Relaxed Core Stability for Hedonic Games with Size-Dependent Utilities

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Abstract

We study relationships between different relaxed notions of core stability in hedonic games. In particular, we study (i) q-size core stable outcomes in which no deviating coalition of size at most q exists and (ii) k-improvement core stable outcomes in which no coalition can improve by a factor of more than k. For a large class of hedonic games, including fractional and additively separable hedonic games, we derive upper bounds on the maximum factor by which a coalition of a certain size can improve in a q-size core stable outcome. We further provide asymptotically tight lower bounds for a large class of hedonic games. Finally, our bounds allow us to confirm two conjectures by Fanelli et al. [20][IJCAI'21] for symmetric fractional hedonic games (S-FHGs): (i) every q-size core stable outcome in an S-FHG is also $\frac{q}{q-1}$-improvement core stable and (ii) the price of anarchy of q-size stability in S-FHGs is precisely $\frac{2q}{q-1}$.

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Supplementary Material *Software*: `https://github.com/DemeulemeesterT/Relaxations-Core-Stability-Alpha-Hedonic-Games`
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1 Introduction

Coalition formation is one of the core topics of multiagent systems and algorithmic game theory. Hedonic games ([19]) constitute the most popular subcase of coalition formation. In a hedonic game, the goal is to divide a set of agents into disjoint coalitions, respecting the preferences of the agents. Over the years, multiple different ways of representing the agents' preferences and multiple different solution concepts emerged. Among the strongest solution concepts is core stability ([9]): a coalition structure is core stable, if no subset of agents could together form a new coalition in which they are all better off than in the original coalition structure. While being a seemingly natural concept, it has been shown that even for very simple preference structures, core stable outcomes may not exist ([2, 3]). Further, in these structures, it is also often computationally intractable to decide whether a core stable

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outcome exists.[1] These results led Fanelli et al. [20] to introduce two natural weakenings of core stability: (i) q-size (core) stability, which requires that no blocking coalitions of size at most q exist, (ii) k-improvement (core) stability, which requires that no blocking coalition exists in which every agent improves by a factor of more than k. For the example of fractional hedonic games ([2]), [20] showed that a 2-size stable outcome and a 2-improvement stable outcome always exist. Further, they also studied the relationship between these two notions and were able to show that a 2-size stable outcome is indeed also always 2-improvement stable.

In this paper, we contribute to this literature in two ways. First, we propose a new class of hedonic games, called α-hedonic games, in which the utility an agent receives from being in a coalition of size m is equal to the sum of the cardinal utilities it ascribes to the other agents in that coalition, multiplied by a factor α_m which depends on the size of that coalition. Several well-studied classes of hedonic games, such as fractional hedonic games [2], modified fractional hedonic games [27], and additively separable hedonic games [9] are a special case of α-hedonic games.

Second, we further study the two weakenings of core stability that were introduced by Fanelli et al. [20]. Our main result quantifies, for any α-hedonic game and for any q-size stable outcome, the maximum factor with which the agents can improve their utility by forming a blocking coalition of size $m \geq q+1$. As a corollary, this allows us to prove two conjectures by Fanelli et al. [20]: (i) every q-size stable outcome is $\frac{q}{q-1}$-improvement stable for fractional hedonic games and (ii) the q-size core price of anarchy (i.e., the worst-case approximation to the social welfare of any q-size core stable outcome) is exactly $\frac{2q}{q-1}$ for fractional hedonic games.

Related Work

Since its inception hedonic games have been a widely studied topic in algorithmic game theory, with several works studying axiomatic or computational properties of hedonic games. For an overview on earlier developments, we refer the reader to the book chapter by Aziz and Savani [1]. In recent years, several new models and optimality notions for hedonic games were introduced and analyzed. Among the most popular of these notions are the aforementioned fractional hedonic games, introduced by Aziz et al. [2] and studied in various forms by, e.g., Bilò et al. [5], Aziz et al. [4], or Carosi et al. [15]. Fractional hedonic games are also related to the model of hedonic diversity games [13, 8, 21] in which agents possess types and derive utility based on the fraction of agents of their own type in their coalition.

The paper closest to ours is the work by Fanelli et al. [20], who introduced the aforementioned notions of q-size and k-improvement core stability for fractional hedonic games. Alternative simplifications of core stability were introduced by Carosi et al. [15], who studied a local variant of core stability for simple fractional hedonic games, i.e., hedonic games in which all utility values are either 0 or 1. In their local variant of core stability, the agents deviating are required to form a clique. For this weakened notion, they show that core stable outcomes always exist and can be computed via improving response dynamics.

Finally, some very recent works on hedonic games include [7, 12, 11] who study various aspects of dynamics, i.e., decentralized processes in which agents perform beneficial changes until a stable outcome is reached, the study of coalition formation with (almost) fixed

[1] For some preference structures, it can even be hard to find a coalition structure in the core, even if it is guaranteed to exist, see Bullinger and Kober [14].

coalition sizes [18, 6, 25, 24], or studies of the complexity of various hedonic games variants [17, 16, 22, 10]. Another interesting direction are the models of altruistic [23] and loyal [14] hedonic games, in which the utilities of agents do not only depend on their own utility, but also on the utility of their friends/agents they are loyal to.

2 Preliminaries

For any $n \in \mathbb{N}^+$ and $\alpha\colon [n] \to \mathbb{R}^+$ an *α-hedonic game* (αHG) consists of a set of *agents* $A = \{a_1, \ldots, a_n\}$ with a *utility function* $u\colon A \times A \to \mathbb{R}$. We restrict ourselves to *symmetric* α-hedonic games (S-αHGs) in this paper, and require that $u(i, j) = u(j, i)$ for all $i, j \in A$. A *coalition* is a subset of A and a *coalition structure* is a partition of A into coalitions. The utility of an agent i in a coalition C is $u_i(C) \coloneqq \sum_{j \in C} \alpha(|C|) \cdot u(i, j)$. We assume that $u(i, i) = 0$. For a coalition structure $\mathcal{C}$ the utility $u_i(\mathcal{C})$ of the coalition structure for agent i is the utility of the coalition agent i belongs to. To simplify notation, for agents a_i and a_j and $C \subseteq A$ we also write $u_i(a_j) \coloneqq u(a_i, a_j)$ and $u_i(C) \coloneqq u_{a_i}(C)$.

The class of α-hedonic games generalizes multiple previously studied hedonic game classes, e.g.,

- *Symmetric additively separable hedonic games* (S-ASHGs) with $\alpha(m) = 1$ for any $m \in \mathbb{N}$.
- *Symmetric fractional hedonic games* (S-FHGs) with $\alpha(m) = \frac{1}{m}$ for any $m \in \mathbb{N}$.
- *Symmetric modified fractional hedonic games* (S-MFHGs) with $\alpha(m) = \frac{1}{m-1}$ for any $m \in \mathbb{N}^+$ and $\alpha(1) = 0$.

A given coalition structure $\mathcal{C}$ is

- *core stable* if for any coalition C it holds that $u_i(C) \leq u_i(\mathcal{C})$ for at least one $i \in C$;
- *q-size core stable* if for any coalition C with $|C| \leq q$ it holds that $u_i(C) \leq u_i(\mathcal{C})$ for at least one $i \in C$;
- *k-improvement core stable* if for any coalition C it holds that $u_i(C) \leq k u_i(\mathcal{C})$ for at least one $i \in C$;
- *(q, k)-core stable* if for any coalition C with $|C| = q$ it holds that $u_i(C) \leq k u_i(\mathcal{C})$ for at least one $i \in C$.

If there is a coalition witnessing a violation to one of these criteria, for instance a coalition C with $u_i(C) > u_i(\mathcal{C})$ for all $i \in C$, we say that C is a *blocking coalition.* In further parts of the paper, we shorten $\alpha(m)$ to α_m to increase readability. Moreover, denote by $\mathbb{1}\colon \mathbb{N} \to \{0, 1\}$ the indicator function such that for any $i \in \mathbb{N}$

$$\mathbb{1}(i) = \begin{cases} 0 & \text{if } i = 0, \\ 1 & \text{else.} \end{cases}$$

Lastly, every S-αHG can be represented by a graph $G(A, E, w)$, where A represents the set of agents, and E contains an undirected edge $\{i, j\}$ between agents i and j with weight $w_{ij} = u(i, j) = u(j, i)$ if $u(i, j) > 0$. Alternatively, given a coalition $C \subseteq A$, we denote the subgraph of $G(A, E, w)$ that is induced by only considering the agents in C by $G(C)$.

Before turning to our result, we present two simple examples, with the second example motivating why we exclusively focus on symmetric instances.

▶ **Example 1.** First, consider the hedonic game induced by the graph on the left of Figure 1, with utilities as indicated by the edges and omitted edges indicating a utility of 0. Here, consider the coalition structure $\{\{a_1, a_2\}, \{a_3, a_4\}\}$. In an S-ASHG, the utility of every agent would be 3, and the coalition structure would be 2-size core stable, but not 3-size core stable, since, for instance, $\{a_1, a_2, a_3\}$ would block. Further, the coalition structure is $(3, \frac{5}{3})$- and

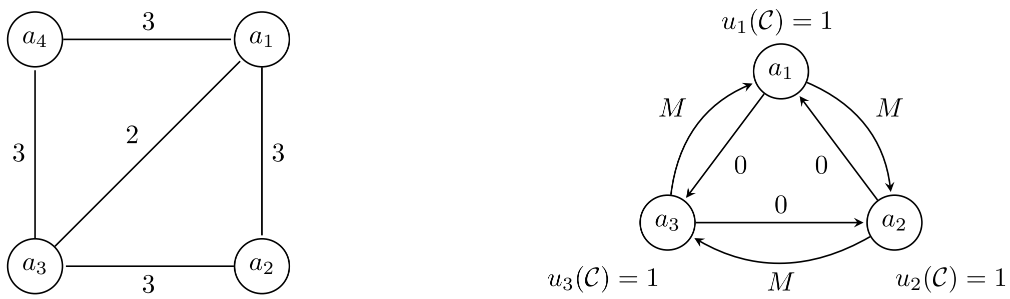

Figure 1 Example of a symmetric hedonic game on the left and of a blocking coalition for an asymmetric game on the left, for which the improvement ratio is unbounded.

$(4, 2)$-core stable and thus also 2-improvement core stable. In an S-FHG, on the other hand, the utility of every agent would be $\frac{3}{2}$ and the coalition $\{a_1, a_2, a_3\}$ would still block. The coalition consisting of all agents, however, would no longer be blocking, since the utility of agent a_2 would be $\frac{6}{4} = \frac{3}{2}$, which was their utility in the original coalition structure. Finally, in an S-MFHG, the utility of every agent would be 3 and the coalition structure would be core stable. Even the coalition $\{a_1, a_2, a_3\}$ would no longer block, since the utility of agent a_1 would be $\frac{5}{2} < 3$.

Secondly, to motivate the choice of symmetric hedonic games, consider the (asymmetric) hedonic game depicted on the right of Figure 1 with all three agents originally being in a coalition structure $\mathcal{C}$ in which they experience utility 1. This coalition structure would be 2-size core stable. However, there is no upper bound on the improvement ratio for the coalition consisting of all three agents, as M goes to infinity. We note that this behaviour can be observed independently of the considered α function.

2.1 Our results

Fanelli et al. [20] conjectured that for fractional hedonic games, every q-size core stable coalition structure is also $\frac{q}{q-1}$-improvement core stable. We refine this conjecture and show that every q-size core stable coalition structure is

$$\left(m, 1 + \frac{\left\lfloor \frac{1}{q-1}(m-2) \right\rfloor}{m} \right) \text{-core stable} \tag{1}$$

for any $m \geq q+1$. As $1 + \frac{\left\lfloor \frac{1}{q-1}(m-2) \right\rfloor}{m} \leq \frac{q}{q-1}$ for any m this implies the conjecture of Fanelli et al. [20]. Further, this result together with the results of Fanelli et al. [20] also allows us to confirm their second conjecture that the *price of anarchy* of q-size stability is exactly $\frac{2q}{q-1}$.

To gain a better intuition of this quite unhandy term, we refer the reader to Table 1 and Figure 2.

In fact, our proof does not only apply to S-FHGs, but to all symmetric α-hedonic games. The more general result that we are able to show is that every q-size core stable outcome in an S-αHG is $(m, f(q, m))$-core stable with

$$f(q, m) = \max \left(1, \left\lfloor \frac{m-1}{q-1} \right\rfloor \frac{\alpha_m}{\alpha_q} + \frac{\mathbb{1}\left((m-1) \bmod (q-1)\right) \alpha_m}{\alpha\left((m-1) \bmod (q-1) + 1\right)} \right).$$

As we discuss in Section 3, this bound is equivalent to Equation (1) for S-FHGs. For S-ASHGs this implies a bound of $f(q, m) = 1 + \left\lfloor \frac{m-2}{q-1} \right\rfloor$.

Table 1 For the given combinations of q and m, the table contains the value $f(q,m)$ derived from Equation (1), such that a q-size core stable is $(m, f(q,m))$-stable in S-FHGs.

q \ m	3	4	5	6	7	8	9	…	$\frac{q}{q-1}$
2	$\frac{4}{3}$	$\frac{6}{4}$	$\frac{8}{5}$	$\frac{10}{6}$	$\frac{12}{7}$	$\frac{14}{8}$	$\frac{16}{9}$		2
3	\	$\frac{5}{4}$	$\frac{6}{5}$	$\frac{8}{6}$	$\frac{9}{7}$	$\frac{11}{8}$	$\frac{12}{9}$	…	$\frac{3}{2}$
4	\	\	$\frac{6}{5}$	$\frac{7}{6}$	$\frac{8}{7}$	$\frac{10}{8}$	$\frac{11}{9}$		$\frac{4}{3}$

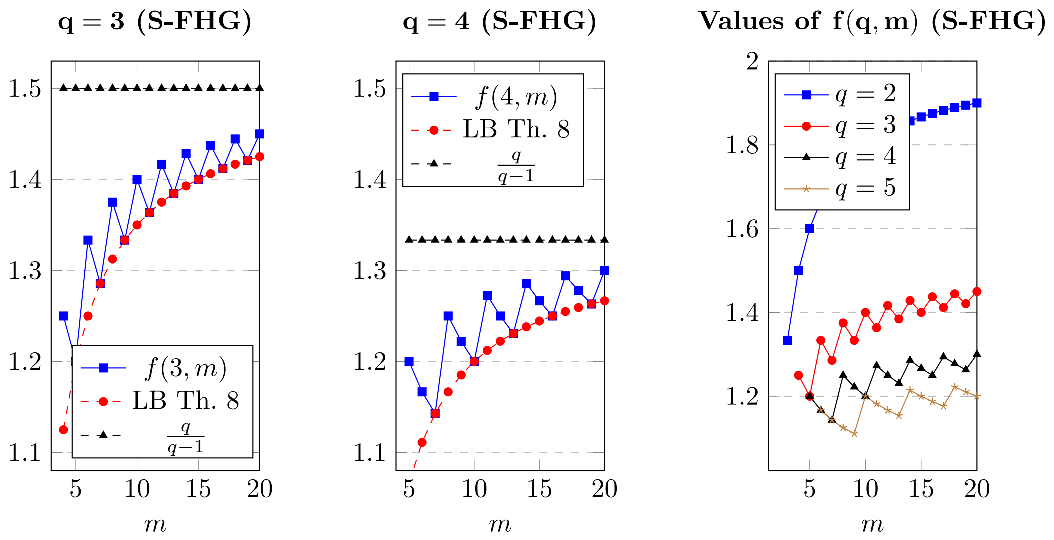

Figure 2 Plotted values of $f(q,m)$ for S-FHGs such that every q-size core stable coalition structure is $(m, f(q,m))$-core stable. In the two leftmost figures, LB indicates the lower bound obtained in Theorem 8, while the black line indicates $\frac{q}{q-1}$, the limit of the upper bound.

Further, in Section 4 we derive lower bounds on these values as well, and show tightness for various combinations of (q,m), and for various types of hedonic games. A summary of our results can be found in Table 3.

3 Main result

We begin with our main result, which quantifies the relationship between the two considered relaxed notions of core stability in symmetric α-hedonic games. As the general proof for symmetric α-hedonic games is quite notationally heavy, we defer the full proof to the supplementary material and only give the proof for symmetric fractional hedonic games here.

▶ **Theorem 2.** *Any q-size core stable coalition structure $\mathcal{C}$ in an S-αHG is $(m, f(q,m))$-core stable, for any integers m, q with $m \geq q+1$, and $f(q,m) =$*

$$\max\left(1, \left\lfloor \frac{m-1}{q-1} \right\rfloor \frac{\alpha_m}{\alpha_q} + \frac{\mathbb{1}\left((m-1) \bmod (q-1)\right)\alpha_m}{\alpha\left((m-1) \bmod (q-1)+1\right)}\right).$$

For fractional hedonic games, this reduces to:

▶ **Corollary 3.** *For S-FHGs, every q-size core stable coalition structure $\mathcal{C}$ is* $\left(m, 1 + \frac{\lfloor \frac{1}{q-1}(m-2) \rfloor}{m}\right)$*-core stable for any $m \geq q+1$.*

Proof. Consider a q-size core stable coalition structure $\mathcal{C}$ and a coalition C of size $m \geq q+1$. For a given coalition $C' \subseteq C$ we let $w(C') = \sum_{a_i \in C'} \left(u_i(C') - 2\frac{q}{q-1} u_i(\mathcal{C})\right)$ denote the *modified social welfare* of coalition C'. Let $\mathcal{C}_{q-1}$ be the set of coalitions of size $q-1$ and consider the weighted hypergraph $(C, \mathcal{C}_{q-1}, w)$. Let $M = \{C_1, \ldots, C_{\lfloor \frac{m-1}{q-1} \rfloor}\}$ be any maximum weight, with regard to w, hypergraph matching, i.e., selection of non-overlapping sets from $\mathcal{C}_{q-1}$, of size $\lfloor \frac{m-1}{q-1} \rfloor$ in this hypergraph. We note that a maximum weight hypergraph matching not of size $\lfloor \frac{m-1}{q-1} \rfloor$ might have a larger weight, due to w being potentially negative. Let C_0 be the set of unmatched agents by this hypergraph matching. The goal of our proof is now to show that there must be an unmatched agent who can only improve by a factor of at most $1 + \frac{\lfloor \frac{1}{q-1}(m-2) \rfloor}{m}$. The set C_0 contains exactly $(m-1) \bmod (q-1) + 1$ agents that are unmatched by M. Let $a_0 \in C_0$. For any $i \in [\lfloor \frac{m-1}{q-1} \rfloor]$ we know that the coalition $\{a_0\} \cup C_i$ is not q-size blocking. Thus, either one of the following two conditions has to hold:

(i) $\sum_{a_j \in C_i} u_0(a_j) \leq q u_0(\mathcal{C})$,

(ii) $\sum_{a_j \in C_i} u_0(a_j) > q u_0(\mathcal{C})$ and there is an $a_\ell \in C_i$ with $\sum_{a_j \in C_i \cup \{a_0\}} u_\ell(a_j) \leq q u_\ell(\mathcal{C})$

If we assume the latter scenario, we first notice that

$$
\begin{aligned}
&\sum_{a_j \in C_i} \frac{u_\ell(a_j)}{q-1} - 2\frac{q}{q-1} u_\ell(\mathcal{C}) \leq \sum_{a_j \in C_i} \frac{u_\ell(a_j)}{q-1} - 2 \sum_{a_j \in C_i \cup \{a_0\}} \frac{u_\ell(a_j)}{q-1} \\
&= 2 \sum_{a_j \in C_i} \frac{u_\ell(a_j)}{q-1} - 2 \sum_{a_j \in C_i} \frac{u_\ell(a_j)}{q-1} - 2\frac{u_0(a_\ell)}{q-1} - \sum_{a_j \in C_i} \frac{u_\ell(a_j)}{q-1} \\
&= -2\frac{u_0(a_\ell)}{q-1} - \sum_{a_j \in C_i} \frac{u_\ell(a_j)}{q-1} = 2 \sum_{a_j \in C_i \setminus \{a_\ell\}} \frac{u_0(a_j)}{q-1} - 2 \sum_{a_j \in C_i} \frac{u_0(a_j)}{q-1} - \sum_{a_j \in C_i} \frac{u_\ell(a_j)}{q-1} \\
&< 2 \sum_{a_j \in C_i \setminus \{a_\ell\}} \frac{u_0(a_j)}{q-1} - 2\frac{q}{q-1} u_0(\mathcal{C}) - \sum_{a_j \in C_i} \frac{u_\ell(a_j)}{q-1} \\
&= 2 \sum_{a_j \in C_i \setminus \{a_\ell\}} \left(\frac{u_0(a_j)}{q-1}\right) - 2\frac{q}{q-1} u_0(\mathcal{C}) - \sum_{a_j \in C_i \setminus \{a_\ell\}} \left(\frac{u_j(a_\ell)}{q-1}\right).
\end{aligned}
$$

For the first inequality, we used the second part of the assumption, while in the second-to-last line we used the first part of the assumption. Further, in the last equality, we used the symmetry of the utilities and the fact that $u_\ell(a_\ell) = 0$. Using this inequality, we can now obtain

$$
\begin{aligned}
&w((C_i \cup \{a_0\}) \setminus \{a_\ell\}) \\
&= \sum_{a_j \in C_i \setminus \{a_\ell\}} \left(u_j(C_i \cup \{a_0\} \setminus \{a_\ell\}) - 2\frac{q}{q-1} u_j(\mathcal{C})\right) \\
&\quad + \sum_{a_j \in C_i \setminus \{a_\ell\}} \left(\frac{u_0(a_j)}{q-1}\right) - 2\frac{q}{q-1} u_0(\mathcal{C}) \\
&= \sum_{a_j \in C_i \setminus \{a_\ell\}} \left(u_j(C_i) + \frac{u_j(a_0)}{q-1} - \frac{u_j(a_\ell)}{q-1} - 2\frac{q}{q-1} u_j(\mathcal{C})\right) \\
&\quad + \sum_{a_j \in C_i \setminus \{a_\ell\}} \left(\frac{u_0(a_j)}{q-1}\right) - 2\frac{q}{q-1} u_0(\mathcal{C})
\end{aligned}
$$

$$= \sum_{a_j \in C_i \setminus \{a_\ell\}} \left(u_j(C_i) - 2\frac{q}{q-1} u_j(\mathcal{C}) \right)$$
$$+ 2 \sum_{a_j \in C_i \setminus \{a_\ell\}} \left(\frac{u_0(a_j)}{q-1} \right) - 2\frac{q}{q-1} u_0(\mathcal{C}) - \sum_{a_j \in C_i \setminus \{a_\ell\}} \left(\frac{u_j(a_\ell)}{q-1} \right)$$
$$> \sum_{a_j \in C_i \setminus \{a_\ell\}} \left(u_j(C_i) - 2\frac{q}{q-1} u_j(\mathcal{C}) \right) + \sum_{a_j \in C_i} \frac{u_\ell(a_j)}{q-1} - 2\frac{q}{q-1} u_\ell(\mathcal{C}) = w(C_i).$$

Hence, we get that $w(C_i \cup \{a_0\} \setminus \{a_\ell\}) > w(C_i)$ and thus the hypergraph matching was not of maximum weight. Therefore, $\sum_{a_j \in C_i} u_0(a_j) \le q u_0(\mathcal{C})$ has to hold for every C_i and every $a_0 \in C_0$.

Next, if $|C_0| = 1$ we know that $(m-1)$ must be divisible by $(q-1)$ and thus we can reformulate our bound as

$$1 + \frac{\left\lfloor \frac{1}{q-1}(m-2) \right\rfloor}{m} = 1 + \frac{\frac{1}{q-1}(m-q)}{m} = \frac{q(m-1)}{m(q-1)}.$$

Hence, by applying the previously calculated bound and by using the observation that $|M| = \frac{m-1}{q-1}$ in the case, we obtain

$$\sum_{a_j \in C} \frac{u_0(a_j)}{m} \le \sum_{C_i \in M} \frac{q}{m} u_0(\mathcal{C}) = u_0(\mathcal{C}) \frac{q(m-1)}{m(q-1)},$$

which implies the result in case $|C_0| = 1$.

If $|C_0| > 1$, there has to be at least one agent a_0 in C_0 with $\sum_{a_i \in C_0} u_0(a_i) \le |C_0| u_0(\mathcal{C}) = ((m-1) \bmod (q-1) + 1) u_0(\mathcal{C})$, since the set C_0 of unmatched agents is non-blocking. Hence, we obtain that

$$\sum_{a_j \in C} \frac{u_0(a_j)}{m} = \sum_{C_i \in M} \sum_{a_j \in C_i} \frac{u_0(a_j)}{m} + \sum_{a_i \in C_0} \frac{u_0(a_i)}{m}$$
$$\le \sum_{C_i \in M} \frac{q}{m} u_0(\mathcal{C}) + \frac{(m-1) \bmod (q-1) + 1}{m} u_0(\mathcal{C})$$
$$= \left\lfloor \frac{m-1}{q-1} \right\rfloor \frac{q}{m} u_0(\mathcal{C}) + \frac{(m-1) \bmod (q-1) + 1}{m} u_0(\mathcal{C})$$
$$= \frac{u_0(\mathcal{C})}{m} \left(\left\lfloor \frac{m-1}{q-1} \right\rfloor q + (m-1) \bmod (q-1) + 1 \right)$$
$$= \frac{u_0(\mathcal{C})}{m} \left(\frac{(m-1) - (m-1) \bmod (q-1)}{q-1} (q-1) \right.$$
$$\left. + \frac{(m-1) - (m-1) \bmod (q-1)}{q-1} + (m-1) \bmod (q-1) + 1 \right)$$
$$= \frac{u_0(\mathcal{C})}{m} \left(m + \frac{(m-1) - (m-1) \bmod (q-1)}{q-1} \right) = \frac{u_0(\mathcal{C})}{m} \left(m + \left\lfloor \frac{m-2}{q-1} \right\rfloor \right).$$

The last step holds since $(m-1) \bmod (q-1) > 0$. Thus, every q-size core stable coalition structure $\mathcal{C}$ is

$$\left(m, 1 + \frac{\left\lfloor \frac{1}{q-1}(m-2) \right\rfloor}{m} \right)\text{-core stable.}$$

Finally, to see that this is equivalent to the bound in Theorem 2 we see that if $(m-1) \bmod (q-1) = 0$ it holds that

$$\left\lfloor \frac{m-1}{q-1} \right\rfloor \frac{\alpha_m}{\alpha_q} + \frac{\mathbb{1}\left((m-1) \bmod (q-1)\right)\alpha_m}{\alpha\left((m-1) \bmod (q-1)+1\right)} = \frac{m-1}{q-1}\frac{q}{m} = 1 + \frac{\left\lfloor \frac{1}{q-1}(m-2) \right\rfloor}{m}$$

and if $(m-1) \bmod (q-1) \neq 0$

$$\left\lfloor \frac{m-1}{q-1} \right\rfloor \frac{\alpha_m}{\alpha_q} + \frac{\mathbb{1}\left((m-1) \bmod (q-1)\right)\alpha_m}{\alpha\left((m-1) \bmod (q-1)+1\right)} = \left\lfloor \frac{m-1}{q-1} \right\rfloor \frac{q}{m} + \frac{(m-1) \bmod (q-1)+1}{m}$$
$$= 1 + \frac{\left\lfloor \frac{1}{q-1}(m-2) \right\rfloor}{m}.$$ ◀

As a corollary we obtain an answer to the conjecture of [20], by confirming that every q-size core stable outcome is also $\frac{q}{q-1}$-improvement stable.

▶ **Corollary 4.** *For S-FHGs, every q-size core stable coalition structure $\mathcal{C}$ is $\frac{q}{q-1}$-improvement stable.*

Proof. This result follows from the observation that

$$1 + \frac{\left\lfloor \frac{1}{q-1}(m-2) \right\rfloor}{m} \leq 1 + \frac{\frac{1}{q-1}(m-2)}{m} \leq 1 + \frac{1}{q-1} = \frac{q}{q-1}$$

holds for all m. Thus, there is no coalition of size $m > q$ in which every agent improves by a factor of more than $\frac{q}{q-1}$. ◀

If we restrict ourselves to the case of *simple fractional hedonic games (SS-FHG)*, i.e., S-FHGs with binary utilities, we can show that this bound is not tight. This proof further provides a strengthening of Theorem 2 by [20].

▶ **Theorem 5.** *For every simple symmetric fractional hedonic game, any 3-size core stable coalition structure $\mathcal{C}$ is*

$$\left(m, \frac{3}{2}\frac{(m-1)}{m}\right)\text{-core stable,}$$

for any integer $m \geq 4$.

Proof. We only prove the result for even values of m, as the result for odd m follows from the general result of Corollary 3. Consider a 3-size core stable coalition structure $\mathcal{C}$ and a blocking coalition C of size $m \geq 4$. Further, we assume that every agent in C improves by more than a factor of k. Because all agents experience a strict improvement by forming C, each agent should be adjacent to at least one edge in the related graph $G(C)$. Since 3-size core stability implies 2-size core stability, there should be at least one agent $a_i \in C$ for whom $u_i(\mathcal{C}) \geq \frac{1}{2}$. Since a_i improves by more than a factor of k it holds that

$$\sum_{a_\ell \in C\setminus\{a_i\}} u(i,\ell) > k \cdot m \cdot u_i(\mathcal{C}) \geq \frac{m}{2}, \tag{2}$$

which implies that a_i should be adjacent to at least $\frac{m}{2}+1$ edges in $G(C)$. Denote the set of agents that are connected to a_i through these edges by $C' \subset C$. For any triplet of agents $\{a_i, a_j, a_k\}$ with $\{a_j, a_k\} \subset C'$, the 3-size core stability of $\mathcal{C}$ implies that either $u_i(\mathcal{C}) \geq \frac{2}{3}$, $u_j(\mathcal{C}) \geq \frac{1}{2}$, or $u_k(\mathcal{C}) \geq \frac{1}{2}$ must hold. If $u_i(\mathcal{C}) \geq \frac{2}{3}$ holds, then because a_i improves in C by a factor of more than k we get that

$$\frac{2}{3}k \leq k \cdot u_i(\mathcal{C}) < \frac{1}{m} \sum_{a_\ell \in C \setminus \{a_i\}} u(i,\ell) \leq \frac{m-1}{m}. \tag{3}$$

Thus, we get that $k < \frac{3}{2}\frac{m-1}{m}$ in this case. Alternatively, assume without loss of generality that $u_j(\mathcal{C}) \geq \frac{1}{2}$. Following the logic from Equation (2), a_j should be adjacent to at least $\frac{m}{2} + 1$ edges in $G(C)$. This implies that there exists an agent $a_\ell \in C'$ such that the agents $\{a_i, a_j, a_\ell\}$ form a triangle, and hence at least one of these three agents should experience a utility of at least $\frac{2}{3}$ in $\mathcal{C}$. Thus, Equation (3) holds for this agent and the result follows. ◀

Finally, as a second corollary of Theorem 2, we also obtain a bound for s-ASHGs.

▶ **Corollary 6.** *For S-ASHGs, every q-size core stable coalition structure $\mathcal{C}$ is $\left(m, 1 + \left\lfloor \frac{m-2}{q-1} \right\rfloor\right)$-core stable for any $m > q$.*

4 Lower Bounds

Next, we focus on proving lower bounds for our setting. We first show a lower bound for the following subclass of S-αHGs, which includes S-FHGs, S-MFHGs, and S-ASHGs.

▶ **Definition 7.** *A function $\alpha \colon [n] \to \mathbb{R}$ is* hospitable *if $\frac{\alpha_q}{\alpha_{q-1}} \geq \frac{q-2}{q-1}$ for all integers $q \geq 2$. Accordingly, an S-αHG is hospitable if α is hospitable.*

The intuition behind hospitable S-αHGs is that the utility of an agent in a coalition of size $q-1$ will never decrease with more than a factor $\frac{q-2}{q-1}$ when an additional agent is added to that coalition. This class includes ASHGs, FHGs, and MFHGs. First, we can show that the bound derived in Theorem 2 is tight for hospitable S-αHGs when $(m-1) \bmod (q-1) = 0$.

▶ **Theorem 8.** *For any hospitable α, there exists an instance of an S-αHG that contains a q-size core stable coalition structure $\mathcal{C}$ which is not (m, δ)-core stable for any $\delta < \frac{\alpha_m(m-1)}{\alpha_q(q-1)}$, with $q, m \in \mathbb{N}$ and $m \geq q+1$.*

Proof. We construct an instance of an S-αHG that is q-size core stable, but which allows for a blocking coalition of size $m \geq q+1$ in which all agents improve with at least a factor $\left(\frac{\alpha_m(m-1)}{\alpha_q(q-1)}\right)$. Given a coalition structure $\mathcal{C}$ in which $u_i(\mathcal{C}) = 1$ for all agents i, and a blocking coalition C of size m, let $u(i,j) = \frac{1}{\alpha_q(q-1)}$ for all agent pairs $\{i,j\} \subset C$. The resulting S-αHG is q-size core stable, since for any coalition C' of size at most q the utility of agent i in C' is at most $\frac{(|C'|-1)\alpha_{|C'|}}{\alpha_q(q-1)} \leq \frac{\alpha_q(q-1)}{\alpha_q(q-1)} = 1$, where the inequality is implied by recursively applying the definition of a hospitable S-αHG. As $u_i(\mathcal{C}) = 1$ and since the utility of agent i in C is $\frac{\alpha_m(m-1)}{\alpha_q(q-1)}$ for all agents $i \in \mathcal{C}$, this implies that the coalition structure $\mathcal{C}$ is not (m, δ)-core stable for any $\delta < \frac{\alpha_m(m-1)}{\alpha_q(q-1)}$. ◀

Since if $(m-1) \bmod (q-1) = 0$ it holds that $\lfloor \frac{m-1}{q-1} \rfloor = \frac{m-1}{q-1}$ and $\mathbb{1}\left((m-1) \bmod (q-1)\right) \alpha_m = 0$, this implies that the bound obtained in Corollary 3 is tight for $(m-1) \bmod (q-1) = 0$ and thus also for $q = 2$. Figure 2 illustrates the tightness of this lower bound for S-FHGs. Further, we show tightness of the bound in Theorem 2 for hospitable S-αHGs when $q = 3$.

▶ **Theorem 9.** *For any hospitable α, there exists an instance of an S-αHG that contains a 3-size core stable coalition structure $\mathcal{C}$ which is not (m, δ)-core stable for any $\delta < f(3, m)$, with $q, m \in \mathbb{N}$ and $m \geq 4$.*

Proof. Note that when $f(3, m) = 1$, the result follows directly. Moreover, when m is odd, so $(m-1) \bmod 2 = 0$, the result follows from Theorem 8. When m is even and when $f(3, m) > 1$, we first observe that

$$f(3, m) = \left\lfloor \frac{m-1}{2} \right\rfloor \frac{\alpha_m}{\alpha_3} + \frac{\mathbb{1}\left((m-1) \bmod 2\right) \alpha_m}{\alpha\left((m-1) \bmod 2 + 1\right)} = \frac{m-2}{2} \frac{\alpha_m}{\alpha_3} + \frac{\alpha_m}{\alpha_2}.$$

Now we assume that we are given m agents $c_1, \dots, c_m$ with $u_i(\mathcal{C}) = 1$ for each c_i. We partition the agents into two sets C_1, C_2 with $C_1 = \{c_1, \dots, c_{\frac{m}{2}}\}$ and $C_2 = \{c_{\frac{m}{2}+1}, \dots, c_m\}$. If two agents c_i and c_j are in the same set, we define $u(i,j) = \frac{1}{\alpha_3} - \frac{1}{\alpha_2}$. Otherwise, we set $u(i,j) = \frac{1}{\alpha_2}$.

For any two agents $\{c_i, c_j\}$ it thus holds that $u_i(\{c_i, c_j\}) \leq \alpha_2 \max(\frac{1}{\alpha_3} - \frac{1}{\alpha_2}, \frac{1}{\alpha_2}) = \max(\frac{\alpha_2}{\alpha_3} - 1, 1) \leq 1$ and, therefore, these two agents do not form a blocking coalition. Further, for any three agents $\{c_i, c_j, c_k\}$ we have two cases: (i) either all three agents come from the same set, then we get that $u_i(\{c_i, c_j, c_k\}) = \frac{2\alpha_3}{\alpha_3} - \frac{2\alpha_3}{\alpha_2} = 2 - \frac{2\alpha_3}{\alpha_2} \leq 2 - \frac{\alpha_3}{\alpha_3} = 1$; (ii) one agent (without loss of generality c_k) has to be from a different partition than the other two; then it holds that $u_i(\{c_i, c_j, c_k\}) = \frac{\alpha_3}{\alpha_3} - \frac{\alpha_3}{\alpha_2} + \frac{\alpha_3}{\alpha_2} = 1$. Hence, this coalition is 3-stable.

Finally, we get that

$$u_i(\{c_1, \dots, c_m\}) = \alpha_m \left(\frac{m-2}{2} \left(\frac{1}{\alpha_3} - \frac{1}{\alpha_2} \right) + \frac{m}{2} \frac{1}{\alpha_2} \right) = \frac{m-2}{2} \frac{\alpha_m}{\alpha_3} + \frac{\alpha_m}{\alpha_2} = f(3, m). ◀$$

Lastly, we provide additional tightness results of the bound in Theorem 2 for S-FHGs and S-ASHGs. We defer the proofs of Theorems 11 and 12 to the supplementary material.

▶ **Theorem 10.** *There exists a q-size core stable coalition structure $\mathcal{C}$ in an S-FHG which is not $(q+1, \delta)$-core stable for any $\delta < \frac{q+2}{q+1}$.*

Proof. Assume we are given $q+1$ agents $a_1, \dots, a_{q+1}$ with $u_i(\mathcal{C}) = 1$. Let the edge weights be such that the edges with weight two form a cycle and all other edges have weight one, i.e., let $u(i,j) = 2$ for all $(i,j) \subset \mathcal{C}$ for which $j = i+1$, let $u(n, 1) = 2$, and let $u(k,l) = 1$ for all other edges. Note that in each subset $C \subset \mathcal{C}$ with $|C| < q+1$ there is at least one agent who is adjacent to at most one edge of weight two to the other agents in C, because the edges with weight two form a cycle over all $q+1$ agents. Hence, for each subset $C \subset \mathcal{C}$ with $|C| < q+1$ there is at least one agent with a utility of at most $\frac{2+|C|-2}{|C|} = 1$, which implies that $\mathcal{C}$ is q-size core stable. Furthermore, the coalition of all $q+1$ agents offers a utility of $\frac{2 \cdot 2 + q - 2}{q+1} = \frac{q+2}{q+1}$. As a result, $\mathcal{C}$ is not $\left(q+1, \frac{q+2}{q+1} - \varepsilon\right)$-core stable for any $\varepsilon > 0$. ◀

▶ **Theorem 11.** *There exists a 4-size core stable coalition structure $\mathcal{C}$ in an S-FHG which is not (m, δ)-core stable, for any $\delta < 1 + \frac{\lfloor \frac{1}{3}(m-2) \rfloor}{m}$, and for any integer $m \geq 5$.*

▶ **Theorem 12.** *There exists a 4-size core stable coalition structure $\mathcal{C}$ in an S-ASHG which is not (m, δ)-core stable, for any $\delta < 1 + \lfloor \frac{m-2}{3} \rfloor$, and for any integer $m \geq 5$.*

While we were not able to show the tightness of Theorem 2 for other values of α, q, and m than the ones described in this section, we found some examples to show the tightness of the result for additional values of (q, m) that are not covered by Theorems 8-12 through the use of the integer linear programming approach described in the supplementary material, as shown in Table 2. A summary of our results can be found in Table 3.

Table 2 Additional values of (q, m) for which we found instances proving the tightness of the bound in Theorem 2 by using the integer linear programming approach described in the supplementary material.

S-FHG	$(5, m \leq 8)$, $(6, m \leq 10)$, $(7, m \leq 10)$, $(8, m \leq 11)$
S-ASHG	$(5, m \leq 8)$, $(6, m \leq 10)$, $(7, m \leq 12)$, $(8, m \leq 13)$, $(9, m \leq 13)$, $(10, m \leq 13)$, $(11, m \leq 13)$, $(12, 13)$

Table 3 Summary and tightness results for the values of $f(q, m)$ such that every q-size core stable coalition structure is $(q, f(q, m))$-stable, with $m \geq q + 1$, for various types of hedonic games. A ✓ indicates that we were able to show tightness of $\mathbf{f(q, m)}$ for this type of hedonic game, while a ✗ indicates that this tightness is still open.

Hedonic Game	$\mathbf{f(q, m)}$	**Tightness proof for...** $(q-1) \mid (m-1)$	$q = 3$	$q = 4$	Other values of (q, m)
S-αHG	see Theorem 2	✗	✗	✗	$\varnothing$
Hospitable S-αHG	see Theorem 2	✓	✓	✗	$\varnothing$
S-FHG	$1 + \frac{1}{m} \left\lfloor \frac{m-2}{q-1} \right\rfloor$	✓	✓	✓	$(q, q+1)$ & Table 2
S-MFHG	1	✓	✓	✓	all combinations
S-ASHG	$1 + \left\lfloor \frac{m-2}{q-1} \right\rfloor$	✓	✓	✓	Table 2

5 Efficiency

In this section, we study the price of anarchy for core-relaxations of symmetric α-hedonic games. Using the same notation as Fanelli et al. [20], we denote the *social welfare* of a coalition structure $\mathcal{C}$ by $\text{SW}(\mathcal{C}) = \sum_{i \in A} u_i(\mathcal{C})$, which is simply the sum of the agents' utilities. Moreover, let $\mathcal{G} = (\alpha, A, u)$ represent an instance of an S-αHG. Given an S-αHG $\mathcal{G}$, let q-SIZE CORE$(\mathcal{G})$ be the set of q-size core stable coalition structures, and let k-IMPR CORE$(\mathcal{G})$ be the set of k-improvement core stable coalition structures. We define the *q-size core price of anarchy* of an S-αHG $\mathcal{G}$ as the ratio between the social welfare of the coalition structure $\mathcal{C}^*(\mathcal{G})$ that maximizes social welfare, and that of the q-size core stable coalition structure with the worst social welfare, i.e., q-SIZE CPoA$(\mathcal{G}) = \max_{\mathcal{C} \in q\text{-SIZE CORE}(\mathcal{G})} \frac{SW(\mathcal{C}^*(\mathcal{G}))}{SW(\mathcal{C})}$. Similarly, we define the *k-improvement core price of anarchy* as k-IMPR CPoA$(\mathcal{G}) = \max_{\mathcal{C} \in k\text{-IMPR CORE}(\mathcal{G})} \frac{SW(\mathcal{C}^*(\mathcal{G}))}{SW(\mathcal{C})}$.

Using Corollary 3, we can extend the results by Fanelli et al. [20] about the q-size core price of anarchy for S-FHGs.

▶ **Corollary 13.** *For any S-FHG $\mathcal{G}$, it holds that q-SIZE CPoA$(\mathcal{G}) \leq \frac{2q}{q-1}$, for any integer $q \geq 2$, and this bound is tight.*

Proof. By Corollary 4, we know that the social welfare of the worst q-size core stable coalition structure is at least the social welfare of the worst $\frac{q}{q-1}$-improvement core stable coalition structure. Moreover, by Theorem 8 by [20], we know that the k-improvement CPoA of an S-FHG is upper bounded by $2k$, for any $k \geq 1$. The tightness of the bound follows from Theorem 9 by [20]. ◀

Next, we show upper bounds on the q-SIZE CPoA$(\mathcal{G})$ and the k-IMPR CPoA$(\mathcal{G})$ for the following subclass of S-αHGs.

▶ **Definition 14.** *A function $\alpha\colon [n] \to \mathbb{R}$ is* decreasing *if $\alpha_q \geq \alpha_{q+1}$ for all integers $q \geq 1$. Accordingly, an S-αHG is decreasing if α is decreasing.*

The class of decreasing S-αHGs is distinct from the class of hospitable S-αHGs defined in Definition 7, but it also contains S-FHGs, S-ASHGs, and S-MFHGs. When restricting our focus to the subclass of decreasing S-αHGs, we can generalize Theorem 8 by [20] to obtain the same upper bound on the k-improvement core price of anarchy.

▶ **Theorem 15.** *For any decreasing S-αHG $\mathcal{G}$ and for every $k \geq 1$, k-IMPR CPoA$(\mathcal{G}) \leq 2k$.*

Proof. The proof is identical to the proof of Theorem 8 by [20]. Using their notation, with the adapted definition that $\mu_i^{>}(C) = \alpha(|C|) \cdot \delta_C^{>}(i)$, the only required alteration to their proof is that equation (2) in their proof should be replaced by:

$$\mu_{i_t}(C_t^*) = \alpha(|C_t^*|) \cdot \delta_{C^*}^{>}(i_t) \geq \alpha(|C^*|) \cdot \delta_{C^*}^{>}(i_t) = \mu_{i_t}^{>}(C^*),$$

which holds, by definition, because we are only considering decreasing S-αHGs. ◀

Lastly, we can use a similar reasoning as in the proof of Corollary 13 and use the bounds from Theorems 3 and 15 to obtain a general upper bound on the q-size core price of anarchy for decreasing S-αHGs.

▶ **Theorem 16.** *For any decreasing S-αHG $\mathcal{G}$, q-SIZE CPoA$(\mathcal{G}) \leq 2 \cdot \max_{m \geq q+1} f(m, q)$.*

Note that this result implies a core price of anarchy of 2 for S-MFHGs, where the core price of anarchy of an S-αHG $\mathcal{G}$ is simply defined as $\max_q$ q-SIZE CPoA$(\mathcal{G})$. As such, we answered an open question by [26], who found a lower bound on the core price of stability of 2 and an upper bound for the core price of anarchy of 4 in S-MFHGs.

▶ **Corollary 17.** *For any S-MFHG, the core price of anarchy is upper bounded by 2.*

6 Conclusion and Outlook

In our paper, we studied hedonic games and the relationship between different relaxed notions of core stability. Most importantly, for a large class of hedonic games, we obtained a general upper bound $f(q, m)$ such that every q-size core stable outcome is $(m, f(q, m))$-core stable. That is, a coalition of size m can deviate at most by a factor of $f(q, m)$. Our bound also allows us to answer a conjecture by [20] that every q-size core stable outcome in symmetric fractional hedonic games is $\frac{q}{q-1}$-improvement core stable. Finally, we also obtain some lower bounds. However, even for fractional and additively separable hedonic games, our bounds are not tight yet. For both, we were only able to show the tightness up to $q = 4$. The smallest case which is unknown (both for fractional and additively separable hedonic games) is the tightness for $q = 5$ and $m = 10$, see Table 2 and Theorem 8. For both kinds of hedonic games, our integer linear programming approach was not able to construct a counterexample, nor show that no counterexample exists. Thus, improving our lower bounds seems like a challenging and interesting venue for future work. Further, it would be interesting to see if the generalization of α-hedonic games, could see application in other areas of hedonic games as well. For instance, it might be interesting to classify for which α-hedonic games certain dynamics converge (see for instance Boehmer et al. [7].)

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