The Geometry of Reachability in Continuous Vector Addition Systems with States

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Abstract

We study the geometry of reachability sets of continuous vector addition systems with states (VASS). In particular we establish that they are "almost" Minkowski sums of convex cones and zonotopes generated by the vectors labelling the transitions of the VASS. We use the latter to prove that short so-called linear path schemes suffice as witnesses of reachability in continuous VASS. Then, we give new polynomial-time algorithms for the reachability problem for linear path schemes. Finally, we also establish that enriching the model with zero tests makes the reachability problem intractable already for linear path schemes of dimension two.

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1 Introduction

Vector Addition Systems with States (VASS, for short) are rich mathematical models for the description of distributed systems, as well as chemical and biological processes, and more [19]. A VASS essentially consists of a finite-state machine whose transitions are labelled with integer vectors. Besides the current state, a configuration of the VASS also comprises the current values of a set of counters. When a transition is taken, the state of the machine changes and the values of the counters are updated by adding to them the vector that labels the transition. VASS arise naturally as an arguably-cleaner model than Petri nets, due to their reachability problem being polytime-interreducible with that of Petri nets.

While VASS are a very expressive model of concurrency that admits algorithmic analysis, the complexity of several associated decision problems is prohibitively high. For instance, the *reachability problem*, which asks "is a given target configuration reachable from a given initial configuration?", was recently proved to be Ackermann-complete [6, 15, 14].

Continuous VASS were introduced by Blondin and Haase [2] as an alternative to *continuous Petri nets* [7] which trade off the ability to encode discrete information in favor of computational and practical benefits. Their only difference compared to VASS concerns how the counters are updated: In continuous VASS, when a transition is taken, the machine is

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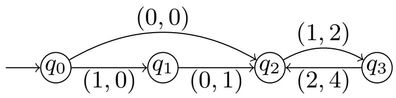

Figure 1 From q_0, with initial counter values $\mathbf{0}$, the state q_2 can be reached with counter values $\{(3i + a, 6i + b) \mid (a, b) \in \{(0, 0), (1, 1)\}, i \in \mathbb{N}\}$; with continuous semantics, it can be reached with counter values $\boldsymbol{x} + \boldsymbol{y}$ where $\boldsymbol{x} \in \{(0, 0)\} \cup \{(a, b) \mid 0 < a, b \leq 1\}$ and $\boldsymbol{y}$ lies on the ray $\{(i, 2i) \mid i \in \mathbb{N}\}$.

allowed to scale the update vector by some scalar $0 < \alpha \leq 1$ before adding it to the current counter values (see Figure 1). In contrast to the situation with "discrete" VASS, the computational complexity of the reachability problem for continuous VASS is relatively low. Namely, in [2] the reachability problem for continuous VASS is shown to be **NP**-complete while the complexity of the same problem for discrete VASS is Ackermann-complete [6, 15, 14].

Despite the relatively low computational complexity, **NP** is not universally considered as tractable. The only subcase previously known to be in **P** was that of cyclic reachability when counters are allowed to hold negative values. It is also worth noting that the aforementioned **NP** upper bound is obtained by encoding the reachability problem into the existential fragment of the first-order theory of the reals with addition and order. It is natural to ask whether more efficient algorithms or encodings into "simpler" logics exist, e.g. linear programming, even if only for particular subcases.

Fixed-dimension VASS. The relatively new Ackermann lower bound for VASS reachability has renewed interest in what could be named the *Bordeaux-Warsaw program*: the study of the computational complexity of the reachability problem for low-dimensional VASS and extensions thereof (see, e.g., [4, 3, 5, 8]). In such cases, there may be efficient algorithms for the problem and, to quote Czerwiński and Orlikowski [6], "it is easier to [design] sophisticated techniques working in a simpler setting [that might] result in finding new techniques useful in much broader generality." For dimensions 1 and 2 (and counter updates encoded in binary) the problem is **NP**-complete [12] and **PSPACE**-complete [1], respectively.

An important structural restriction on VASS which is often used as an intermediate step in establishing upper bounds is that of *flatness*, i.e. disallowing nested cycles. In fact, the upper bounds for dimensions 1 and 2 mentioned above can be seen as a consequence of such VASS being effectively flattable [16]. A further restriction consists of asking that the set of all runs of the VASS can be represented by a single regular expression $\pi_0 \chi_1^+ \ldots \pi_{n-1} \chi_n^+ \pi_n$ over the transitions. Such VASS are called *linear path schemes* (LPS, for short). Linear path schemes played an essential role in [1], where it is shown that for any path that witnesses reachability, there exists a linear path scheme that also witnesses reachability.

VASS variants. In this work we study continuous VASS. For complexity matters, we assume all counter updates are encoded in binary. As decision problems, we focus on reachability (via a path that might make the counters negative); nonnegative reachability, i.e. reachability via a path that keeps the counters nonnegative at all times; and zero-test reachability, corresponding to reachability with the added constraint that some states can only be visited with value zero for a designated counter. We summarize known and new results in Table 1. Below, we give a textual account of the complexity bounds from the table.

(Discrete) Reachability. The **NP**-hardness bound for LPS can be shown using a simple reduction from the SubsetSum problem with multiplicities, i.e. summands can be added more than once. The latter is known to be **NP**-complete, see e.g. [9, Proposition 4.1.1]. The upper bound for the general case is folklore and is proven in [11] even with *resets*.

Continuous reachability. The **NP**-hardness bound for flat VASS is stated in [2, Lemma 4.13(a)] for nonnegative reachability but the reduction establishes it for reachability as well. The upper bound for the general case follows from [2, Corollary 4.10]. Membership in **P** for LPS can be derived from [2, Theorem 4.15] which states that continuous cyclic reachability is in **P**. In this work, we give an alternative algorithm for continuous cyclic reachability and present a full decision procedure for continuous reachability for LPS.

Nonnegative reachability. For fixed dimension d, only an $F_{d+O(1)}$ upper bound is known [15]. **NP**-hardness for LPS follows from the same proof as for reachability since the construction has no negative updates. Finally, the **NP** upper bound for flat VASS is folklore: (nonnegative) reachability in flat VASS can be encoded into existential Presburger Arithmetic (PA), a theory whose decidability is **NP**-complete (see, e.g., [10]).

Continuous nonnegative reachability. The **NP** upper and lower bounds for the general and flat cases follow from (the proofs of) the same results in the continuous reachability case. For the **P** upper bound, however, one cannot rely on [2, Theorem 4.15]. In fact, cyclic reachability (for general dimensions) is **NP**-hard in the continuous nonnegative case [2, Lemma 4.13(b)]. This is, thus, the first novel complexity bound we establish.

Zero-test reachability. The **NP**-hardness bound for LPS is a consequence of reachability being a subcase of zero-test reachability. The matching upper bound for flat VASS is an extension of the classical encoding into PA which accounts for linear constraints imposed by the zero tests on cycles. Finally, the general model is also known as Minsky machines and its reachability problem was proven undecidable by Minsky himself [17].

Continuous zero-test reachability. The **NP**-hardness for flat VASS is a consequence of reachability being a subcase of zero-test reachability. For LPS, the lower bound is novel and points to continuous zero-test reachability not being a suitable approximation of the discrete case. The general case is undecidable in dimension 4 or higher [2, Theorem 4.17].

Our contributions. Our main contribution is a geometrical understanding of the reachability sets of continuous VASS (see Theorem 1, Theorem 5, and Theorem 7). The latter allows us (1) to prove that short LPS suffice as witnesses of (nonnegative) reachability (see Theorem 11 and Theorem 18), and (2) to give new algorithms for the reachability problem for LPS (see Theorem 13 and Theorem 19) via encodings of their reachability sets into tractable theories. Namely, we stay within linear programming solutions to enable efficient implementation of our algorithms. Finally, we establish that zero-test reachability for LPS is **NP**-hard even in dimension 2 (Theorem 21).

Table 1 Summary of computational complexity results for the reachability problem for VASS of fixed dimension. We write lower bounds for simpler cases and upper bounds for more general ones. New results are shown in green (upper) and red (lower bounds).

	Discrete			Continuous		
Problem	General	Flat	LPS	General	Flat	LPS
Reachability	in **NP**	=	**NP**-hard	in **NP**	**NP**-hard	in **P**
Nonneg. reach.	in Ackermann	in **NP**	**NP**-hard	in **NP**	**NP**-hard	in **P**
Zero-test reach.	Undecidable	in **NP**	**NP**-hard	Undecidable	**NP**-hard	**NP**-hard

2 Preliminaries

In this work, $\mathbb{Q}_{>0}$ denotes the set of all strictly positive rational numbers; and $\mathbb{Q}_{\geq 0}$, all nonnegative ones – including 0. Similarly, $\mathbb{N}$, i.e. the set of all natural numbers, includes 0, however $\mathbb{N}_{>0}$ does not.

Let d be a positive integer. For any $\boldsymbol{x} \in \mathbb{Q}^d$, and $r \in \mathbb{R}$, we define the *open ball of radius r around x* as usual: $B_r(\boldsymbol{x}) = \{\boldsymbol{y} \in \mathbb{Q}^d : \|\boldsymbol{x}-\boldsymbol{y}\|_2 < r\}$. Let $X \subseteq \mathbb{Q}^d$. Then, the *interior* of X is $\mathrm{int}(X) = \{\boldsymbol{x} \in X \mid \exists \varepsilon > 0, B_\varepsilon(\boldsymbol{x}) \subseteq X\}$; the *closure* of X is $\mathrm{cl}(X) = \{\boldsymbol{x} \in \mathbb{Q}^d \mid \forall \varepsilon > 0,\ B_\varepsilon(\boldsymbol{x}) \cap X \neq \emptyset\}$; and the *boundary* of X is $\mathrm{bd}(X) = \mathrm{cl}(X) \setminus \mathrm{int}(X)$. Finally, we define the *relative interior of* X, $\mathrm{relint}(X)$, as its interior with respect to its embedding into its own affine hull as follows: $\mathrm{relint}(X) = \{\boldsymbol{x} \in X \mid \exists \varepsilon > 0, B_\varepsilon(\boldsymbol{x}) \cap \mathrm{aff}(X) \subseteq X\}$.

Let $G \subseteq \mathbb{Q}^d$ be a set of (generating) vectors. We write $\mathrm{cone}(G)$ to denote the *(rational convex) cone* $\{\sum_{i=0}^k a_i \boldsymbol{g_i} \mid k \in \mathbb{N}, \boldsymbol{g_i} \in G, a_i \in \mathbb{Q}_{\geq 0}\}$. The *(linear) span* of G is defined as follows: $\mathrm{span}(G) = \{\sum_{i=0}^k a_i \boldsymbol{g_i} \mid k \in \mathbb{N}, \boldsymbol{g_i} \in G, a_i \in \mathbb{Q}\}$. Finally, the *affine hull* $\mathrm{aff}(G)$ of G is the set $\{\sum_{i=0}^k a_i \boldsymbol{g_i} \mid k \in \mathbb{N}, \boldsymbol{g_i} \in G, a_i \in \mathbb{Q}, \sum_{i=0}^k a_i = 1\}$. (In particular, note that if $\boldsymbol{0} \in G$ then $\mathrm{aff}(G) = \mathrm{span}(G) = \mathrm{span}(H)$, for some $H \subseteq G$ with cardinality at most d.)

2.1 Continuous VASS

Let d be a positive integer. A *continuous VASS* $\mathcal{V}$ of dimension d is a tuple (Q, T, ℓ) where Q is a finite set of states, $T \subseteq Q \times Q$ is a finite set of transitions, and $\ell : T \to \mathbb{Q}^d$ assigns an *update label* to every transition.

Paths and runs. A *configuration* $c \in Q \times \mathbb{Q}^d$ is a tuple consisting of a state and the concrete values of the d *counters* of the VASS. We denote the configuration $(p, \boldsymbol{x})$ by $p(\boldsymbol{x})$.

A *path* π is a sequence $(p_1, p_2)(p_2, p_3) \ldots (p_{n-1}, p_n) \in T^*$ of transitions. We write $|\pi|$ to denote the length of the path, i.e. $|\pi| = n-1$. A *run* ρ is a sequence $q_1(\boldsymbol{x_1})q_2(\boldsymbol{x_2}) \ldots q_n(\boldsymbol{x_n})$ of configurations such that for all $1 \leq i < n$ we have: $(q_i, q_{i+1}) \in T$ and $\boldsymbol{x_i} + \alpha_i \cdot \ell(q_i, q_{i+1}) = \boldsymbol{x_{i+1}}$ for some $\alpha_i \in \mathbb{Q}$ with $0 < \alpha_i \leq 1$. Often, we refer to the α_i as the *coefficients of the run*. We say ρ *induces* the path $(q_1, q_2) \ldots (q_{n-1}, q_n)$. Conversely, we sometimes say a run is *lifted* from a path. For instance, π can be lifted to a run $p_1(\boldsymbol{y_1}) \ldots p_n(\boldsymbol{y_n})$ by fixing $p_1(\boldsymbol{y_1})$ as initial configuration and by choosing adequate coefficients α_i for all transitions.

As a more concrete example, consider the path $(q_0, q_1), (q_1, q_2), (q_2, q_3)$ in Figure 1, whose transitions are labelled by $(1, 0)$ and $(0, 1)$. Starting from the configuration $(0, 0)$ and using the coefficients $\alpha_1 = 0.3$ and $\alpha_2 = 0.5$ this path lifts to the run $q_0(0, 0)q_1(0.3, 0)q_2(0.3, 0.5)$.

We consider continuous VASS in a setting where only nonnegative counter values are allowed, denoted $\mathbb{Q}_{\geq 0}$VASS; and one which allows negative counters, denoted $\mathbb{Q}$VASS.

Reachability. Let $p(\boldsymbol{x})$ and $q(\boldsymbol{y})$ be two configurations. We say $q(\boldsymbol{y})$ is *reachable* from $p(\boldsymbol{x})$, denoted $p(\boldsymbol{x}) \xrightarrow{*} q(\boldsymbol{y})$, if there exists a run whose first and last configurations are $p(\boldsymbol{x})$ and $q(\boldsymbol{y})$ respectively. For a path π, we write $p(\boldsymbol{x}) \xrightarrow{\pi} q(\boldsymbol{y})$ if, additionally, such a run exists which can be lifted from π. Given a configuration $p_1(\boldsymbol{x})$ and a state q, we define the *reachability set* of a path $\pi = (p_1, p_2) \ldots (p_{n-1}, p_n)$ or a set P of paths below.

$$\mathrm{Reach}^{\boldsymbol{x}}(\pi) = \{\boldsymbol{y} \in \mathbb{Q}^d \mid p_1(\boldsymbol{x}) \xrightarrow{\pi} p_n(\boldsymbol{y})\} \qquad \mathrm{Reach}^{\boldsymbol{x}}(P) = \bigcup_{\pi \in P} \mathrm{Reach}^{\boldsymbol{x}}(\pi)$$

If $\boldsymbol{x} = \boldsymbol{0}$ then we write simply $\mathrm{Reach}(\pi)$ and $\mathrm{Reach}(P)$.

3 The Geometry of $\mathbb{Q}$VASS Reachability Sets

In this section we discuss the geometry of the reachability sets in continuous VASS of dimension d. We first discuss paths and cycles separately. Then, we show that for solving the reachability problem, we only need to take short *linear path schemes* into consideration.

3.1 The geometry of reachability sets for cycles

For this section, we fix a cycle $\chi = (p_1, p_2) \ldots (p_m, p_{m+1})$, with $p_1 = p_{m+1}$. We study the geometry of the set $\mathrm{Reach}(\chi^+)$, where χ^+ stands for $\{\chi^k \mid k \in \mathbb{N}_{>0}\}$.

Intuitively, following χ allows us to add a scaled version of each transition vector along χ arbitrarily many times, with the proviso that the scaling is strictly positive (the restriction to scale up to 1 disappears since we can repeatedly take the cycle). Thus, we can intuitively reach the interior of a cone, i.e. a positive linear combination of the vectors along χ. For example, a cycle with vectors $(1, 0)$ and $(0, 1)$ will allow us to reach $\{(x, y) \mid x > 0, y > 0\}$. However, this intuition needs to be formalized carefully to account for linear dependencies between the vectors. This may render the cone not full-dimensional, i.e. its linear span may be a strict subspace of the vector space it is in. That would mean that the interior of the cone is empty. However, in such cases, the reachability set still is a "flattened" version of the interior, namely the relative interior of the cone.

3.1.1 From cycles to cones

We formalize our intuition by proving that $\mathrm{Reach}(\chi^+)$ is the relative interior of the cone generated by $G(\chi) = \{\ell(p_i, p_{i+1}) \mid 1 \le i \le m\}$.

Indeed, all points $\boldsymbol{x} \in \mathrm{Reach}(\chi^+)$ can be obtained as positive linear combinations of generators. To any such $\boldsymbol{x}$, we can add or subtract any generating vector and stay within $\mathrm{cone}(G(\chi))$, as long as it is sufficiently scaled down. Conversely, if one can add and subtract suitably scaled versions of all generating vectors to a point $\boldsymbol{x} \in \mathrm{cone}(G(\chi))$, and remain within $\mathrm{cone}(G(\chi))$, then it must be in the (relative) interior of $\mathrm{cone}(G(\chi))$.

▶ **Theorem 1.** *Let $G(\chi)$ be as defined above. Then,* $\mathrm{Reach}(\chi^+) = \mathrm{relint}(\mathrm{cone}(G(\chi)))$.

Note that we consider the set $G(\chi)$ of all labels of transitions from χ, ignoring the fact that multiple transitions can have the same label. This is justified by the following lemma for the cycle χ with generator $G(\chi)$.

▶ **Lemma 2.** *We have that* $\mathrm{relint}(\mathrm{cone}(\{\lambda_1 \ell(p_i, p_{i+1}) \mid 1 \le i \le m, \lambda_i \in \mathbb{Q}_{>0}\}))$ *is equal to* $\{\sum_{i=1}^{m} a_i \boldsymbol{g_i} \mid a_i \in \mathbb{Q}_{>0}\}$.

A concrete case: dimension 2. For intuition, we state a consequence of Theorem 1 in dimension 2. For $d = 2$, a cone C is *trivial* if there exists $\boldsymbol{v} \in \mathbb{Q}^d$ such that C is a subset of the line $\{r \cdot \boldsymbol{v} \mid r \in \mathbb{Q}\}$, and otherwise it is full-dimensional. For a trivial cone, its relative interior is either the entire cone (if the cone is the entire line), or the cone without $\mathbf{0}$, if the cone is a ray. It is easy to see that for a cycle χ whose vector labellings $G(\chi)$ are co-linear to $\boldsymbol{v}$, the reachability set of χ^+ in the continuous semantics is $\mathrm{cone}(G(\chi))$, excluding any end-points (since the only possible end-point is $\mathbf{0}$ if $\mathrm{cone}(G(\chi))$ is a ray). For full-dimensional cones, we can take any positive combination of the generators, but since no element of the generators can be taken zero times, the reachability set excludes the boundary. See Figure 2 (left) for a visualization. In the following statement we write $\mathrm{int}(G(\chi))$ and $\mathrm{bd}(G(\chi))$ for $\mathrm{int}(\mathrm{cone}(G(\chi)))$ and $\mathrm{bd}(\mathrm{cone}(G(\chi)))$ respectively.

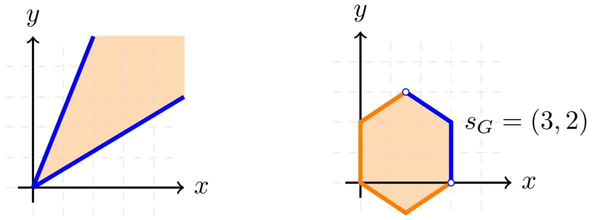

Figure 2 On the left: a cone with its boundary in blue; on the right: a zonotope with $G = \{(1.5,1),(1.5,-1),(0,2)\}$, where $\mathrm{adj}(G)$ is drawn in blue.

▶ **Corollary 3.** *Let $G(\chi)$ be as above. In dimension $d = 2$, one of the following holds.*
- *Either* $\mathrm{cone}(G(\chi))$ *is trivial and* $\mathrm{Reach}(\chi^+)$ *is* $\mathrm{cone}(G(\chi))$*, excluding any end-points;*
- *or* $\mathrm{cone}(G(\chi))$ *is nontrivial and* $\mathrm{Reach}(\chi^+) = \mathrm{int}(G(\chi))$*.*

3.2 The geometry of reachability sets for paths

For this section, we fix a path $\pi = (p_1, p_2)\dots(p_m, p_{m+1})$. Similarly to the case of cycles, we establish a connection between $\mathrm{Reach}(\pi)$ and a type of polytope known as a zonotope.

Intuitively, since we now have a path that is taken once (rather than a cycle), the restriction that the scaling is at most 1 comes into play, and limits us. Moreover, multiplicities of linearly dependent vectors along the path also matter.

Zonotopes. Let $G = \{\boldsymbol{g_1}, \dots, \boldsymbol{g_k}\} \subseteq \mathbb{Q}^d$ be a finite set of (generating) vectors. We write $\mathrm{zono}(G)$ to denote the *zonotope*[1] $\{\sum_{j=1}^{k} a_i \boldsymbol{g_i} \mid a_i \in \mathbb{Q}, 0 \leq a_i \leq 1\}$.

Following our interior-based approach for cycles, we study the reachable part of the boundary of a zonotope. We define s_G as the sum of all vectors in G, or $s_G = \sum_{\boldsymbol{g} \in G} \boldsymbol{g}$. Additionally, we define the set $\mathrm{adj}(G)$ of *faces* of $\mathrm{zono}(G)$ that are *adjacent to* s_G below. A face of $\mathrm{zono}(G)$ is any nonempty intersection of $\mathrm{zono}(G)$ with a half-space H such that none of the interior points of $\mathrm{zono}(G)$ lies on the boundary of H.

▶ **Definition 4** (Adjacent Sets)**.** *We define* $\mathrm{adj}(G)$ *as the union of* $\{s_G\}$ *and all* $\boldsymbol{x} \in \mathbb{Q}^d$ *on the relative interior of a face of* $\mathrm{zono}(G)$ *that contains* s_G*. (See Figure 2 for intuition.)*

Observe that $\mathrm{adj}(G) = \emptyset$ whenever $s_G \in \mathrm{int}(\mathrm{zono}(G))$.

3.2.1 From paths to zonotopes

We show $\mathrm{Reach}(\pi)$ has a close relation with a zonotope whose generator is derived from the *multiset* $M = [\ell(p_i, p_{i+1}) \mid 1 \leq i \leq m]$. Intuitively, we obtain from M a generator $G(\pi)$ by summing together co-linear vectors that are in the same "orientation". For example, the vectors $(1,0)$ and $(2,0)$ along a single path have the same effect as $(3,0)$, but $(1,0)$ and $(-1,0)$ have distinct effects. Technically, this is captured by grouping together vectors that are in the cones of each other. More formally, choose $M' \subseteq M \setminus \{\boldsymbol{0}\}$ such that for all $\boldsymbol{u} \in M \setminus \{\boldsymbol{0}\}$ there is a unique vector $\boldsymbol{u}' \in M'$ such that $\boldsymbol{u} \in \mathrm{cone}(\boldsymbol{u}')$. Then, define $G(\pi)$ as follows: $G(\pi) = \left\{ \sum_{\boldsymbol{u} \in M \cap \mathrm{cone}(\boldsymbol{u}')} \boldsymbol{u} \;\middle|\; \boldsymbol{u}' \in M' \right\}$.

[1] Zonotope is the standard term, but since we do not use any of its special properties, the reader may view this as a standard polytope.

We show that the reachability set of π can be computed by taking $\mathrm{zono}(G(\pi))$, and removing its boundary except for faces adjacent to s_G. The intuition behind this is similar to that of cycles: one can add strictly positive-scaled versions of the generating vectors, and therefore boundary elements that are obtained using 0-scales are unreachable. However, in zonotopes there are additional boundary faces that are obtained by capping the scale at 1 on some elements, and these are the faces adjacent to s_G (with s_G itself being the reachable vector where all elements are scaled to 1).

▶ **Theorem 5.** *Let $G(\pi)$ be as above. Then,* $\mathrm{Reach}(\pi) = \mathrm{relint}(\mathrm{cone}(G)) \cup \mathrm{adj}(G(\pi))$.

As with cycles, compared to the multiset of all path labels, we restrict our attention to a simpler set G of vectors. The following result connecting them is almost immediate.

▶ **Lemma 6.** *Let $\boldsymbol{x} \in \mathbb{Q}^d$. Then, there exists $(a_1, \ldots, a_n) \in \mathbb{Q}^n$ such that $\boldsymbol{x} = \sum_{i=1}^n a_i \boldsymbol{g_i}$ and $0 < a_i \leq 1$, for all $1 \leq i \leq n$, if and only if $\boldsymbol{x} \in \mathrm{Reach}(\pi)$.*

3.3 The geometry of reachability sets for linear path schemes

A linear path scheme (LPS, for short) is a regular expression $\pi_0\chi_1^+\pi_1 \ldots \chi_n^+\pi_n$ over the transitions. Importantly, the π_i are (possibly empty) paths; the χ_i are cycles; and $\pi_0\chi_1\pi_1 \ldots \chi_n\pi_n$ is a valid path. Each LPS determines an infinite set $\{\pi_0\chi_1^{k_1} \ldots \chi_n^{k_n}\pi_n \mid k_1, \ldots, k_n \in \mathbb{N}_{>0}\}$ of paths that follows each of the paths exactly once and each of the cycles an arbitrary number of times. For this section, we fix a linear path scheme $\sigma = \pi_0\chi_1^+\pi_1 \ldots \chi_n^+\pi_n$.

3.3.1 From LPS to cones and a zonotope

From previous developments in this work, the reader might already believe that the reachability set of an LPS can be shown to be the Minkowski sum of suitable subsets of convex cones and zonotopes. It transpires that one can further simplify this and obtain a characterization which involves a single zonotope and a single cone.

Below, we write $G(\pi)$ to denote the generator of the zonotope for the path $\pi_0\pi_1 \ldots \pi_n$ as defined in Subsubsection 3.2.1; for each $1 \leq i \leq n$, we write $G(\chi_i)$ to denote the generator of the convex cone for the cycle χ_i as defined in Subsubsection 3.1.1.

▶ **Theorem 7.** $\mathrm{Reach}(\sigma)$ *is*[2] $\mathrm{relint}(\mathrm{zono}(G(\pi))) \cup \mathrm{adj}(G(\pi)) + \mathrm{relint}\left(\mathrm{cone}\left(\bigcup_{i=1}^n G(\chi_i)\right)\right)$.

To prove the theorem, we establish two intermediate results. The first one, together with Theorem 5, already yields the first (Minkowski) summand. The result below follows immediately from the definitions and commutativity of the Minkowski sum.

▶ **Lemma 8.** *We have that* $\mathrm{Reach}(\sigma) = \mathrm{Reach}(\pi_0\pi_1 \ldots \pi_n) + \sum_{i=1}^n \mathrm{Reach}(\chi_i^+)$.

The next result allows us to group the sums of cycle reachability sets into a single convex cone. Indeed, Theorem 1 and an induction on the following lemma yield the last summand from Theorem 7.

▶ **Lemma 9.** *Let C and C' be cones with generators G and G' respectively. Then, $C + C' =$* $\mathrm{cone}(G \cup G')$ *and* $\mathrm{relint}(C + C') = \mathrm{relint}(C) + \mathrm{relint}(C')$.

[2] To avoid clutter, we omit some parentheses: union has higher precedence than Minkowski sum.

4 The Complexity of $\mathbb{Q}$VASS Reachability

In this section, we use our results concerning the geometry of reachability sets to give an **NP** decision procedure for the reachability problem.

▶ **Theorem 10.** *Given a $\mathbb{Q}$VASS of dimension d, and two configurations $p(\boldsymbol{x})$ and $q(\boldsymbol{y})$, determining whether $p(\boldsymbol{x}) \xrightarrow{*} q(\boldsymbol{y})$ is in* **NP**.

First, without loss of generality, we assume $\boldsymbol{x} = \boldsymbol{0}$. Indeed, $p(\boldsymbol{x}) \xrightarrow{*} q(\boldsymbol{y})$ if and only if $p(\boldsymbol{0}) \xrightarrow{*} q(\boldsymbol{y} - \boldsymbol{x})$. In the following we prove that if $p(\boldsymbol{0}) \xrightarrow{*} q(\boldsymbol{x})$ then there is an LPS σ such that $p(\boldsymbol{0}) \xrightarrow{\pi} q(\boldsymbol{x})$ for some $\pi \in \sigma$ and σ has size polynomial on the number of transitions $|T|$ and on the dimension d. Then, we show that checking whether $p(\boldsymbol{0}) \xrightarrow{*} q(\boldsymbol{x})$ under a given linear path scheme is decidable in polynomial time. It follows that to check whether a configuration is reachable, in a general $\mathbb{Q}$VASS, one can guess a polynomial-sized LPS and check whether the corresponding configuration is reachable in it.

4.1 Short linear path schemes suffice

Presently, we argue that for any path we can find an LPS with a number of cycles that is polynomial in the number of transitions of the $\mathbb{Q}$VASS and the dimension such that all paths and cycles are simple, the set of transitions in the LPS is the same as that in the path, and the reachability set of the LPS includes that of the path.

For convenience, we define the *support* of a path $\pi = t_1 \dots t_n$ as the set of all transitions that are present in the path: $[\![\pi]\!] = \{t_i \mid 1 \leq i \leq n\}$. For an LPS $\sigma = \pi_0 \chi_1^+ \dots \chi_n^+ \pi_n$, its support is $[\![\sigma]\!] = [\![\pi_0]\!] \cup \bigcup_{i=1}^n ([\![\chi_i]\!] \cup [\![\pi_i]\!])$.

▶ **Theorem 11.** *Let π be an arbitrary path. Then, there exists a linear path scheme $\sigma = \pi_0 \chi_1^+ \pi_1 \dots \chi_n^+ \pi_n$, with $n \leq |T|$, such that all π_i and χ_i are simple paths and cycles, respectively, $[\![\pi]\!] = [\![\sigma]\!]$, and $\mathrm{Reach}(\pi) \subseteq \mathrm{Reach}(\sigma)$.*

Most properties of the LPS in the result above follow from considering an LPS with minimal length, with the length of an LPS defined as $|\sigma| = |\pi_0| + \sum_{i=1}^n |\chi_i| + |\pi_i|$. The main technical hurdle is thus the upper bound on the number of cycles. Our approach is to remove cycles whose support is covered by other cycles. The result below, which follows directly from Theorem 1 and Lemma 9, gives us that flexibility. As in Subsection 3.3, we write $G(\chi)$ to denote the generator of the convex cone for χ, i.e. $G(\chi) = \{\ell(t) \mid t \in [\![\chi]\!]\}$.

▶ **Lemma 12.** *Let χ be a cycle and C be a set of cycles. If $[\![\chi]\!] \subseteq \bigcup_{\theta \in C} [\![\theta]\!]$ then $\mathrm{Reach}(\chi^+) + \sum_{\theta \in C} \mathrm{Reach}(\theta^+) = \sum_{\theta \in C} \mathrm{Reach}(\theta^+)$.*

Hence, to check whether a configuration is reachable in a general $\mathbb{Q}$VASS, one can guess a polynomial-sized LPS and check whether the corresponding configuration is reachable in it. To conclude membership in **NP**, it remains to argue that the latter check can be realized in polynomial time.

4.2 Reachability for linear path schemes is tractable

In this section, we show that determining whether a configuration is reachable via a linear path scheme is decidable in polynomial time.

▶ **Theorem 13.** *Given LPS σ and $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Q}^d$, determining whether $\boldsymbol{y} \in \mathrm{Reach}^{\boldsymbol{x}}(\sigma)$ is in* **P**.

Based on Theorem 7 and the geometric characterizations of the reachability sets of cycles and paths, it suffices to show how to determine whether there exist $\boldsymbol{z}, \boldsymbol{c} \in \mathbb{Q}^d$ in a zonotope and a cone, respectively, such that $\boldsymbol{y} = \boldsymbol{z} + \boldsymbol{c}$. To do so, we make Lemma 6 and Lemma 2 effective by encoding them into systems of linear inequalities with *strict-inequality constraints*. It is known that the feasibility problem for linear programs is decidable in polynomial time even in the presence of strict inequalities (see, e.g., [18, Ch. 8.7.1]).

Henceforth, we fix an LPS $\sigma = \pi_0 \chi_1^+ \pi_1 \dots \chi_n^+ \pi_n$. We also adopt the notation from Subsection 3.3: $G(\pi)$ denotes the generator of the zonotope for the path $\pi_0 \pi_1 \dots \pi_n$; and $G(\chi_i)$ the generator of the cone for χ_i for each $1 \leq i \leq n$.

4.2.1 Encoding the zonotope

Let $G(\pi) = \{\boldsymbol{g_1}, \dots, \boldsymbol{g_m}\}$. We now define the matrix $\boldsymbol{A} \in \mathbb{Q}^{d \times (m+d)}$ and the vector $\boldsymbol{a} \in \mathbb{Q}^d$:

$$\boldsymbol{A} = \begin{pmatrix} (\boldsymbol{g_1})_1 & (\boldsymbol{g_2})_1 & \dots & (\boldsymbol{g_m})_1 & -1 & 0 & \dots & 0 \\ (\boldsymbol{g_1})_2 & (\boldsymbol{g_2})_2 & \dots & (\boldsymbol{g_m})_2 & 0 & -1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (\boldsymbol{g_1})_d & (\boldsymbol{g_2})_d & \dots & (\boldsymbol{g_m})_d & 0 & \dots & 0 & -1 \end{pmatrix} \text{ and } \boldsymbol{a} = (0, \dots, 0), \tag{1}$$

Further, we define the matrix $\boldsymbol{B} \in \mathbb{Q}^{m \times (m+d)}$ and the vector $\boldsymbol{b} \in \mathbb{Q}^m$ as:

$$\boldsymbol{B} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} & \dots & \boldsymbol{0} \end{pmatrix} \text{ and } \boldsymbol{b} = (1, 1, \dots, 1), \tag{2}$$

where $\boldsymbol{I}$ is the $m \times m$ identity matrix and $\boldsymbol{0}$ is the $m \times 1$ zero vector. Finally, we define $\boldsymbol{C} \in \mathbb{Q}^{m \times (m+d)}$ and $\boldsymbol{c} \in \mathbb{Q}^m$ as follows.

$$\boldsymbol{C} = \begin{pmatrix} -\boldsymbol{I} & \boldsymbol{0} & \dots & \boldsymbol{0} \end{pmatrix} \text{ and } \boldsymbol{c} = (0, 0, \dots, 0) \tag{3}$$

▶ **Lemma 14.** *$\boldsymbol{Az} = \boldsymbol{a} \wedge \boldsymbol{Bz} \leq \boldsymbol{b} \wedge \boldsymbol{Cz} < \boldsymbol{c}$ has a solution $(\boldsymbol{\alpha}, \boldsymbol{y}) \in \mathbb{Q}^{m+d}$ iff $\boldsymbol{y} \in \mathrm{Reach}(\pi)$.*

This follows immediately from the fact that, by construction, the system has a solution if and only if there exists $(\alpha_1, \dots, \alpha_m) \in (0, 1]$ such that $\boldsymbol{y} = \sum_{i=1}^m \alpha_i \boldsymbol{g_i}$, and Lemma 6.

4.2.2 Encoding the cone

The encoding for the code is even simpler, but requires we adapt our notation slightly.

Let $\bigcup_{i=1}^n G(\chi_i) = \{\boldsymbol{g_1}, \dots, \boldsymbol{g_m}\}$. We define the matrix $\boldsymbol{A} \in \mathbb{Q}^{d \times (m+d)}$ and vector $\boldsymbol{a} \in \mathbb{Q}^d$ as in Equation 1; and $\boldsymbol{C} \in \mathbb{Q}^{m \times (m+d)}$ and $\boldsymbol{c} \in \mathbb{Q}^d$ as in Equation 3.

▶ **Lemma 15.** *$\boldsymbol{Az} = \boldsymbol{a} \wedge \boldsymbol{Cz} < \boldsymbol{c}$ has a solution $(\boldsymbol{\alpha}, \boldsymbol{y}) \in \mathbb{Q}^{m+d}$ iff $\boldsymbol{y} \in \sum_{i=1}^n \mathrm{Reach}(\chi_i^+)$.*

This time the lemma follows from Lemma 2 and because, by construction, the system has a solution if and only if there are $(\alpha_1, \dots, \alpha_m)$ nonnegative such that $\boldsymbol{y} = \sum_{i=1}^m \alpha_i g_i$.

Proof of Theorem 13. The result follows from the fact that Lemma 8 can be made effective by encoding it into a master system of linear inequalities for the zonotope and the cycle. ◀

5 The Complexity of $\mathbb{Q}_{\geq 0}$VASS Reachability

We now give an **NP** decision procedure for the reachability problem for $\mathbb{Q}_{\geq 0}$VASS.

▶ **Theorem 16.** *Given a $\mathbb{Q}_{\geq 0}$VASS of dimension d, and configurations $p(\boldsymbol{x})$ and $q(\boldsymbol{y})$, determining whether $p(\boldsymbol{x}) \xrightarrow{*} q(\boldsymbol{y})$ is in* **NP***.*

The structure of our argument is similar to the one in Section 4: we prove that short LPS suffice and then we prove reachability is tractable for LPS. The first part is considerably more complicated for $\mathbb{Q}_{\geq 0}$VASS and it is based on the fact that short LPS exist for $\mathbb{Q}$VASS. For this reason, we need additional notation: We write $p(\boldsymbol{x}) \xrightarrow{*} q(\boldsymbol{y})$ to denote that $q(\boldsymbol{y})$ is reachable from $p(\boldsymbol{x})$ in a $\mathbb{Q}_{\geq 0}$VASS and instead use $p(\boldsymbol{x}) \overset{*}{\dashrightarrow} q(\boldsymbol{y})$ to denote that $q(\boldsymbol{y})$ is reachable from $p(\boldsymbol{x})$ with respect to $\mathbb{Q}$VASS semantics (i.e. when allowing negative counter values). Similarly, we write $\mathrm{Reach}^{\boldsymbol{x}}(\cdot)$ for reachability sets w.r.t. $\mathbb{Q}$VASS and $\mathrm{Reach}^{\boldsymbol{x}}_{\geq 0}(\cdot)$ for that w.r.t. $\mathbb{Q}_{\geq 0}$VASS. We make repeated use of the following result by Blondin and Haase.

▶ **Lemma 17** (From [2, Proposition 4.5]). *There exists a path π such that $q(\boldsymbol{x}) \xrightarrow{\pi} q(\boldsymbol{y})$ if and only if there exist paths π_1, π_2, π_3 such that:*

1. $q(\boldsymbol{x}) \overset{\pi_2}{\dashrightarrow} q(\boldsymbol{y})$*;*
2. $q(\boldsymbol{x}) \xrightarrow{\pi_1} q(\boldsymbol{x'})$*, for some* $\boldsymbol{x'} \in \mathbb{Q}^d_{\geq 0}$*;*
3. $q(\boldsymbol{y'}) \xrightarrow{\pi_3} q(\boldsymbol{y})$*, for some* $\boldsymbol{y'} \in \mathbb{Q}^d_{\geq 0}$*; and*
4. $[\![\pi]\!] = [\![\pi_1]\!] = [\![\pi_2]\!] = [\![\pi_3]\!]$.

Moreover, if item 1–item 4 hold, then $\boldsymbol{y} \in \mathrm{Reach}^{\boldsymbol{x}}_{\geq 0}(\pi_1 \pi_2^+ \pi_3)$.

Intuitively, $q(\boldsymbol{x}) \xrightarrow{\pi} q(\boldsymbol{y})$ if the following conditions hold: first, we obviously need $q(\boldsymbol{x}) \overset{\pi_2}{\dashrightarrow} q(\boldsymbol{y})$, and second, we need $q(\boldsymbol{x})$ to allow some "wiggle room" using the same transitions as π and while keeping the counters nonnegative. Similarly, there should be wiggle room to reach $q(\boldsymbol{y})$ with nonnegative counters. The lemma also shows that these conditions are necessary.

5.1 Short linear path schemes suffice

We start by proving that short LPS suffice.

▶ **Theorem 18.** *Let π be an arbitrary path such that $p(\boldsymbol{x}) \xrightarrow{\pi} q(\boldsymbol{y})$. Then, there exists a linear path scheme $\sigma = \pi_0 \chi_1^+ \pi_1 \ldots \chi_n^+ \pi_n$, with:*

- $n \leq |Q|$,
- π_i *is a simple path for all* $0 \leq i \leq n$,
- $|\chi_i| \leq 4|Q|(|T| + d + 2)(|T| + 1)$ *for all* $1 \leq i \leq n$*, and*
- $\boldsymbol{y} \in \mathrm{Reach}^{\boldsymbol{x}}_{\geq 0}(\sigma)$.

Let us we outline our approach to prove this. Consider a path π such that $p(\boldsymbol{x}) \xrightarrow{\pi} q(\boldsymbol{y})$. We decompose π into $\pi = \tau_0 \theta_1 \tau_1 \cdots \theta_n \tau_n$ where the τ_i are simple paths separated by at most $|Q|$ (not necessarily simple) cycles θ_i. We would now like to replace each cycle θ with a short LPS χ^+, as per the third item in Theorem 18. Note that we cannot readily do so using Theorem 11, as it does not guarantee nonnegative reachability. We thus take a more elaborate approach, as follows.

Consider a cycle θ. By applying Lemma 17, we can replace θ by an LPS $\pi_1 \pi_2^+ \pi_3$. We now apply Theorem 11 to π_2, thus obtaining an LPS $\sigma = \mu_0 \zeta_1^+ \mu_1 \cdots \zeta_m^+ \mu_m$ such that $\mathrm{Reach}^{\boldsymbol{z}}(\pi_2) \subseteq \mathrm{Reach}^{\boldsymbol{z}}(\sigma)$ for all $\boldsymbol{z}$ (note that nonnegativity is no longer maintained). Recall that our goal is to represent π_2^+ as an LPS, so we cannot use σ^+, as it is not an LPS. Instead, we show that by looking at the concrete path $\overline{\sigma} = \mu_0 \zeta_1 \mu_1 \cdots \zeta_m \mu_m$ we have $\mathrm{Reach}^{\boldsymbol{z}}(\pi_2^+) \subseteq \mathrm{Reach}^{\boldsymbol{z}}(\overline{\sigma}^+)$, and $\overline{\sigma}^+$ is an LPS.

The next challenge is to plug back $\overline{\sigma}^+$ as part of an LPS for θ. To do so, we need to find LPS for π_1 and π_3. We show that this is possible. We can now use Lemma 17 again, in the opposite direction, to conclude that $\pi_1 \pi_2^+ \pi_3$ can be described by an LPS with appropriate bounds, retaining nonnegative reachability.

5.2 Reachability for linear path schemes is tractable

As in the $\mathbb{Q}$VASS case, here we are able to prove that determining whether a configuration is reachable via an LPS is decidable in polynomial time.

▶ **Theorem 19.** *Given LPS σ and $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Q}^d_{\geq 0}$, determining if $\boldsymbol{y} \in \mathrm{Reach}^{\boldsymbol{x}}_{\geq 0}(\sigma)$ is in* **P**.

Once more, our argument relies on an encoding into a system of linear inequalities. However, in contrast to $\mathbb{Q}$VASS, the encoding is slightly less elegant. For each path $\pi = (p_1, p_2) \ldots (p_m, p_{m+1})$ occurring inside σ, instead of encoding an (affine) zonotope into a system of linear equalities, we focus directly on the m steps of the path.

Let $\boldsymbol{b_i} = \ell(p_i, p_{i+1})$ for all $1 \leq i \leq m$. We will use the $\boldsymbol{b_i}$ instead of the zonotope basis $G(\pi)$ of Subsection 4.2. We now adapt the system from Subsection 4.2 to account for the nonnegativity of partial sums induced by the path prefixes. Recall $\boldsymbol{A}$, $\boldsymbol{a}$, $\boldsymbol{B}$, $\boldsymbol{b}$, $\boldsymbol{C}$, and $\boldsymbol{c}$ as defined in Equation 1, Equation 2, and Equation 3. We introduce $d + md$ new variables – the first d account for an initial vector $\boldsymbol{x}$, and the remaining md represent the intermediate values of the path after each transition.

Intuitively, we obtain from the above matrices new ones, denoted $\boldsymbol{A'}$, $\boldsymbol{a'}$, $\boldsymbol{B'}$, $\boldsymbol{b'}$, $\boldsymbol{C'}$ so that $\boldsymbol{A'}$ includes the constraints $\bigwedge_{k=1}^{d} \bigwedge_{n=1}^{m} \left(\boldsymbol{x}_k + \sum_{j=1}^{n} (\boldsymbol{b_j})_k z_j\right) = z_{m+2d+(n-1)d+k} \geq 0$. We thus have the following.

▶ **Lemma 20.** *For a path π, the system $\boldsymbol{A'z} = \boldsymbol{a'} \wedge \boldsymbol{B'z} \leq \boldsymbol{b'} \wedge \boldsymbol{C'z} < \boldsymbol{c'}$ of linear inequalities has a solution $(\boldsymbol{\alpha}, \boldsymbol{y}, \boldsymbol{x}, \boldsymbol{\iota}) \in \mathbb{Q}^{m+2d+md}$ if and only if $p_1(\boldsymbol{x}) \xrightarrow{\pi} p_{m+1}(\boldsymbol{y})$.*

Proof of Theorem 19. By Lemma 20, it suffices to argue that nonnegative reachability via a cycle χ^+ of σ can also be encoded into a system of linear inequalities. For this, we make use of the "if" direction of Lemma 17, which for χ^+ amounts to $p(\boldsymbol{x}) \xrightarrow{\chi^+} p(\boldsymbol{y})$ iff (1) $p(\boldsymbol{x}) \overset{\chi+}{\dashrightarrow} p(\boldsymbol{y})$, (2) $p(\boldsymbol{x}) \xrightarrow{\chi} p(\boldsymbol{x'})$ for some $\boldsymbol{x'} \in \mathbb{Q}_{\geq 0}$, (3) $p(\boldsymbol{y'}) \xrightarrow{\chi} p(\boldsymbol{y})$ for some $\boldsymbol{y'} \in \mathbb{Q}_{\geq 0}$. Condition 1 can be encoded in a system of linear inequalities by Lemma 15, and conditions 2 and 3 can be encoded in such systems too as per Lemma 20. We can now conjoin the systems for the paths and cycles to obtain a master system of linear inequalities of polynomial size. ◀

6 The Complexity of Reachability with Zero Tests

We now argue that the reachability problem for continuous VASS with *zero tests* is **NP**-hard, already for LPS of dimension 2. For convenience, we state the result for $\mathbb{Q}$VASS. However, we note that the same proof establishes the result for $\mathbb{Q}_{\geq 0}$VASS.

We start by formally defining the model. A continuous VASS $\mathcal{V}$ of dimension 2 with zero tests is a tuple (Q, t, ℓ, Z_1, Z_2), where $\mathcal{V}' = (Q, t, \ell)$ is a continuous VASS and $Z_i \subseteq Q$ for $i = 1, 2$. A run $\rho = q_1(\boldsymbol{x}_1) \ldots q_n(\boldsymbol{x}_n)$ of such a VASS is a run of $\mathcal{V}'$ such that, additionally, for all $1 \leq i \leq n$ we have that if $q_i \in Z_j$, for some $j \in \{1, 2\}$, then $(\boldsymbol{x_i})_j = 0$. That is, any run that reaches a state in Z_j must be such that the the value of the j-th counter is 0 then.

▶ **Theorem 21.** *For every $d \in \mathbb{N}, d \geq 2$, given a $\mathbb{Q}$VASS (or $\mathbb{Q}_{\geq 0}$VASS) of dimension d, and two configurations $p(\boldsymbol{x})$ and $q(\boldsymbol{y})$, determining whether $\mathbf{p}(\boldsymbol{x}) \xrightarrow{*} q(\boldsymbol{y})$ is* **NP***-hard, even for linear path schemes of dimension* 2.

We reduce from the PrimeCover problem: Given a set X of prime numbers and a collection S of subsets of X, with $T \in \mathbb{N}$, determine whether there is a subset $S' \subseteq S$ such that $\prod_{s' \in S'} \prod_{p \in s'} p = T$. It is straightforward to prove PrimeCover is **NP**-hard by reduction from the the ExactCover problem [13].

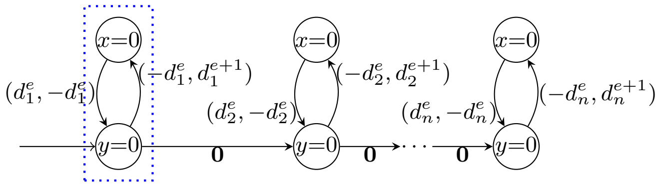

Figure 3 In the dotted blue box, a multiplier gadget is shown: the state above is an element of Z_1 (noted by $x = 0$) while the state below is an element of Z_2 (noted by $y = 0$); The whole LPS encodes an instance of PrimeCover with $S = \{s_1, s_2, \ldots, s_n\}$ – recall that $d_i = \prod_{p \in c_i} p$.

Now, for each $s \in S$ we create multiplier gadgets as depicted in Figure 3 where $d = \prod_{p \in s} p$ and $e = \lceil \log_2(\prod_{s \in S} \prod_{p \in s} p) \rceil + 1$, and we link them in an LPS with transitions (q_i, q_{i+1}), for $1 \leq i \leq |S|$, labelled with $(0, 0)$ updates (see Figure 3). We claim that PrimeCover has a positive answer if and only if $q_1(1, 0) \xrightarrow{*} q_{|C|}(T, 0)$ in the constructed LPS.

7 Conclusion

We gave geometrical characterizations for the reachability sets of continuous VASS and their flat and LPS restrictions. Using these, we showed that polynomial-sized LPS suffice as witnesses of reachability and that reachability in linear path schemes is tractable. In addition, we sharpened hardness results in the presence of zero tests: it is **NP**-hard already for dimension two.

References

1 Michael Blondin, Matthias Englert, Alain Finkel, Stefan Göller, Christoph Haase, Ranko Lazic, Pierre McKenzie, and Patrick Totzke. The reachability problem for two-dimensional vector addition systems with states. *J. ACM*, 68(5):34:1–34:43, 2021. doi:10.1145/3464794.

2 Michael Blondin and Christoph Haase. Logics for continuous reachability in Petri nets and vector addition systems with states. In *2017 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–12. IEEE, 2017.

3 Wojciech Czerwinski, Slawomir Lasota, Ranko Lazic, Jérôme Leroux, and Filip Mazowiecki. Reachability in fixed dimension vector addition systems with states. In Igor Konnov and Laura Kovács, editors, *31st International Conference on Concurrency Theory, CONCUR 2020, September 1-4, 2020, Vienna, Austria (Virtual Conference)*, volume 171 of *LIPIcs*, pages 48:1–48:21. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2020. doi:10.4230/LIPIcs.CONCUR.2020.48.

4 Wojciech Czerwinski, Slawomir Lasota, Christof Löding, and Radoslaw Piórkowski. New pumping technique for 2-dimensional VASS. In Peter Rossmanith, Pinar Heggernes, and Joost-Pieter Katoen, editors, *44th International Symposium on Mathematical Foundations of Computer Science, MFCS 2019, August 26-30, 2019, Aachen, Germany*, volume 138 of *LIPIcs*, pages 62:1–62:14. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2019. doi:10.4230/LIPIcs.MFCS.2019.62.

5 Wojciech Czerwinski, Slawomir Lasota, and Lukasz Orlikowski. Improved lower bounds for reachability in vector addition systems. In Nikhil Bansal, Emanuela Merelli, and James Worrell, editors, *48th International Colloquium on Automata, Languages, and Programming, ICALP 2021, July 12-16, 2021, Glasgow, Scotland (Virtual Conference)*, volume 198 of *LIPIcs*, pages 128:1–128:15. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2021. doi:10.4230/LIPIcs.ICALP.2021.128.

6 Wojciech Czerwinski and Lukasz Orlikowski. Reachability in vector addition systems is Ackermann-complete. In *62nd IEEE Annual Symposium on Foundations of Computer Science, FOCS 2021, Denver, CO, USA, February 7-10, 2022*, pages 1229–1240. IEEE, 2021. doi:10.1109/FOCS52979.2021.00120.

7 René David and Hassane Alla. *Discrete, continuous, and hybrid Petri nets*, volume 1. Springer, 2010.

8 Alain Finkel, Jérôme Leroux, and Grégoire Sutre. Reachability for two-counter machines with one test and one reset. In Sumit Ganguly and Paritosh K. Pandya, editors, *38th IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2018, December 11-13, 2018, Ahmedabad, India*, volume 122 of *LIPIcs*, pages 31:1–31:14. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2018. doi:10.4230/LIPIcs.FSTTCS.2018.31.

9 Christoph Haase. *On the complexity of model checking counter automata.* PhD thesis, University of Oxford, UK, 2012. URL: http://ora.ox.ac.uk/objects/uuid:f43bf043-de93-4b5c-826f-88f1bd4c191d.

10 Christoph Haase. A survival guide to presburger arithmetic. *ACM SIGLOG News*, 5(3):67–82, 2018. doi:10.1145/3242953.3242964.

11 Christoph Haase and Simon Halfon. Integer vector addition systems with states. In Joël Ouaknine, Igor Potapov, and James Worrell, editors, *Reachability Problems – 8th International Workshop, RP 2014, Oxford, UK, September 22-24, 2014. Proceedings*, volume 8762 of *Lecture Notes in Computer Science*, pages 112–124. Springer, 2014. doi:10.1007/978-3-319-11439-2_9.

12 Christoph Haase, Stephan Kreutzer, Joël Ouaknine, and James Worrell. Reachability in succinct and parametric one-counter automata. In Mario Bravetti and Gianluigi Zavattaro, editors, *CONCUR 2009 – Concurrency Theory, 20th International Conference, CONCUR 2009, Bologna, Italy, September 1-4, 2009. Proceedings*, volume 5710 of *Lecture Notes in Computer Science*, pages 369–383. Springer, 2009. doi:10.1007/978-3-642-04081-8_25.

13 Richard M Karp. Reducibility among combinatorial problems. In *Complexity of computer computations*, pages 85–103. Springer, 1972.

14 Jérôme Leroux. The reachability problem for petri nets is not primitive recursive. In *62nd IEEE Annual Symposium on Foundations of Computer Science, FOCS 2021, Denver, CO, USA, February 7-10, 2022*, pages 1241–1252. IEEE, 2021. doi:10.1109/FOCS52979.2021.00121.

15 Jérôme Leroux and Sylvain Schmitz. Reachability in vector addition systems is primitive-recursive in fixed dimension. In *34th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2019, Vancouver, BC, Canada, June 24-27, 2019*, pages 1–13. IEEE, 2019. doi:10.1109/LICS.2019.8785796.

16 Jérôme Leroux and Grégoire Sutre. On flatness for 2-dimensional vector addition systems with states. In Philippa Gardner and Nobuko Yoshida, editors, *CONCUR 2004 – Concurrency Theory, 15th International Conference, London, UK, August 31 – September 3, 2004, Proceedings*, volume 3170 of *Lecture Notes in Computer Science*, pages 402–416. Springer, 2004. doi:10.1007/978-3-540-28644-8_26.

17 Marvin Lee Minsky. *Computation.* Prentice-Hall Englewood Cliffs, 1967.

18 Christos H. Papadimitriou and Kenneth Steiglitz. *Combinatorial Optimization: Algorithms and Complexity.* Prentice-Hall, 1982.

19 Sylvain Schmitz. The complexity of reachability in vector addition systems. *ACM SigLog News*, 3(1):4–21, 2016.