 for an example of this. When $C = V^{\otimes n}$ is the tensor-product of a single-qubit gate, V, we will use the term "V-rotated" as opposed to "$V^{\otimes n}$-rotated".

The utility of considering a C-rotated Hamiltonian is that in addition to preserving locality, the NLTS property is also preserved.

▶ **Lemma 4.** *If $\{\mathcal{H}^{(n)}\}$ is a family of ϵ_0-NLTS local Hamiltonians and $C = \{C_n\}$ is a family of constant-depth circuits, then $\{\mathcal{H}^{(n)}\}^C$ is also ϵ_0-NLTS.*

Proof. Suppose that $\{\mathcal{H}^{(n)}\}^C$ is not NLTS. By definition, for every $\epsilon > 0$ there is an n and constant-depth circuit $U_{\epsilon,n}$ such that $U_{\epsilon,n}\,|0\rangle^{\otimes n}$ is an ϵ-low-energy state of $C_n^\dagger \mathcal{H}^{(n)} C_n$, i.e.

$$\langle 0|^{\otimes n}\, U_{\epsilon,n}^\dagger C_n^\dagger \mathcal{H}^{(n)} C_n U_{\epsilon,n}\,|0\rangle^{\otimes n} < \lambda_{\min}(C_n^\dagger \mathcal{H}^{(n)} C_n) + \epsilon.$$

Since C_n is a unitary operator the minimum eigenvalues of $\mathcal{H}^{(n)}$ and $C_n^\dagger \mathcal{H}^{(n)} C_n$ are equal. Defining $|\psi_{\epsilon_0,n}\rangle \equiv C_n U_{\epsilon_0,n}\,|0\rangle^{\otimes n}$ we have

$$\langle \psi_{\epsilon_0,n}|\, \mathcal{H}^{(n)}\, |\psi_{\epsilon_0,n}\rangle < \lambda_{\min}(\mathcal{H}^{(n)}) + \epsilon_0,$$

i.e. $|\psi_{\epsilon_0,n}\rangle$ is an ϵ_0-low-energy state of $\mathcal{H}^{(n)}$. Since $C_n U_{\epsilon_0,n}$ is a constant-depth circuit this implies that $\mathcal{H}^{(n)}$ has a low-energy trivial state, contradicting the assumption of ϵ_0-NLTS. ◀

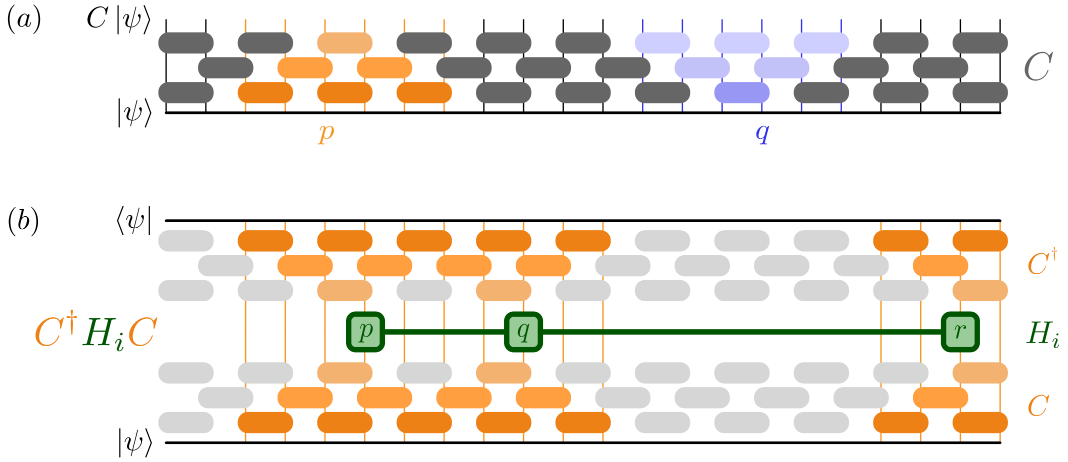

Figure 1 (a) Consider a constant-depth circuit, C. The [blue] highlighted gates on the right of the figure represent the "lightcone" of qubit q, i.e. the set of gates that can be traced back to q. The [orange] highlighted gates on the left of the figure represent the gates in the "reverse-lightcone" of qubit p, i.e. the gates that will ultimately affect p.
(b) Consider a single k-local Hamiltonian term, $\mathcal{H}_i$, that acts only on qubits p, q, and r. When conjugating $\mathcal{H}_i$ with C, any gate not in the reverse-lightcone of one of p, q, or r will cancel with its inverse. The number of qubits in the reverse-lightcone of any one qubit is $\leq 2^d$ where d is the depth of C, so $C^\dagger \mathcal{H}_i C$ will be at most $k2^d$-local. Note that we have only drawn a 2D geometrically-local circuit here, whereas this upper bound holds for a constant-depth circuit with arbitrary connectivity.

3 NLCS from CSS codes

We will show that rotating by the tensor product of a single-qubit gate is sufficient to turn most CSS Hamiltonians into NLCS Hamiltonians, including the quantum Tanner codes used in [4]. In particular, we consider the single-qubit gate $D \equiv e^{-i\frac{\pi}{8}Y}$ and rotate a CSS Hamiltonian by $D^{\otimes n}$. For a local Hamiltonian, $\mathcal{H}^{(n)}$, we will denote its ***D*-rotated version** by $\tilde{\mathcal{H}}^{(n)} \equiv D^{\dagger\otimes n}\mathcal{H}^{(n)}D^{\otimes n}$. We denote the ***D*-rotated projector** associated with a Pauli element $S \in \mathcal{P}_n$ by $\tilde{\Pi}_S \equiv D^{\otimes n}\Pi_S D^{\dagger\otimes n}$. By definition, we have $\tilde{\Pi}_S = \tilde{\Pi}_S\mid_{\mathrm{N}(S)} \otimes \mathbb{I}_{[n]\setminus\mathrm{N}(S)}$, where $\tilde{\Pi}_S\mid_{\mathrm{N}(S)} = D^{\otimes\,\mathrm{wt}(S)}\Pi_S\mid_{\mathrm{N}(S)} D^{\dagger\otimes\mathrm{wt}(S)}$. Note that we have not explicitly included D in the above notations since D will refer exclusively to $e^{-i\frac{\pi}{8}Y}$, throughout.

We have the following result:

▶ **Theorem 5.** *Let $\{\mathcal{H}_{\mathcal{S}_n}\}$ be a family of CSS Hamiltonians associated with a family of quantum (CSS) LDPC codes, $\{\langle\mathcal{S}_n\rangle\}$. Suppose for every n a constant fraction, $\alpha > 0$, of the generators $S \in \mathcal{S}_n$ have odd weight. Then $\{\tilde{\mathcal{H}}_{\mathcal{S}_n}\}$ is a family of NLCS Hamiltonians.*

We prove this by giving local lower bounds on the energies of D-rotated projectors associated with CSS generators. As a technical requirement, these lower bounds only hold when the weight of a generator is odd.

Recall that, up to a permutation of the qubits, the generators of a CSS code can be written as either $\bar{X}\otimes\mathbb{I}$ or $\bar{Z}\otimes\mathbb{I}$, where $\bar{X} \equiv X^{\otimes k}$ and $\bar{Z} \equiv Z^{\otimes k}$. First consider what happens to the projectors $\Pi_{\bar{X}}$ and $\Pi_{\bar{Z}}$ when rotating by D:

▷ Claim 6.

$$\tilde{\Pi}_{\bar{X}} = \frac{\mathbb{I} - \mathrm{H}^{\otimes k}}{2}, \qquad \tilde{\Pi}_{\bar{Z}} = \frac{\mathbb{I} - (-X\,\mathrm{H}\,X)^{\otimes k}}{2}.$$

These identities are derived in Appendix A.2. The local lower bounds will be a result of the following:

▶ **Lemma 7.** *If k is odd, then for every k-qubit stabilizer state, $|\eta\rangle$, we have $\left|\langle\eta|\,\mathrm{H}^{\otimes k}\,|\eta\rangle\right| \leq \frac{1}{\sqrt{2}}$. On the other hand, if k is even then there exists a k-qubit stabilizer state, $|\eta_0\rangle$, with $\langle\eta_0|\,\mathrm{H}^{\otimes k}\,|\eta_0\rangle = 1$.*

The proof will use the following result on the geometry of stabilizer states:

▶ **Fact 8** (Corollary 3 of [15]). *Let $|\zeta\rangle, |\xi\rangle$ be two stabilizer states. If $|\langle\zeta|\xi\rangle| \neq 1$, then $|\langle\zeta|\xi\rangle| \leq \frac{1}{\sqrt{2}}$.*

Proof of Lemma 7. Since H is a Clifford gate, $\mathrm{H}^{\otimes k}\,|\eta\rangle$ is a stabilizer state. We will show that $\left|\langle\eta|\,\mathrm{H}^{\otimes k}\,|\eta\rangle\right| \neq 1$ in the case of odd k, which by Fact 8 will imply the bound.

Recall that $|\eta\rangle\langle\eta| = \frac{1}{|G|}\sum_{g\in G} g$, where $G \equiv \mathrm{Stab}(|\eta\rangle)$. We have two cases:

(1) (Every $S \in G$ contains an $\mathbb{I}$ or a Y in some position) In this case, we calculate

$$\begin{aligned}
\langle\eta|\,\mathrm{H}^{\otimes k}\,|\eta\rangle &= \mathrm{Tr}\Big[|\eta\rangle\langle\eta|\,\mathrm{H}^{\otimes k}\Big], \\
&= \frac{1}{|G|}\sum_{g\in G}\mathrm{Tr}\Big[g\,\mathrm{H}^{\otimes k}\Big], \\
&= \frac{1}{|G|}\sum_{g\in G}\prod_{i\in[k]}\mathrm{Tr}[g_i\,\mathrm{H}], \\
&= 0,
\end{aligned}$$

where the last line follows since $g_j \in \{\mathbb{I}, Y\}$ for some j, and $\mathrm{Tr}[\mathrm{H}] = \mathrm{Tr}[Y\,\mathrm{H}] = 0$.

(2) (There is an $S \in G$ which consists of only X's and Z's) Consider the case when k is odd. Since $\mathrm{wt}(S) = k$, S contains either (1) an odd number of X's and an even number of Z's or (2) an even number of X's and an odd number of Z's. We focus on the former situation; the latter is similar.
Note that $\left|\langle\eta|\,\mathrm{H}^{\otimes k}\,|\eta\rangle\right| = 1$ if and only if $\mathrm{H}^{\otimes k}\,|\eta\rangle$ and $|\eta\rangle$ have the same stabilizer group. Since S stabilizes $|\eta\rangle$, $\mathrm{H}^{\otimes k}\,S\,\mathrm{H}^{\otimes k}$ stabilizes $\mathrm{H}^{\otimes k}\,|\eta\rangle$. We know how H conjugates Pauli operators: $X \mapsto Z$, $Z \mapsto X$, and $Y \mapsto -Y$. By assumption, S has an odd number of X's and an even number of Z's, so $\mathrm{H}^{\otimes k}\,S\,\mathrm{H}^{\otimes k}$ will have an even number of X's and an odd number of Z's. Therefore, we have that $S\cdot(\mathrm{H}^{\otimes k}\,S\,\mathrm{H}^{\otimes k}) = -(\mathrm{H}^{\otimes k}\,S\,\mathrm{H}^{\otimes k})\cdot S$, which implies S and $\mathrm{H}^{\otimes k}\,S\,\mathrm{H}^{\otimes k}$ cannot both be elements of the same stabilizer group. Hence, $\mathrm{Stab}(|\eta\rangle) \neq \mathrm{Stab}(\mathrm{H}^{\otimes k}\,|\eta\rangle)$ and $\left|\langle\eta|\,\mathrm{H}^{\otimes k}\,|\eta\rangle\right| \neq 1$.

Since in both cases $\left|\langle\eta|\,\mathrm{H}^{\otimes k}\,|\eta\rangle\right| \neq 1$, by Fact 8 we must have that $\left|\langle\eta|\,\mathrm{H}^{\otimes k}\,|\eta\rangle\right| \leq \frac{1}{\sqrt{2}}$ when k is odd. We note that the above proof will not work for even k, since it can be the case that all stabilizers have an even number of X's and Z's (or both odd). In this case $\mathrm{H}^{\otimes k}$ will be in the normalizer of G, and the two stabilizer groups may be equal.

We can easily find an example with even k where no non-trivial upper bound can be found. Note that $|\Phi^+\rangle \equiv \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is a $+1$ eigenstate of $\mathrm{H}^{\otimes 2}$, so for even k define $|\eta_0\rangle \equiv |\Phi^+\rangle^{\otimes k/2}$. ◀

We can now prove the local lower bound on odd-weight CSS generators.

▶ **Lemma 9.** *For every k-qubit stabilizer state, $|\eta\rangle$, $\langle\eta|\,\tilde{\Pi}_{\bar{X}}\,|\eta\rangle \geq c_k$ and $\langle\eta|\,\tilde{\Pi}_{\bar{Z}}\,|\eta\rangle \geq c_k$, where $c_k = 0$ if k is even and $c_k = \sin^2(\frac{\pi}{8})$ if k is odd.*

Proof. Let $|\eta\rangle$ be a k-qubit stabilizer state. We first consider $\langle\eta|\,\tilde{\Pi}_{\bar{X}}\,|\eta\rangle$:

$$\langle\eta|\,\tilde{\Pi}_{\bar{X}}\,|\eta\rangle \equiv \langle\eta|\,D^{\dagger\otimes k}\left(\frac{\mathbb{I}-\bar{X}}{2}\right)D^{\otimes k}\,|\eta\rangle\,, \tag{4}$$

$$\text{(By Claim 6)} \qquad = \langle\eta|\,\frac{\mathbb{I}-\mathrm{H}^{\otimes k}}{2}\,|\eta\rangle \tag{5}$$

$$= \frac{1}{2}\Big(1-\langle\eta|\,\mathrm{H}^{\otimes k}\,|\eta\rangle\Big)\,. \tag{6}$$

The bound follows from Lemma 7, since $\sin^2(\frac{\pi}{8}) = \frac{1}{2}(1-\frac{1}{\sqrt{2}})$.

For $\langle\eta|\,\tilde{\Pi}_{\bar{Z}}\,|\eta\rangle$, we have:

$$\langle\eta|\,\tilde{\Pi}_{\bar{Z}}\,|\eta\rangle \equiv \langle\eta|\,D^{\dagger\otimes k}\left(\frac{\mathbb{I}-\bar{Z}}{2}\right)D^{\otimes k}\,|\eta\rangle\,, \tag{7}$$

$$\text{(By Claim 6)} \qquad = \langle\eta|\,\frac{\mathbb{I}-(-X\,\mathrm{H}\,X)^{\otimes k}}{2}\,|\eta\rangle \tag{8}$$

$$= \frac{1}{2}\left(1-\langle\eta|\,(-X\,\mathrm{H}\,X)^{\otimes k}\,|\eta\rangle\right), \tag{9}$$

$$= \frac{1}{2}\Big(1-(-1)^k\,\langle\zeta|\,\mathrm{H}^{\otimes k}\,|\zeta\rangle\Big) \tag{10}$$

where $|\zeta\rangle \equiv X^{\otimes k}\,|\eta\rangle$ is another stabilizer state since $X = X^\dagger$ is in the Clifford group. The bound follows again from Lemma 7. ◀

Lemma 9 implies the following lower bound for n-qubit stabilizer states.

▶ **Lemma 10.** *Let $S \in \mathcal{P}_n$ be a tensor product of only Pauli X and $\mathbb{I}$ or only Pauli Z and $\mathbb{I}$. Denote $k = \mathrm{wt}(S)$. For every n-qubit stabilizer state, $|\eta\rangle$, $\langle\eta|\,\tilde{\Pi}_S\,|\eta\rangle \geq c_k$.*

Proof. Recall that $\tilde{\Pi}_S = \tilde{\Pi}_S\,|_{\mathrm{N}(S)} \otimes \mathbb{I}_{[n]\setminus\mathrm{N}(S)}$, so

$$\langle\eta|\,\tilde{\Pi}_S\,|\eta\rangle = \mathrm{Tr}\big[\eta_{\mathrm{N}(S)}\tilde{\Pi}_S\,|_{\mathrm{N}(S)}\big], \tag{11}$$

where $\eta_{\mathrm{N}(S)} \equiv \mathrm{Tr}_{-\mathrm{N}(S)}[|\eta\rangle\!\langle\eta|]$ is the reduced state of $|\eta\rangle$ on $\mathrm{N}(S) \subset [n]$. Since $\eta_{\mathrm{N}(S)}$ is the reduced state of a Clifford state, by Claim 3 there are pure stabilizer states on k qubits, $\{|\eta_j\rangle\}$ such that $\eta_{\mathrm{N}(S)} = \sum_j p_j\,|\eta_j\rangle\!\langle\eta_j|$. The lower bound follows by applying Lemma 9 to each $\langle\eta_j|\,\tilde{\Pi}_S\,|_{\mathrm{N}(S)}\,|\eta_j\rangle$. ◀

We can now prove Theorem 5.

▶ **Theorem 5.** *Let $\{\mathcal{H}_{\mathcal{S}_n}\}$ be a family of CSS Hamiltonians associated with a family of quantum (CSS) LDPC codes, $\{\langle\mathcal{S}_n\rangle\}$. Suppose for every n a constant fraction, $\alpha > 0$, of the generators $S \in \mathcal{S}_n$ have odd weight. Then $\{\tilde{\mathcal{H}}_{\mathcal{S}_n}\}$ is a family of NLCS Hamiltonians.*

Proof. By definition, $\tilde{\mathcal{H}}_{\mathcal{S}_n} = \frac{1}{|\mathcal{S}_n|}\sum_{S\in\mathcal{S}_n}\tilde{\Pi}_S$ where $\tilde{\Pi}_S$ is the D-rotated projector associated with $S \in \mathcal{S}_n$. Let ψ be a stabilizer state on n qubits. We will directly lower-bound the energy of ψ.

By definition, $\psi = \sum_j p_j\,|\varphi_j\rangle\!\langle\varphi_j|$, where each $|\varphi_j\rangle$ is a pure stabilizer state on n qubits. We have:

$$
\begin{aligned}
\operatorname{Tr}\big[\psi\tilde{\mathcal{H}}_{\mathcal{S}_n}\big] &= \sum_j p_j \langle\varphi_j|\, \tilde{\mathcal{H}}_{\mathcal{S}_n} \,|\varphi_j\rangle\,, && (12)\\
&= \frac{1}{|\mathcal{S}_n|}\sum_{S\in\mathcal{S}_n}\sum_j p_j \langle\varphi_j|\, \tilde{\Pi}_S \,|\varphi_j\rangle\,, && (13)\\
\text{(By Lemma 10)} \qquad &\geq \frac{1}{|\mathcal{S}_n|}\sum_{S\in\mathcal{S}_n} c_{\mathrm{wt}(S)} \sum_j p_j, && (14)\\
\text{(Definition of } c_k\text{)} \qquad &= \frac{1}{|\mathcal{S}_n|}\sum_{S\in\mathcal{S}_n:\mathrm{wt}(S),\ \mathrm{odd}} \sin^2\left(\frac{\pi}{8}\right), && (15)\\
&= \alpha\sin^2\left(\frac{\pi}{8}\right), && (16)
\end{aligned}
$$

where the last line follows by assumption $\alpha|\mathcal{S}_n|$ terms of $\mathcal{S}_n$ have odd weight. Since this holds for all stabilizer states, ψ, we have that $\{\tilde{\mathcal{H}}_{\mathcal{S}_n}\}$ is ϵ-NLCS with $\epsilon = \alpha\sin^2(\frac{\pi}{8}) = \Omega(1)$. ◀

We now turn to our main result, the existence of a simultaneous NLTS and NLCS family of local Hamiltonians. Recall the NLTS result of [4]:

▶ **Theorem** (Theorem 5 of [4], simplified). *There exists a constant $\epsilon_0 > 0$ and an explicit family of CSS Hamiltonians associated with a family of quantum LDPC codes, $\{\langle\mathcal{S}_n\rangle\}$, which is ϵ_0-NLTS.*

In order to use our Theorem 5, we require that a constant fraction of the stabilizer generators in $\mathcal{S}_n$ have an odd weight. It is not immediately clear that this would be true for the quantum Tanner codes from [20]. However, we have the following result:

▷ Claim 11. There exists an explicit family of CSS codes satisfying the conditions of Theorem 5 of [4] such that every stabilizer-generator has odd weight.

Section 4 of the Full Version is dedicated to proving Claim 11. The proof is rather straightforward and relies on the random choice of local codes in the construction of quantum Tanner codes. Essentially, we show that the local codes of the two component *classical* Tanner codes of a quantum Tanner code can be chosen such that all of the parity-checks of the global codes have odd weight. This implies that all of the stabilizer-generators of the quantum Tanner code also have odd weight.

With Claim 11, we are now prepared to prove the main result of our paper.

▶ **Theorem 12.** *Let $\{\mathcal{H}^{(n)}\}$ be the family of CSS Hamiltonians from Claim 11. The D-rotated version, $\{\tilde{\mathcal{H}}^{(n)}\}$, is a family of simultaneous NLTS and NLCS local Hamiltonians.*

Proof. Since $\{\mathcal{H}^{(n)}\}$ satisfies the conditions of Theorem 5 of [4] it is a valid local Hamiltonian, and it is ϵ_0-NLTS for some constant $\epsilon_0 > 0$. Since $D^{\otimes n}$ is a depth-$\mathcal{O}(1)$ circuit by Lemma 4 the D-rotated family $\{\tilde{\mathcal{H}}^{(n)}\}$ is also ϵ_0-NLTS.

By Claim 11, all of the stabilizer terms of $\mathcal{H}^{(n)}$ have odd weight for every n. Thus, by Theorem 5 $\{\tilde{\mathcal{H}}^{(n)}\}$ is ϵ_1-NLCS for $\epsilon_1 \equiv \sin^2(\frac{\pi}{8})$. Letting $\epsilon' \equiv \min\{\epsilon_0, \epsilon_1\}$, we have that $\{\tilde{\mathcal{H}}^{(n)}\}$ is both ϵ'-NLTS and ϵ'-NLCS. ◀

4 Future work

(1) The most immediate problem raised by this work is to show that rotating arbitrary CSS Hamiltonians by $(e^{-i\frac{\pi}{8}Y})^{\otimes n}$ yields NLCS Hamiltonians. We have shown this when a constant fraction of the stabilizer generators have odd weight, which is a technical requirement of our proof technique. Nonetheless, we believe all $e^{-i\frac{\pi}{8}Y}$-rotated CSS Hamiltonians are NLCS. A first step would be to show this for $\mathcal{H} \equiv \frac{1}{n}\sum |11\rangle\langle 11|_{i,i+1} = \frac{1}{n}\sum \frac{1}{2}(\mathbb{I} - Z_i Z_{i+1})$, which has only even weight stabilizer generators.

(2) NLACS Hamiltonians are an implication of either NLSS or the quantum PCP conjecture together with $\mathsf{NP} \neq \mathsf{QMA}$ (see Diagram 3), so we believe they exist. In Appendix B we give a self-contained proof that the simple D-rotated zero Hamiltonian, $\tilde{\mathcal{H}}_0 = \frac{1}{n}\sum (e^{i\frac{\pi}{8}Y}|1\rangle\langle 1| e^{-i\frac{\pi}{8}Y})_i$, is NLCS, and in Appendix B.1, we give a sharp lower-bound on the energy of states produced by Clifford + 1 T gate under $\tilde{\mathcal{H}}_0$. We also conjecture a sharp lower-bound on the energy for states prepared by Clifford + t T gates, for any $t \leq n$.

(3) We hope that our techniques may lead to local Hamiltonians which satisfy NLSS. Consider the zero Hamiltonian, $\mathcal{H}_0 = \frac{1}{n}\sum |1\rangle\langle 1|_i$, and a family of Haar-random low-depth circuits, $C = \{C_n\}$. The unique ground-state of the local Hamiltonian $C\mathcal{H}_0C^\dagger$ is exactly $C|0^n\rangle$,[10] which is not sampleable (as defined in Section 2) unless $\mathsf{P} = \#\mathsf{P}$ [9, 21]. We hope that the same is true for states of low-enough constant energy, but new techniques would be necessary to show this. If true, $C\mathcal{H}_0C^\dagger$ would be an NLSS Hamiltonian unless $\mathsf{P} = \#\mathsf{P}$. Analogously to our result for simultaneous NLTS and NLCS, one may hope that rotating arbitrary CSS Hamiltonians by random low-depth circuits could also yield simultaneous NLTS and NLSS. However, there are many unresolved prerequisites needed to show this. For example, for a CSS Hamiltonian, $\mathcal{H}$, every ground-state of $C\mathcal{H}C^\dagger$ has the form $C|\psi\rangle$ for a codestate $|\psi\rangle$. It is not a fortiori true that applying a random low-depth circuit to codestates of a CSS code will result in a state that is not sampleable, so it is not clear that even the ground-space of such a Hamiltonian is not sampleable.

(4) It is important to note that the technique of rotating Hamiltonians by a constant-depth circuit, while potentially useful for establishing NLSS, seemingly cannot provide certain other prerequisites of the quantum PCP conjecture. For example, Fact 2 says that the energies of locally-approximable states can be computed in NP, and so the quantum PCP conjecture implies the following (assuming $\mathsf{NP} \neq \mathsf{QMA}$):

▶ **Conjecture 13** (No Low-energy Locally-approximable States (NLLS))**.** *There exists a family of local Hamiltonians, $\mathcal{H}^{(n)}$, and a constant $\epsilon > 0$ such that all ϵ-low-energy states of $\mathcal{H}^{(n)}$ are not locally-approximable.*

A closely-related conjecture ("no low-lying classically-evaluatable states" conjecture) was very recently stated in [22].[11] Rotating by a constant-depth circuit preserves the NLLS property in the same way that it preserves the NLTS property, thus ruling out the use of rotating Hamiltonians in solving the NLLS conjecture.

[10] Note that we typically denote rotating by C as $C^\dagger\mathcal{H}C$, not $C\mathcal{H}C^\dagger$. We have swapped the order here so that the ground state is $C|0^n\rangle$, as opposed to $C^\dagger|0^n\rangle$.

[11] Note that these conjectures would not imply LH-$\epsilon \notin \mathsf{NP}$ as it would not rule out Hamiltonians whose ground-state energies have indirect NP-witnesses. [10] constructs such witnesses for certain commuting Hamiltonians.

Furthermore, for any CSS Hamiltonian rotated by a constant-depth circuit, which includes every construction considered in this paper, the local Hamiltonian problem is contained in NP. To see this, note that every C-rotated CSS Hamiltonian has a ground state of the form $C^\dagger |\varphi\rangle$ for some stabilizer state $|\varphi\rangle$. Such states are locally-approximable since the local density matrices can be efficiently calculated by using a combination of the local density matrix calculation for trivial states and stabilizer states.

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A Omitted proofs

A.1 Mixed Clifford states

▶ **Definition 14.** *Let G be a stabilizer group, $P = P_1 \otimes \cdots \otimes P_n \in \mathcal{P}_n$ be any Pauli operator, and $A \subseteq [n]$ be any subset of n qubits. We define the set $G_{A,P}$ to be*

$$G_{A,P} \equiv \left\{ g_A \;\middle|\; g \in G,\ g_j = P_j \text{ for all } j \notin A \right\},$$

where g_A denote the restriction of g to A (note that g_A acts on $|A|$ qubits, not n qubits).

$G_{A,P}$ can be thought of as all of the elements of G which are equal to P outside of the subset A, though we consider the restriction of these elements to A only (including overall phases). By abuse of notation we will denote $G_{i,P} \equiv G_{\{i\},P}$ and $G_{-A,P} \equiv G_{[n]\setminus A,P}$ for $i \in [n]$. We denote the special case of $G_{A,\mathbb{I}}$ by G_A. $G_A \equiv \{g_A \mid g \in G \text{ and } \mathrm{N}(g) \subseteq A\} \cup \{\mathbb{I}_A\}$ is the set of all elements in G which act non-trivially only on qubits in A.

Claim 3 is immediate from the following two well-known facts.

▶ **Fact 15.** *Let $G \leq \mathcal{P}_n$ be a stabilizer group and $\mathcal{C}$ the associated codespace. $\frac{1}{|G|}\sum_{g\in G} g$ is the projector onto $\mathcal{C}$. If $|G| = 2^n$, then $\frac{1}{2^n}\sum_{g\in G} g = |\psi\rangle\langle\psi|$, where $|\psi\rangle$ is the stabilizer state associated with G. Otherwise, $|G| = 2^{n-r}$ for $r > 0$ and there are 2^r logical basis states of $\mathcal{C}$. Let $\{|\bar{x}\rangle\}$ denote the logical computational basis states for $\mathcal{C}$. Then*

$$\frac{1}{2^{n-r}} \sum_{g\in G} g = \sum_{x\in\mathbb{F}_2^r} |\bar{x}\rangle\langle\bar{x}|\,.$$

▶ **Fact 16.** *Suppose $|\psi\rangle$ is a stabilizer state on N qubits with stabilizer group G and let A be a subset of the qubits of size n. By Fact 15 we can write $|\psi\rangle\langle\psi| = \frac{1}{2^N}\sum_{g\in G} g$. The local state on A, $\psi \equiv \mathrm{Tr}_{-A}[|\psi\rangle\langle\psi|]$, is equal to*

$$\psi = \frac{1}{2^n} \sum_{\hat{g}\in G_A} \hat{g}.$$

▷ Claim 3. If ψ is a Clifford state, then it is also a stabilizer state.

Proof. By definition, there is a pure Clifford state $|\psi\rangle$ on $N \geq n$ qubits and a subset A of n qubits such that $\psi = \mathrm{Tr}_{-A}[|\psi\rangle\langle\psi|]$. Let $G \equiv \mathrm{Stab}(|\psi\rangle)$, and let G_A be defined as in Fact 16. By definition, G_A is an abelian subgroup of $\mathcal{P}_n$ not containing $-\mathbb{I}$, and so it is a valid stabilizer group. Let $|G_A| = 2^{n-r}$. We have

$$\text{(By Fact 16)} \qquad \psi = \frac{1}{2^r 2^{n-r}} \sum_{\hat{g} \in G_A} \hat{g}, \tag{17}$$

$$\text{(By Fact 15)} \qquad = \frac{1}{2^r} \sum_{x \in \mathbb{F}_2^r} |\bar{x}\rangle\langle\bar{x}| \,. \tag{18}$$

Since each $|\bar{x}\rangle\langle\bar{x}|$ is a stabilizer state on n qubits and $\sum_{x\in\mathbb{F}_2^r} \frac{1}{2^r} = 1$, the statement is proven. ◁

A.2 Rotated projectors

Return to Claim 6.

▷ Claim 6.

$$\tilde{\Pi}_{\bar{X}} = \frac{\mathbb{I} - \mathrm{H}^{\otimes k}}{2}, \qquad \tilde{\Pi}_{\bar{Z}} = \frac{\mathbb{I} - (-X\,\mathrm{H}\,X)^{\otimes k}}{2}.$$

Proof. We will show that $D^\dagger X D = \mathrm{H}$ and $D^\dagger Z D = -X\,\mathrm{H}\,X$. As $\Pi_{\bar{X}} \equiv (\frac{1}{2})(\mathbb{I} - X)$ and $\Pi_{\bar{Z}} \equiv (\frac{1}{2})(\mathbb{I} - Z)$, the result follows.

$$\begin{aligned}
D^\dagger X D &= \left(\cos\left(\frac{\pi}{8}\right)\mathbb{I} + \sin\left(\frac{\pi}{8}\right) ZX\right) X \left(\cos\left(\frac{\pi}{8}\right)\mathbb{I} + \sin\left(\frac{\pi}{8}\right) XZ\right), \\
&= \left(\cos\left(\frac{\pi}{8}\right) X + \sin\left(\frac{\pi}{8}\right) Z\right) \left(\cos\left(\frac{\pi}{8}\right)\mathbb{I} + \sin\left(\frac{\pi}{8}\right) XZ\right), \\
&= \cos^2\left(\frac{\pi}{8}\right) X + \sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right) Z + \sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right) Z - \sin^2\left(\frac{\pi}{8}\right) X, \\
&= \cos\left(\frac{\pi}{4}\right) Z + \sin\left(\frac{\pi}{4}\right) X, \\
&= \frac{1}{\sqrt{2}}(Z + X), \\
&= \mathrm{H}\,.
\end{aligned}$$

$$\begin{aligned}
D^\dagger Z D &= \left(\cos\left(\frac{\pi}{8}\right)\mathbb{I} + \sin\left(\frac{\pi}{8}\right) ZX\right) Z \left(\cos\left(\frac{\pi}{8}\right)\mathbb{I} + \sin\left(\frac{\pi}{8}\right) XZ\right), \\
&= \left(\cos\left(\frac{\pi}{8}\right) Z - \sin\left(\frac{\pi}{8}\right) X\right) \left(\cos\left(\frac{\pi}{8}\right)\mathbb{I} + \sin\left(\frac{\pi}{8}\right) XZ\right), \\
&= \cos^2\left(\frac{\pi}{8}\right) Z - \sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right) X - \sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right) X - \sin^2\left(\frac{\pi}{8}\right) Z, \\
&= \cos\left(\frac{\pi}{4}\right) Z - \sin\left(\frac{\pi}{4}\right) X, \\
&= \frac{1}{\sqrt{2}}(Z - X), \\
&= -X\,\mathrm{H}\,X.
\end{aligned}$$

◁

B A simple NLCS Hamiltonian

The goal of this section is to demonstrate the existence of a simple family of NLCS Hamiltonians.

▶ **Definition 17.** *The* ***zero Hamiltonian****, $\mathcal{H}_0^{(n)}$ is defined as*

$$\mathcal{H}_0^{(n)} \equiv \frac{1}{n}\sum_{i=1}^{n} |1\rangle\!\langle 1|_i \otimes \mathbb{I}_{[n]\setminus\{i\}} .$$

Note that $\mathcal{H}_0^{(n)}|x\rangle = \frac{|x|}{n}|x\rangle$ for all $x \in \mathbb{F}_2^n$. In particular, the unique ground state of $\mathcal{H}_0^{(n)}$ is $|0\rangle^{\otimes n}$ with energy 0. For $n = 1$ we have $\mathcal{H}_0^{(1)} = |1\rangle\!\langle 1|$, so we can write the zero Hamiltonian on n qubits as

$$\mathcal{H}_0^{(n)} \equiv \frac{1}{n}\sum_{i=1}^{n} \mathcal{H}_0^{(1)} \otimes \mathbb{I}_{[n]\setminus\{i\}} .$$

▶ Remark. Define a set of stabilizer generators, $\mathcal{S}_n \equiv \{Z_1, \dots, Z_n\}$ where Z_i is a Pauli Z on qubit i and identity elsewhere. The zero Hamiltonian is the CSS Hamiltonian associated with $\langle \mathcal{S}_n\rangle$, since $|1\rangle\!\langle 1| = \frac{\mathbb{I}-Z}{2}$. The results of this section are a direct corollary of the results in Section 3. ⌟

Let $D \equiv e^{-i\frac{\pi}{8}Y}$. We define the D-rotated zero Hamiltonian as

$$\tilde{\mathcal{H}}_0^{(n)} \equiv \frac{1}{n}\sum_{i=1}^{n} \tilde{\mathcal{H}}_0^{(1)} \otimes \mathbb{I}_{[n]\setminus\{i\}},$$

where $\tilde{\mathcal{H}}_0^{(1)} = D^\dagger |1\rangle\!\langle 1| D$. We will prove the D-rotated zero Hamiltonian is NLCS by demonstrating a simple lower bound on the energy of stabilizer states for each local term. Since the reduced state of every stabilizer state is a convex combination of stabilizer states by Claim 3, these "local" lower bounds imply a global lower bound for all stabilizer states. We have the following local energy bound. Note that

▶ **Lemma 18.** *If $|\eta\rangle$ is a single-qubit stabilizer state, then $\langle\eta|\tilde{\mathcal{H}}_0^{(1)}|\eta\rangle \geq \sin^2(\frac{\pi}{8})$.*

Proof. By definition $\tilde{H}_0^{(1)} = D^\dagger |1\rangle\!\langle 1| D$, so $\langle\eta|\tilde{H}_0^{(1)}|\eta\rangle = |\langle 1| D|\eta\rangle|^2$. As

$$D = \cos\left(\frac{\pi}{8}\right)\mathbb{I} - i\sin\left(\frac{\pi}{8}\right)Y = \cos\left(\frac{\pi}{8}\right)\mathbb{I} + \sin\left(\frac{\pi}{8}\right)XZ,$$

we have

$$D = \begin{bmatrix} \cos\left(\frac{\pi}{8}\right) & -\sin\left(\frac{\pi}{8}\right) \\ \sin\left(\frac{\pi}{8}\right) & \cos\left(\frac{\pi}{8}\right) \end{bmatrix}.$$

There are only six single-qubit stabilizer states and it is easy to verify that the minimum value of $|\langle 1| D|\eta\rangle|^2$ is $\sin^2(\frac{\pi}{8})$. ◀

▶ **Corollary 19.** *If η is a single-qubit mixed stabilizer state, then $\mathrm{Tr}\left[\eta\tilde{\mathcal{H}}_0^{(1)}\right] \geq \sin^2(\frac{\pi}{8})$.*

Proof. By definition, $\eta = \sum_j p_j |\varphi_j\rangle\!\langle\varphi_j|$, where each $|\varphi_j\rangle$ is a pure stabilizer state on a single qubit. The lower bound follows by applying Lemma 18 to each $|\varphi_j\rangle$. ◀

We now have the following global lower bound.

▶ **Lemma 20.** *If $|\eta\rangle$ is an n-qubit stabilizer state, then $\langle\eta|\,\tilde{\mathcal{H}}_0^{(n)}\,|\eta\rangle \geq \sin^2(\frac{\pi}{8})$.*

Proof. By definition, $\tilde{\mathcal{H}}_0^{(n)} = \frac{1}{n}\sum_{i=1}^{n} \tilde{\mathcal{H}}_0^{(1)}\,|_i \otimes \mathbb{I}_{[n]\setminus\{i\}}$, so

$$\langle\eta|\,\tilde{\mathcal{H}}_0^{(n)}\,|\eta\rangle = \frac{1}{n}\sum_{i=1}^{n}\mathrm{Tr}\Big[\eta_i\tilde{\mathcal{H}}_0^{(1)}\Big],$$

where $\eta_i \equiv \mathrm{Tr}_{-i}[|\eta\rangle\langle\eta|]$ is the reduced state of $|\eta\rangle$ on qubit i. Since η_i is the reduced density matrix of a Clifford state, by Claim 3 it is also a stabilizer state. The bound follows by applying Corollary 19 to each term in the summation. ◀

▶ **Proposition 21.** $\{\tilde{\mathcal{H}}_0^{(n)}\}$ *is a family of NLCS Hamiltonians.*

Proof. By definition, $\psi = \sum_j p_j\,|\varphi_j\rangle\langle\varphi_j|$, where each $|\varphi_j\rangle$ is a pure stabilizer state on n qubits. The lower bound follows by applying Lemma 20 to each $|\varphi_j\rangle$. Thus, every n-qubit stabilizer state has energy at least $\sin^2(\frac{\pi}{8})$ with respect to $\tilde{\mathcal{H}}_0^{(n)}$, which implies $\tilde{\mathcal{H}}_0^{(n)}$ is ϵ-NLCS with $\epsilon = \sin^2(\frac{\pi}{8})$. ◀

B.1 Towards NLACS

There are several notions of how "non-Clifford" a state is, the number of T gates being a common one. The notion we consider here is the number of arbitrary Pauli-rotation gates, $e^{i\theta P}$ for $\theta \in [0, 2\pi)$ and $P \in \mathcal{P}_n$, as it encapsulates the T gate count.[12]

▶ **Lemma 22.** *Let C be a quantum circuit on n-qubits containing polynomially many Clifford gates and at most t arbitrary Pauli-rotation gates, $e^{i\theta_j P_j'}$. There exist t Pauli operators, $\{P_j\} \subset \mathcal{P}_n$ and a stabilizer state $|\varphi\rangle$ such that*

$$C\,|0\rangle^{\otimes n} = \prod_{j\in[t]}\Big[e^{i\theta_j P_j}\Big]\,|\varphi\rangle\,, \tag{19}$$

where by convention $C\,|0\rangle^{\otimes n} = |\varphi\rangle$ if $t = 0$.

Proof. By definition we can decompose C as

$$C = C_t e^{i\theta_t P_t'} C_{t-1} \ldots e^{i\theta_2 P_2'} C_1 e^{i\theta_1 P_1'} C_0, \tag{20}$$

where each C_ℓ is a Clifford circuit.

For every $j \in [t]$ we have $e^{i\theta_j P_j'} = \cos(\theta_j)\,\mathbb{I} + i\sin(\theta_j)P_j'$. Since Clifford gates normalize the Pauli group, for every Clifford circuit, C', and every Pauli operator, $P' \in \mathcal{P}_n$, there is another Pauli operator, $P'' \in \mathcal{P}_n$, such that $C'(\cos\theta\,\mathbb{I} + i\sin\theta P') = (\cos\theta\,\mathbb{I} + i\sin\theta P'')C'$. Thus, we can move each Clifford circuit, C_ℓ, past all of the Pauli-rotation gates by changing only the individual Pauli operators via the conjugation relations of C_ℓ.

Ultimately, we can rewrite C as

$$C = e^{i\theta_t P_t} \ldots e^{i\theta_2 P_2} e^{i\theta_1 P_1} C_t \ldots C_1 C_0, \tag{21}$$

for t Pauli operators, $\{P_t\}$, as desired. ◀

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[12] The T gate is equal to $\mathrm{T} = \cos\left(\frac{\pi}{8}\right)\mathbb{I} + i\sin\left(\frac{\pi}{8}\right)Z = e^{i\frac{\pi}{8}Z}$.

Proposition 21 shows that the D-rotated zero Hamiltonian, $\tilde{\mathcal{H}}_0 = \frac{1}{n}\sum\left(D^\dagger |1\rangle\langle 1| D\right)_i$, is $\sin^2\left(\frac{\pi}{8}\right)$-NLCS. It is natural to question if $\tilde{\mathcal{H}}_0$ is also ϵ-NLACS for some appropriate constant ϵ. In this section we will prove an explicit lower-bound on all states prepared by Clifford gates + at most 1 Pauli-rotation gate:

$$\langle\psi| \tilde{\mathcal{H}}_0^{(n)} |\psi\rangle \geq \left(1 - \frac{1}{n}\right)\sin^2\left(\frac{\pi}{8}\right). \tag{22}$$

In fact, there is numerical evidence suggesting the following lower bound for an arbitrary number of Pauli-rotation gates, though we have been unable to prove it analytically:

▶ **Conjecture 23.** *Let $|\psi\rangle$ be an n-qubit state prepared by a Clifford circuit plus at most t Pauli-rotation gates. For the D-rotated zero-Hamiltonian, $\tilde{\mathcal{H}}_0^{(n)}$, the energy of $|\psi\rangle$ is lower-bounded as*

$$\langle\psi| \tilde{\mathcal{H}}_0^{(n)} |\psi\rangle \geq \left(1 - \frac{t}{n}\right)\sin^2\left(\frac{\pi}{8}\right). \tag{23}$$

In particular, if there is a constant $\beta \in [0, 1)$ such that $t \leq \beta n$ for all sufficiently large n, then the energy of $|\psi\rangle$ is lower-bounded by $(1-\beta)\sin^2\left(\frac{\pi}{8}\right) > 0$, a constant.

By Lemma 22, the most general such state is a stabilizer state with t Pauli-rotation gates applied to it and no intermediate circuits between them. The intuition behind Conjecture 23 is that the only way to reduce the energy of a stabilizer state is to "undo" one of the D gates conjugating the Hamiltonian. For instance, to produce a state with sub-constant energy one could apply $n - o(n)$ D gates to $|0\rangle^{\otimes n}$.

We note also that is in unclear what, if any, similar lower bound could be shown for an arbitrary D-rotated CSS Hamiltonian (as considered in Theorem 5). We leave this as an open problem, as well. For now, we consider the case of $t = 1$ for the D-rotated zero Hamiltonian.

First, recall the following definition.

▶ **Definition 14.** *Let G be a stabilizer group, $P = P_1 \otimes \cdots \otimes P_n \in \mathcal{P}_n$ be any Pauli operator, and $A \subseteq [n]$ be any subset of n qubits. We define the set $G_{A,P}$ to be*

$$G_{A,P} \equiv \left\{ g_A \;\middle|\; g \in G,\ g_j = P_j \text{ for all } j \notin A \right\},$$

where g_A denote the restriction of g to A (note that g_A acts on $|A|$ qubits, not n qubits).

The following lemma gives an explicit description of the local density matrices of states with at most 1 Pauli-rotation gate.

▶ **Lemma 24.** *Let $|\psi\rangle = e^{i\theta P}|\varphi\rangle$ for $P \in \mathcal{P}_n$, $\theta \in [0, 2\pi)$, and let $|\varphi\rangle$ be a stabilizer state with $G \equiv \mathrm{Stab}(|\varphi\rangle)$. For $A \subset [n]$ we can write $\psi_A \equiv \mathrm{Tr}_{-A}[|\psi\rangle\langle\psi|]$ as*

$$\psi_A = \frac{1}{2^{|A|}}\sum_{\hat{g}\in G_A}\left(\cos^2(\theta)\hat{g} + \sin^2(\theta)P_A\hat{g}P_A\right) + \frac{1}{2^{|A|}}\sum_{g'\in G_{A,P}} i\sin(\theta)\cos(\theta)[P_A, g']. \tag{24}$$

The left part of this expression can be thought of as the stabilizer part of ψ_A, as it is the convex combination of two stabilizer states, and the right hand part can be thought of as the non-stabilizer part, as it equals zero if $P \in G$ or if $P_A = \mathbb{I}$.

Proof. Since $|\varphi\rangle$ is a stabilizer state there is a stabilizer group G with $|G| = 2^n$ such that $|\varphi\rangle\langle\varphi| = \frac{1}{2^n}\sum_{g\in G} g$. Using the exponential of Pauli matrices we have

$$\psi = \frac{1}{2^n}\sum_{g\in G}(\cos(\theta)\,\mathbb{I} + i\sin(\theta)P)g(\cos(\theta)\,\mathbb{I} - i\sin(\theta)P), \tag{25}$$

$$= \frac{1}{2^n}\sum_{g\in G}\cos^2(\theta)g + \sin^2(\theta)PgP + i\sin(\theta)\cos(\theta)Pg - i\sin(\theta)\cos(\theta)gP, \tag{26}$$

$$= \frac{1}{2^n}\sum_{g\in G}\Big(\cos^2(\theta)g + \sin^2(\theta)PgP\Big) + \frac{1}{2^n}\sum_{g\in G}\Big(i\sin(\theta)\cos(\theta)(Pg - gP)\Big). \tag{27}$$

Consider tracing out all qubits outside of the set A. The only Pauli group element with nonzero trace is $\mathbb{I}$, which has trace 2. For the left term in Equation (27), we have

$$\frac{1}{2^n}\sum_{g\in G}\Big(\cos^2(\theta)\,\mathrm{Tr}_{-A}[g] + \sin^2(\theta)\,\mathrm{Tr}_{-A}[PgP]\Big) \tag{28}$$

$$= \frac{1}{2^n}\sum_{g\in G}\Big(\cos^2(\theta)g_A\prod_{j\in[n]\setminus A}\mathrm{Tr}[g_j] + \sin^2(\theta)P_A g_A P_A\prod_{j\in[n]\setminus A}\mathrm{Tr}[P_j g_j P_j]\Big), \tag{29}$$

$$= \frac{1}{2^n}\sum_{g\in G}\Big(\cos^2(\theta)g_A + \sin^2(\theta)P_A g_A P_A\Big)\Big(\prod_{j\in[n]\setminus A}\mathrm{Tr}[g_j]\Big), \tag{30}$$

$$= \frac{1}{2^{|A|}}\sum_{\hat{g}\in G_A}\Big(\cos^2(\theta)\hat{g} + \sin^2(\theta)P_A\hat{g}P_A\Big), \tag{31}$$

where the last line follows since only those $g \in G$ which are identity outside of A will have nonzero trace, and the product of the individual traces when non-zero is $2^{n-|A|}$.

Similarly, for the right term in Equation (27) we have

$$\frac{1}{2^n}\sum_{g\in G}\Big(i\sin(\theta)\cos(\theta)\,\mathrm{Tr}_{-A}[Pg - gP]\Big), \tag{32}$$

$$= \frac{1}{2^n}\sum_{g\in G}\Big(i\sin(\theta)\cos(\theta)[P_A, g_A]\Big)\Big(\prod_{j\in[n]\setminus A}\mathrm{Tr}[P_j g_j]\Big), \tag{33}$$

$$= \frac{1}{2^{|A|}}\sum_{g'\in G_{A,P}} i\sin(\theta)\cos(\theta)[P_A, g'], \tag{34}$$

where the last line follows again since the trace will be non-zero only if $g_j = P_j$ for all $j \notin A$. ◀

▶ **Lemma 25.**

$$\langle\psi|\,\tilde{\mathcal{H}}_0^{(n)}\,|\psi\rangle \geq \left(1 - \frac{1}{n}\right)\sin^2\left(\frac{\pi}{8}\right). \tag{35}$$

Proof. By Lemma 22 there is a Pauli operator, P, and an n-qubit Clifford state $|\varphi\rangle$ such that $|\psi\rangle = e^{i\theta P}|\varphi\rangle$. Let $G \equiv \mathrm{Stab}(|\varphi\rangle)$.

Recall that by definition $\tilde{\mathcal{H}}_0^{(n)} = \frac{1}{n}\sum_{i=1}^n \tilde{\mathcal{H}}_0^{(1)}\,|_i \otimes \mathbb{I}_{[n]\setminus\{i\}}$, so

$$\langle\psi|\,\tilde{\mathcal{H}}_0^{(n)}\,|\psi\rangle = \frac{1}{n}\sum_{i=1}^n \mathrm{Tr}\Big[\psi_i\tilde{\mathcal{H}}_0^{(1)}\Big], \tag{36}$$

where $\psi_i \equiv \mathrm{Tr}_{-i}[|\psi\rangle\langle\psi|]$ is the reduced state of $|\psi\rangle$ on qubit i. We will show that at most one of the terms in this summation can be 0, and that the remainder of the terms are lower-bounded by $\sin^2\left(\frac{\pi}{8}\right)$.

By Lemma 24 we can write the reduced state as

$$\psi_i = \frac{1}{2}\sum_{\hat{g}\in G_i}\left(\cos^2(\theta)\hat{g} + \sin^2(\theta)P_i\hat{g}P_i\right) + \frac{1}{2}\sum_{g'\in G_{i,P}} i\sin(\theta)\cos(\theta)[P_i, g']. \tag{37}$$

We proceed in cases:

(1) If $P \in G$, $P_i = \mathbb{I}$, $G_{i,P} = \emptyset$, or $G_{i,P} = \{\mathbb{I}\}$ then ψ_i is a stabilizer state, so $\mathrm{Tr}\left[\psi_i\tilde{\mathcal{H}}_0^{(1)}\right] \geq \sin^2\left(\frac{\pi}{8}\right)$.

(2) Suppose the four conditions from Case **I.** do not hold. It must be that $G_{i,P} = \{\mathbb{I}, P^\star\}$ for some $P^\star \in \mathcal{P}_1 \setminus \{\mathbb{I}, P_i\}$; $P^\star$ cannot be P_i as this would imply $P \in G$. Note that $G_{i,P}$ cannot be any larger as this would contradict the fact G is a stabilizer group. We now consider cases for G_i.

(a) If $G_i = \{\mathbb{I}\}$, then ψ_i can be written as

$$\psi_i = \frac{1}{2}\mathbb{I} + \frac{1}{2}i\sin(\theta)\cos(\theta)[P_i, P^\star], \tag{38}$$

$$= \frac{1}{2}\mathbb{I} + \frac{1}{4}\sin(2\theta)\sigma, \tag{39}$$

since $P_i \neq P^\star$ and $2i[P_i, P^\star] = \sigma$ for some non-identity Pauli. The desired bound holds by direct computation over $\sigma \in \mathcal{P} \setminus \{\pm\mathbb{I}\}$.

(b) If G_i is non-trivial then $G_i = \{\mathbb{I}, P^\star\}$ since it must commute with the $g \in G$ which satisfies $g_i = P^\star$ and $g_{-i} = P_{-i}$ (which exists since we are in Case **II.**) Since $P^\star \notin \{\mathbb{I}, P_i\}$ we can write ψ_i as

$$\psi_i = \frac{1}{2}\mathbb{I} + \frac{1}{2}\left(\cos^2(\theta) - \sin^2(\theta)\right)P^\star + \frac{1}{2}i\sin(\theta)\cos(\theta)[P_i, P^\star], \tag{40}$$

$$= \frac{1}{2}\mathbb{I} + \frac{1}{2}\cos(2\theta)P^\star + \frac{1}{4}\sin(2\theta)i[P_i, P^\star]. \tag{41}$$

By direct computation we have the following:

(i) If $P_i \neq Y$ then $\mathrm{Tr}\left[\psi_i\tilde{\mathcal{H}}_0^{(1)}\right] \geq \sin^2\left(\frac{\pi}{8}\right)$ regardless of θ.

(ii) If $P_i = Y$ and $P^\star \neq Z$ then $\mathrm{Tr}\left[\psi_i\tilde{\mathcal{H}}_0^{(1)}\right] \geq \sin^2\left(\frac{\pi}{8}\right)$ regardless of θ.

(iii) If $P_i = Y$ and $P^\star = Z$ then $\mathrm{Tr}\left[\psi_i\tilde{\mathcal{H}}_0^{(1)}\right] \geq 0$ with possible equality.

To recap the cases, ψ_i can have energy less than $\sin^2\left(\frac{\pi}{8}\right)$ only if (1) $P_i = Y$, (2) $Z_i \in G$, and (3) there is a $g \in G$ such that $g_i = Z$ and $g_{-i} = P_{-i}$, i.e. g and P agree on every qubit except i.

We must show that at most one qubit can satisfy all three of these condition for a given $P \in \mathcal{P}_n$ and stabilizer group G. Suppose there are two such qubits, i and j, which satisfy (1) $P_i = P_j = Y$, (2) $Z_i, Z_j \in G$, and (3) there exist $g, h \in G$ such that $g_i = h_j = Z$, $g_{-i} = P_{-i}$, and $h_{-j} = P_{-j}$. By condition (3) $g_i = Z$ and $g_j = Y$ and by condition (2) $Z_j \in G$, but this implies that $gZ_j = -Z_j g$, which contradicts the fact that G is abelian. Thus, at most a single qubit can satisfy the conditions required for the reduced state ψ_i to have energy less than $\sin^2\left(\frac{\pi}{8}\right)$, which implies the desired lower bound. ◀