Parameterized Complexity of Domination Problems Using Restricted Modular Partitions

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Abstract

For a graph class $\mathcal{G}$, we define the $\mathcal{G}$-modular cardinality of a graph G as the minimum size of a vertex partition of G into modules that each induces a graph in $\mathcal{G}$. This generalizes other module-based graph parameters such as neighborhood diversity and iterated type partition. Moreover, if $\mathcal{G}$ has bounded modular-width, the W[1]-hardness of a problem in $\mathcal{G}$-modular cardinality implies hardness on modular-width, clique-width, and other related parameters. Several FPT algorithms based on modular partitions compute a solution table in each module, then combine each table into a global solution. This works well when each table has a succinct representation, but as we argue, when no such representation exists, the problem is typically W[1]-hard. We illustrate these ideas on the generic (α, β)-domination problem, which is a generalization of known domination problems such as Bounded Degree Deletion, k-Domination, and α-Domination. We show that for graph classes $\mathcal{G}$ that require arbitrarily large solution tables, these problems are W[1]-hard in the $\mathcal{G}$-modular cardinality, whereas they are fixed-parameter tractable when they admit succinct solution tables. This leads to several new positive and negative results for many domination problems parameterized by known and novel structural graph parameters such as clique-width, modular-width, and `cluster`-modular cardinality.

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1 Introduction

Modular decompositions of graphs have played an important role in algorithms since their inception [23]. In the world of parameterized complexity [11, 13], Gajarský et. al. [22] proposed the notion of *modular-width*, or mw for short, which can be defined as the maximum degree of a prime node in the modular decomposition tree of G. Unlike other structural parameters such as treewidth [4], mw can be bounded on certain classes of dense graphs, making it comparable to the clique-width (cw) parameter [10]. In fact, cw is at most mw plus two, and mw can sometimes be arbitrarily larger than cw. It is known that several problems that are hard in cw are fixed-parameter tractable (FPT) in mw, with popular examples including Hamiltonian Cycle, Graph Coloring [16, 18, 22], and Metric Dimension [2, 5]. In particular in [22], the main technique used to design such algorithms is dynamic programming over the modular decomposition. In essence, the values of an optimal solution are found recursively in each module of the graph G, which are then combined into a solution for the graph itself, often using small integer linear programs based on the prime graph of G. Another technique was recently introduced in [19], where the authors show that the number of potential maximal cliques of a graph is at most $O^*(1.73^{mw})$, a fact that can be combined with results of [6, 20] to obtain FPT algorithms for a family of problems.

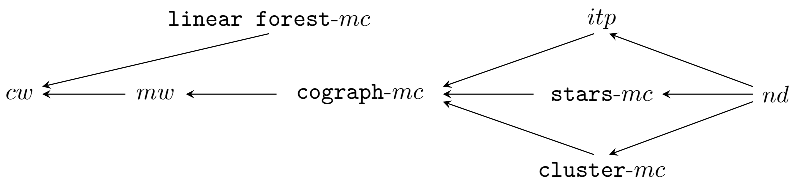

Figure 1 Relation among the parameters clique-width (*cw*), modular-width (*mw*), `cograph`-modular cardinality (`cograph`-*mc*), iterated type partitions (*itp*), `stars`-modular cardinality (`stars`-*mc*), `cluster`-modular cardinality (`cluster`-*mc*), neighborhood diversity (*nd*), and `linear forest`-modular cardinality (`linear forest`-*mc*). Arrows indicate generalizations, e.g. modular-width(*mw*) generalizes `cograph`-modular cardinality and thus is bounded by (a function of) `cograph`-modular cardinality.

Despite these efforts, there are still several problems that are known to be hard on *cw*, for instance Max-Cut [17] and Bounded Degree Deletion [3, 24], but unknown to be hard or FPT in *mw*. Recently, an XP algorithm is given for k-Domination in parameter treewidth (*tw*) [34], but the W-hardness for k-Domination in parameter *tw*, *cw* or *mw* is unknown. Also, the authors of [22] conclude with the question of whether Edge Dominating Set and Partition into Triangles are FPT in *mw*, which are 10 years-old questions that are still unanswered. One promising direction to gain knowledge and tools for *mw* algorithms is to study some of its related parameters. We consider such variants in which the graph must be decomposed into modules that each induces a subgraph belonging to a specific graph class. Notable examples include neighborhood diversity (*nd*) [31], in which each module must induce an edgeless graph or a clique, and iterated type partition (*itp*) [9], in which each module must induce a cograph. This idea was also used in [25] from which we borrow our terminology, where a partition into modules inducing cliques is used to obtain linear kernels for the cluster editing problem. Meanwhile, as Knop wrote in [29], 'another important task in this area is to understand the boundary between modular-width on one side, and neighborhood diversity, twin-cover number, and clique-width on the other side'.

Our work resides in this boundary, as we propose to generalize the above ideas by restricting modules to a given graph class $\mathcal{G}$. That is, we define the $\mathcal{G}$-modular cardinality of a graph G, denoted $\mathcal{G}$-$mc(G)$, as the size of the smallest partition of its vertices into modules that each induces a subgraph in $\mathcal{G}$. If $\mathcal{G}$ has bounded modular-width (e.g. cographs), the hardness of a problem in $\mathcal{G}$-modular cardinality implies its hardness in *mw* (and thus *cw*). On the other hand, FPT techniques for $\mathcal{G}$-*mc* may shed light towards developing better algorithms for *mw*. Also, by considering graph classes of unbounded modular-width (e.g. paths, grids), $\mathcal{G}$-*mc* may be incomparable with *mw* or even *cw*, leading to FPT algorithms for novel types of graphs. To the best of our knowledge, such a generalization had not been studied, although it is worth mentioning that in [27], the authors propose a similar concept for treewidth, where some bags of a tree decomposition are allowed to be in some graph class.

Our contributions. We first establish that if $\mathcal{G}$ is hereditary and has bounded *mw*, then $mw(G)$ is at most $\mathcal{G}$-$mc(G)$ for a graph $G \notin \mathcal{G}$, allowing the transfer of hardness results. We then show that for many graph classes, namely those that are *easily mergeable*, there is a polynomial-time algorithm to compute $\mathcal{G}$-*mc* and obtain a corresponding modular partition.

We then introduce techniques to obtain W[1]-hardness results and FPT algorithms for the $\mathcal{G}$-*mc* parameter. In essence, we argue that the dynamic programming technique on *mw* algorithms works well when a small amount of information from each module is sufficient to obtain a solution for the whole graph (for instance, the algorithms of [22] require only a

Table 1 Our results for the (α, β)-Linear Degree Domination ((α, β)-LDD) on different parameters (the mark * is implied by [3, 24]). Results in boldface are those proved directly in this paper (other entries are implied by these results). Recall that $\alpha = 0$ is equivalent to the k-Domination, $\alpha = 1$ to the Bounded Degree Deletion (BDD), and $\alpha \in (0, 1)$ to the α-Domination. Not shown in the table: BDD is FPT in parameters `linear forest`-*mc* and `binary forest`-*mc*.

(α, β)-LDD problem	$\alpha = 0$ (k-dom)	$\alpha \in (0, 1)$ (α-dom $+\beta$)	$\alpha = 1$ (BDD)	Parameters
β is in the input	W[1]-h	W[1]-h	W[1]-h*	*cw*
	W[1]-h	W[1]-h	W[1]-h	*mw*
	W[1]-h	W[1]-h	W[1]-h	`cograph`-*mc*
	W[1]-h	W[1]-h	W[1]-h	*itp*
	W[1]-h	W[1]-h	**W[1]-h**	`stars`-*mc*
	open	open	**FPT**	`cluster`-*mc*
	FPT	**FPT**	**FPT**	*nd*
β is any constant	**FPT**	W[1]-h	**FPT**	*cw*
	FPT	W[1]-h	FPT	*mw*
	FPT	W[1]-h	FPT	`cograph`-*mc*
	FPT	W[1]-h	FPT	*itp*
	FPT	**W[1]-h**	FPT	`stars`-*mc*
	FPT	open	FPT	`cluster`-*mc*
	FPT	FPT	FPT	*nd*

single integer from each module). Such *succinct solution tables* from each module can often be combined using integer programs with few integer variables [21, 28, 33]. Conversely, when too much information is required from each module (e.g. linear in the size of the modules) to obtain a final solution, we are unable to use integer programming and this typically leads to W[1]-hardness. This occurs when *arbitrary solution tables* are possible in each module.

We use a large class of domination problems to illustrate these techniques. Specifically, for a real number $\alpha \in [0, 1]$ and integer β (possible negative), we introduce the (α, β)-Linear Degree Domination problem. Given a graph G and an integer q, this problem asks for a $X \subseteq V(G)$ of size at most q such that, for each $v \in V(G) \setminus X$, we have $|N(v) \cap X| \geq \alpha |N(v)| + \beta$. In other words, each unchosen vertex has at least a fraction α of its neighbors dominating it, plus some number β. The problem is equivalent to the Bounded Degree Deletion (BDD) [3, 24] if $\alpha = 1$ and $\beta \leq 0$; equivalent to the k-Domination [8, 32] if $\alpha = 0$ and $\beta \geq 1$; and equivalent to the α-Domination [1, 12, 14, 35] if $\alpha \in (0, 1]$ and $\beta = 0$.

Table 1 illustrates the main results of this paper. The hardness results follow from a more general result on arbitrary solution tables (Theorem 7). It is slightly technical, so we describe its high-level implication on the case $\alpha = 1$ (BDD). In this problem, a possible solution table is a function $f : [n] \to \mathbb{N}$ such that $f(i)$ is the minimum maximum degree achievable in G after deleting i vertices. The theorem states that for graph class $\mathcal{G}$, if for any such f we can construct a graph in $\mathcal{G}$ whose solution table is f, then BDD is W[1]-hard in $\mathcal{G}$-modular cardinality. We show that the class of disjoint stars satisfies this property, which implies several other hardness results. On the other hand, several positive results for the BDD make use of succinct solution tables. In essence, when f can be represented by a constant number of linear functions, or by a convex function, then we can use integer programming to merge these tables and obtain positive results. Finally, additional results not shown in the table can be deduced easily from this technique. We show that BDD is FPT in `linear forest`-modular cardinality and `binary forest`-modular cardinality as parameters, which are of interest since they are incomparable with modular-width. Due to space constraints, we refer the reader to the full version of this paper [30] for definitions and complete proofs.

2 Preliminary notions

For an integer n, denote $[n] = \{1, \ldots, n\}$. The maximum degree of a graph G is denoted by $\Delta(G)$. $\overline{G}$ denotes the *complement* graph of G. The neighborhood of $v \in V(G)$ is $N(v)$. The set of connected components of a graph G is denoted $CC(G)$. For $X \subseteq V(G)$, $G[X]$ denotes the subgraph of G induced by X and $G - X = G[V(G) \setminus X]$. If $X = \{v\}$, we may write $G - v$. Slightly abusing notation, we may also write $v \in G$ instead of $v \in V(G)$, $|G|$ instead of $|V(G)|$, and $X \cap G$ instead of $X \cap V(G)$.

A *graph class* $\mathcal{G}$ is a (possibly infinite) set of graphs containing at least one non-empty graph. We say that $\mathcal{G}$ is *hereditary* if, for any $G \in \mathcal{G}$, any induced subgraph of G is also in $\mathcal{G}$. Note that if $\mathcal{G}$ is hereditary, the graph consisting of an isolated vertex is in $\mathcal{G}$. We say that $\mathcal{G}$ is a *polynomial-time recognition graph class* if there is a polynomial-time algorithm that decides whether a given graph G is in $\mathcal{G}$. Some popular graph classes that we will use throughout this paper: $\mathcal{I}$ is the set of all edgeless graphs; $\mathcal{K}$ is the set of all complete graphs; `cluster` is the set of graphs in which every connected component induces a complete graph; `stars` is the set of graphs in which every connected component is a star graph; `cograph` is the set of cographs, where a cograph is either a single vertex, or a graph obtained by applying either a join or a disjoint union of two cographs [7]. Observe that $\mathcal{I} \subseteq$ `stars` $\subseteq$ `cograph`.

Modular parameters. For a graph $G = (V, E)$, a *module* of G is a $M \subseteq V$ such that for every $v \in V \setminus M$, either $M \subseteq N(v)$ or $M \cap N(v) = \emptyset$. The empty set, V, and every $\{v\}$ for $v \in V$ are called the *trivial modules*. In a *prime* graph, all modules are trivial. A *factor* is a subgraph induced by a module. A module M is *strong* if for any module M' of G, either $M' \subseteq M$, $M \subseteq M'$, or $M \cap M' = \emptyset$. M is *maximal* if $M \subsetneq V$ and there is no module M' such that $M \subsetneq M' \subsetneq V$. A partition $\mathcal{M}$ of $V(G)$ is called a *modular partition* if every element of $\mathcal{M}$ is a module of G. If $\mathcal{M}$ only contains maximal strong modules, then it is a *maximal modular partition*. This partition is unique. Two modules M and M' are *adjacent* in G if every vertex of M is adjacent to every vertex of M', and *non-adjacent* otherwise. For a modular partition $\mathcal{M}$ of V, the *quotient graph* $G_{/\mathcal{M}}$ is defined by $V(G_{/\mathcal{M}}) = \{v_M : M \in \mathcal{M}\}$ and $v_{M_1}v_{M_2} \in E(G_{/\mathcal{M}})$ if and only if M_1, M_2 are adjacent.

We call $MD(G)$ the *modular decomposition tree* of G, in which each vertex v_M corresponds to a strong module M. More specifically, each leaf $v_{\{v\}}$ of the inclusion tree corresponds to a vertex v of G and the root vertex v_V corresponds to V. Moreover, for any two strong modules M and M', M' is a proper subset of M if and only if $v_{M'}$ is a descendant of v_M in $MD(G)$. An internal vertex v_M of $MD(G)$ is *parallel* if $G[M]$ is disconnected, is *series* if $\overline{G[M]}$ is disconnected, and is *prime* otherwise. The *modular-width* of G is the maximum number of children of a prime vertex in $MD(G)$ (see [26] for more). Our variant follows.

▶ **Definition 1.** *Let $\mathcal{G}$ be a graph class. For a graph G (not necessarily in $\mathcal{G}$), a module M of G is a $\mathcal{G}$-*module *if $G[M]$ belongs to $\mathcal{G}$. A modular partition $\mathcal{M} = \{M_1, \ldots, M_k\}$ of a graph G is called a $\mathcal{G}$-*modular partition *if each M_i is a $\mathcal{G}$-module. The $\mathcal{G}$-*modular cardinality *of G, denoted by $\mathcal{G}$-mc(G), is the minimum cardinality of a $\mathcal{G}$-modular partition of G.*

The neighborhood diversity (*nd*) is equivalent to the $(\mathcal{K} \cup \mathcal{I})$-modular cardinality. The iterated type partition (*itp*) parameter [9] is the number of vertices of the graph obtained through the following process: start with the smallest modular partition into cliques and edgeless graphs; contract each module into a single vertex; repeat until no more contraction is possible. It can be shown that the remaining vertices represent modules that are cographs. Thus, $itp(G)$ is not smaller than the `cograph`-modular cardinality of $\mathcal{G}$.

3 Properties and tractability of $\mathcal{G}$-modular cardinality

In this section, we state two basic results regarding $\mathcal{G}$-*mc*. First, *mw* is not larger than $\mathcal{G}$-*mc* for graph classes of bounded modular-width. This allows hardness results on $\mathcal{G}$-*mc* to also apply to *mw*. Second, $\mathcal{G}$-*mc* is polynomial-time computable for "easily mergeable" graph classes. The first is realtively easy using modular partitions and induction.

▶ **Theorem 2.** *Let $\mathcal{G}$ be an hereditary graph class and define $\omega_\mathcal{G} := \max_{H \in \mathcal{G}} mw(H)$. Then for any graph G, $mw(G) \leq \max(\mathcal{G}\text{-}mc(G), \omega_\mathcal{G})$.*

Let us note that the above bound is tight, in the sense that $mw(G)$ can be at least as large as either $\mathcal{G}\text{-}mc(G)$ or $\omega(G)$. For instance, if $G \notin \mathcal{G}$ is a prime graph, then $mw(G) = \mathcal{G}\text{-}mc(G)$, and if $G = \arg\max_{H \in \mathcal{G}} mw(H)$, then $mw(G) = \omega_\mathcal{G}$.

Let us now state the second result. A graph G is a $\mathcal{G}$-*join* if $\overline{G}$ is disconnected and $G[C] \in \mathcal{G}$ for each $C \in CC(\overline{G})$. Likewise, G is a $\mathcal{G}$-*union* if G is disconnected and $G[C] \in \mathcal{G}$ for each $C \in CC(G)$. If G is a $\mathcal{G}$-join (resp. $\mathcal{G}$-union), a $\mathcal{G}$-*merge* is a $\mathcal{G}$-modular partition $\mathcal{M}$ of G such that for each $C \in CC(\overline{G})$ (resp. $C \in CC(G)$), there is some $M \in \mathcal{M}$ that contains C. We say that $\mathcal{M}$ is a *minimum* $\mathcal{G}$-merge if no other $\mathcal{G}$-merge has a size strictly smaller than $\mathcal{M}$. We say that a graph class $\mathcal{G}$ is *easily mergeable* if there exists a polynomial time algorithm that, given a graph G such that G is either a $\mathcal{G}$-join or a $\mathcal{G}$-union, outputs a minimum $\mathcal{G}$-merge of G. We say that $\mathcal{G}$ is *trivially mergeable* if, for any $\mathcal{G}$-join or $\mathcal{G}$-union G, one of $\{V(G)\}, CC(G)$, or $CC(\overline{G})$ is a minimum $\mathcal{G}$-merge of G.

▶ **Theorem 3.** *Suppose that $\mathcal{G}$ is a hereditary graph class. Suppose further that $\mathcal{G}$ is polynomial-time recognizable and easily mergeable. Then a $\mathcal{G}$-modular partition of G of minimum size can be obtained in polynomial time.*

The above implies that for $\mathcal{G} \in \{\mathcal{I}, \mathcal{K}, \texttt{cograph}, \texttt{cluster}\}$, a $\mathcal{G}$-modular partition of minimum size of a graph G can be computed in polynomial time. It is not hard to show that $\mathcal{I}, \mathcal{K}$, and `cograph` are trivially mergeable. For `cluster`, if G is a `cluster`-join, then $G \in \texttt{cluster}$. If G is a `cluster`-union, one may show that we can merge all the connected components of $\overline{G}$ that contain one clique into a single clique, and the other connected components are left intact.

Note that not all graph classes are equally easy to merge. Consider the class "graphs with at most 100 vertices". Merging such graphs amounts to solving a bin packing problem with a fixed capacity of 100, and current polynomial-time algorithms require time in the order of n^{100}. This is easily mergeable nonetheless, and it would be interesting to find graph classes that are hereditary and polynomial-time recognizable, but not easily mergeable.

4 Hardness of domination problems with arbitrary solution tables

Let us recall our generic domination problem of interest. For $\alpha \in [0, 1]$ and $\beta \in \mathbb{Z}$ we define:

The (α, β)-Linear Degree Domination problem
Input: a graph $G = (V, E)$ and a non-negative integer q;
Question: does there exist a subset $X \subseteq V$ of size at most q such that for every $v \in V \setminus X$, $|N(v) \cap X| \geq \alpha|N(v)| + \beta$?

In the above, the vertex set X is called a (α, β)-*linear degree dominating set* of G. In addition, for convenience, we also call X the deletion part of G. For any vertex $v \in V(G-X)$, the degree of v in X, denoted by $\deg(v, G, X)$, equals the number of vertices in $X \cap V(G)$ adjacent to v. The minimum degree of $G - X$ in X equals $\min\{\deg(v, G, X) : v \in G - X\}$, which is denoted by $\delta(G - X, X)$.

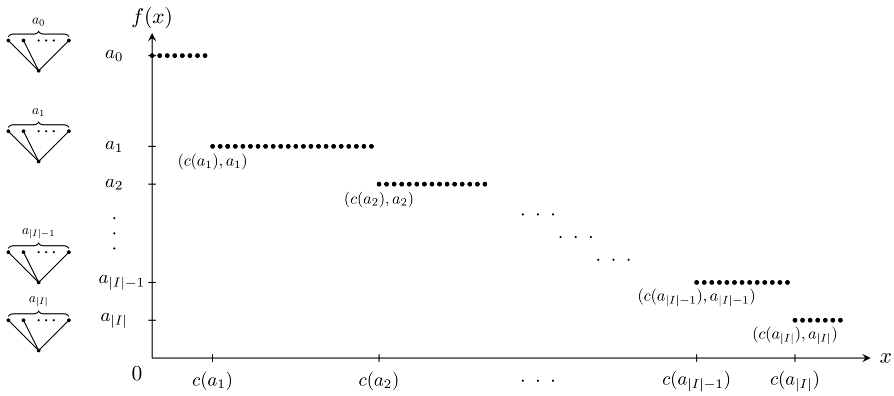

Figure 2 The degree deletion function of a (a_0, I, c)-degree deletion graph. The x-axis represents the number of vertices to delete, while the y-axis represents the minimum maximum degree achievable by deleting x vertices. The first point of the $(i+1)$-th horizontal vertex set is $(c(a_i), a_i)$ for $0 \le i \le |I|$, and the last point of the $(i+1)$-th horizontal vertex set is $(c(a_{i+1}) - 1, a_i)$ for $0 \le i \le |I| - 1$. Disjoint stars are used as an example here since the graph class `stars` admits arbitrary deletion tables. The number of stars with a_i leaves is $c(a_{i+1}) - c(a_i)$ for $0 \le i \le |I| - 1$. The degree deletion process removes the internal vertices of stars from large to small.

Our goal is to formalize the intuition that graph classes with arbitrary solution tables lead to W[1]-hardness in $\mathcal{G}$-*mc*. For (α, β)-Linear Degree Domination, this takes the form of *arbitrary deletion tables* for $\alpha \in (0, 1]$ and *arbitrary retention tables* for $\alpha = 0$.

▶ **Definition 4.** *Let (a_0, I, c) be a triple with $\{a_0\} \cup I \subseteq \mathbb{N}$ and $c : \mathbb{N} \to \mathbb{N}$. We call c a cost function. We say that (a_0, I, c) is* decreasing valid *if, by listing the elements $a_1, \ldots, a_{|I|}$ of I in decreasing order, we have $a_0 > a_1 > \ldots > a_{|I|} > 1$, and $0 = c(a_0) < c(a_1) < \ldots < c(a_{|I|})$.*

For a decreasing valid triple (a_0, I, c), we say that a graph $G = (V, E)$ is a (a_0, I, c)-degree deletion graph *if all of the following conditions hold:*
1. *G has maximum degree a_0 and at most $(a_0 c(a_{|I|}))^{10}$ vertices;*
2. *for any $a_i \in I$, there exists $X \subseteq V$ of size $c(a_i)$ such that $G - X$ has maximum degree a_i;*
3. *for any $a_i \in I$ and any $X \subseteq V$ of size strictly less than $c(a_i)$, $G - X$ has maximum degree at least a_{i-1}.*

In addition, if $G \in$ `stars` and satisfies the above three conditions, then we say that G is a (a_0, I, c)-degree deletion star graph.

We say that a graph class $\mathcal{G}$ *admits arbitrary degree deletion tables* if, for any decreasing valid triple (a_0, I, c), one can construct in time polynomial in $a_0 c(a_{|I|})$ a graph $G \in \mathcal{G}$ such that G is a (a_0, I, c)-degree deletion graph. Note that the size of G is only required to be a polynomial function of $a_0 c(a_{|I|})$, but we fix it to $(a_0 c(a_{|I|}))^{10}$ for convenience. For an integer $x \in [0, |V|]$, we call $f(x) = \min\{\Delta(G - X) : |X| = x\}$ the *degree deletion function* of G, where X is a subset of V. Figure 2 demonstrates the degree deletion function of a (a_0, I, c)-degree deletion graph. The intuition behind degree deletion graphs is that their deletion function has a stepwise behavior with many steps.

The above notion of an arbitrary solution table works well for $\alpha \in (0, 1]$. For $\alpha = 0$, we need to replace "deletion" with "retention". This is useful for the $\alpha = 0$ case, where the set X must contain at least β neighbors of each unchosen vertex. Hence, the steps of the table describe, for each number x of vertices to include in X, the maximum possible $\delta(G - X, X)$ that can be achieved with a subset of size x.

▶ **Definition 5.** *Let (a_0, I, c) be a triple with $\{a_0\} \cup I \subseteq \mathbb{N}$ and $c : \mathbb{N} \to \mathbb{N}$. We call c a cost function. We say that (a_0, I, c) is* increasing valid *if, by listing the elements $a_1, \ldots, a_{|I|}$ of I in decreasing order, we have $a_0 > a_1 > \ldots > a_{|I|} > 100$,*[1] *and $c(a_0) > c(a_1) > \ldots > c(a_{|I|}) = 0$.*

For a increasing valid triple (a_0, I, c), we say that a graph $G = (V, E)$ is a (a_0, I, c)-degree retention graph above (p, l), *where p and l are positive integers and $l < 100$, if all of the following conditions hold:*

1. *G has maximum degree a_0, at most $(a_0 c(a_0))^{10}$ vertices, and p vertices of degree less than l;*
2. *for any $a_i \in \{a_0\} \cup I$, there exists $X \subseteq V$ of size $p + c(a_i)$ such that the minimum degree of $G - X$ in X is a_i;*
3. *for any $a_i \in \{a_0\} \cup I$ and any $X \subseteq V$ of size strictly less than $p + c(a_i)$, the minimum degree of $G - X$ in X is at most a_{i+1}, where here we define $a_{|I|+1} = l$.*

We say that a graph class $\mathcal{G}$ *admits arbitrary degree retention tables* if, for any increasing valid triple (a_0, I, c), there exist integers p and l such that one can construct in time polynomial in $c(a_0)a_0$ a graph $G \in \mathcal{G}$ such that G is a (a_0, I, c)-degree retention graph above (p, l). Note that condition 2 of Definition 5 imply that $p + c(a_0) < |V| \leq (c(a_0)a_0)^{10}$. For any integer $x \in [0, |V| - 1]$, we call $f(x) = \max\{\delta(G - X, X) : |X| = x\}$ the *degree retention function* of G, where X is a subset of V. Importantly, `stars` admit both types of tables.

▶ **Proposition 6.** *The graph class* `stars` *admits arbitrary deletion tables and arbitrary retention tables.*

▶ **Theorem 7.** *The (α, β)-*Linear Degree Domination *problem is W[1]-hard in the following cases:*

1. *$\alpha = 0$, β is in the input, and the parameter is the $(\mathcal{G} \cup \mathcal{I})$-modular cardinality, where $\mathcal{G}$ is any graph class that admits arbitrary degree retention tables;*
2. *α is any fixed constant in the interval $(0, 1)$, β is any fixed constant in $\mathbb{Z}$, and the parameter is the* `stars`*-modular cardinality;*
3. *$\alpha = 1$, β is in the input, and the parameter is the $(\mathcal{G} \cup \mathcal{I})$-modular cardinality, where $\mathcal{G}$ is any graph class that admits arbitrary degree deletion tables.*

The proof of the Theorem 7 is quite long and can be found in [30]. The proof of the three cases all use the same construction and the same set of claims, but most claims require an argument for each case. The next section illustrates the main techniques on the case $\alpha = 1$. Using the relationship demonstrated in Figure 1 we have the following.

▶ **Corollary 8.** *The (α, β)-*Linear Degree Domination *problem parameterized by either clique-width, modular-width,* `cograph`*-modular cardinality, iterated type partition, and* `stars`*-modular cardinality, is W[1]-hard in the following cases:*

1. *$\alpha = 0$, β is in the input;*
2. *α is any fixed constant in the interval $(0, 1)$, β is any fixed constant in $\mathbb{Z}$;*
3. *$\alpha = 1$, β is in the input.*

In particular, Bounded Degree Deletion *problem, k-*Domination *problem, and α-*Domination *problem are W[1]-hard in all these parameters.*

One can show that these problems are in XP parameterized by `stars`-modular cardinality. Indeed, one can choose, for each stars module M, a number in $[|M|] \cup \{0\}$ of vertices to delete in M, and apply these deletions greedily. There are at most $|V(G)|^{\mathcal{G}\text{-}mc(G)}$ combinations

[1] In fact, any large constant here is enough for our proof in this paper, we fix it to be 100 for convenience.

of choices, and it suffices to try each of them. This works because the greedy strategy is applicable to stars, and it remains to study the XP complexity for other parameters. The next proposition is not related to the above hardness results, but allows us to fill in some of the gaps that the above leaves in our results table. In section 6, we will also show that for $\alpha = 1$ and β in the input, the problem is FPT in `cluster`-modular cardinality.

▶ **Proposition 9.** *The (α, β)-Linear Degree Domination problem is FPT in the following cases:*
1. *$\alpha \in [0, 1]$, β is in the input, and the parameter is the neighborhood diversity;*
2. *$\alpha \in \{0, 1\}$, β is a constant, and the parameter is the clique-width.*

The first case requires some work, whereas the second is a simple application of Courcelle's theorem [10], as the problem admits a constant-length MSO_1 formula.

5 $(1, \beta)$-Linear Degree Domination (BDD)

We sketch the proof of the W[1]-hardness of $(1, \beta)$-Linear Degree Domination problem, which is enough to demonstrate the main idea of the reduction technique. Recall that we assume that $\alpha = 1$ and $\beta < 0$, which is the Bounded Degree Deletion problem. That is, we must delete at most q vertices such that the resulting subgraph has maximum degree at most $|\beta|$. Furthermore, in the $(1, \beta)$-Linear Degree Domination problem, we also call X, the $(1, \beta)$-linear degree dominating set of G, the deletion part of G.

We provide a reduction from the Symmetric Multicolored Clique problem, which we define as follows. A symmetric multicolored graph $G = (V^1 \cup V^2 \ldots \cup V^k, E)$ is a connected graph such that, for all distinct $i, j \in [k]$,
1. $V^i = \{v^i_1, \ldots, v^i_n\}$, where $n \geq k$;
2. all the vertices of V^i are colored by color i;
3. if $v^i_r v^j_s \in E(G)$, then $v^j_s v^i_r \in E(G)$ as well.

Then, for the Symmetric Multicolored Clique problem, the input is a symmetry multicolored graph G and an integer k, and the objective is to decide whether G contains a k-clique with vertices of all k colors. We also call $v^i_r v^j_s$ and $v^i_s v^j_r$ symmetry edges.

The reduction in [15, Lemma 1], which proves the W[1]-hardness of the multicolored clique problem, actually produces a symmetric multicolored graph. Hence, Symmetric Multicolored Clique is W[1]-hard. We now sketch the following.

▶ **Lemma 10.** *Case 3 for the W[1]-hardness results in Theorem 7 is correct.*

Let (G, k) be an instance of Symmetric Multicolored Clique, where $G = (V^1 \cup V^2 \cup \ldots \cup V^k, E)$. Without loss of generality, suppose $k \geq 100$, otherwise, the problem can be solved in polynomial time. We will construct a corresponding instance (H, β, q) of $(1, \beta)$-Linear Degree Domination, where H is a graph whose $\mathcal{G}$-modular cardinality will be bounded by $O(k^2)$, $\beta = -(nk)^{10000}$, and q is the maximum allowed size of X, the desired deletion part of H (to be specified later). Before proceeding, we will make use of a 2-sumfree-set, which is a set of positive integers in which every couple of elements has a distinct sum. That is, I is a 2-sumfree set if, for any $(a, b), (a', b') \in I \times I$, $a + b = a' + b'$ if and only if $\{a, b\} = \{a', b'\}$ (note that $a = b$ is possible). It is known that one can construct in time $O(n^3)$ a 2-sumfree-set I of cardinality n in which the maximum value is n^4, which can be achieved with a greedy procedure (this is because $(n+1)^4 - n^4 > n^3$ and $a_i + a_j - a_r$ has at most n^3 different values). We thus assume that we have built a 2-sumfree set $I = \{a_1, \ldots, a_n\}$ for H, where $n^4 \geq a_1 > \ldots > a_n \geq 1$. Without loss of generality, we may multiply each $a_i \in I$ by an

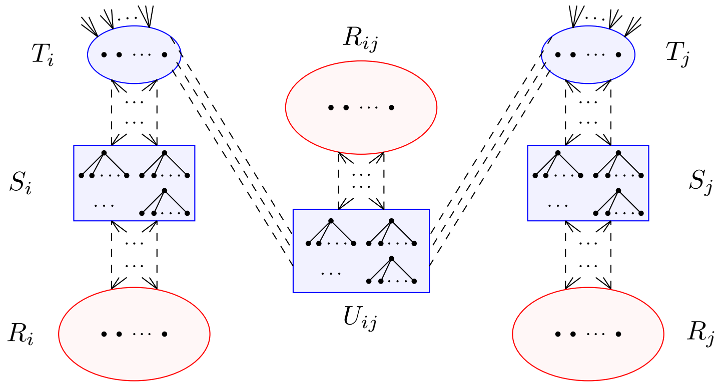

Figure 3 Illustration of the construction of H.

integer r, where here we choose $r = 2(k-1)^2k^3$. Then, we have that $n^4r \geq a_1 > \ldots > a_n \geq r$ for the updated I. Moreover, for any distinct pair $(a,b),(a',b') \in I \times I$, we have that the absolute value of $a+b-a'-b'$ is at least r.

For each color class $V^i = \{v_1^i, \ldots, v_n^i\}$ of G, we provide a bijection f_i from V^i to I, such that $f_i(v_s^i) = a_s$ for every $s \in [n]$. Clearly, we can use $f_i^{-1}(a_s)$ to denote the unique vertex v_s^i of V^i associated with $a_s \in I$. We have that, for all $s,t \in [n]$, each pair of $a_s, a_t \in I$ has a unique sum. For distinct i,j and any $u \in V^i, w \in V^j$ such that $uw \in E(G)$, if $f_i(u) + f_j(w) = a_s + a_t$, then edge uw is either $v_s^i v_t^j$ or $v_t^i v_s^j$. Moreover, for any distinct color classes V^i, V^j, edges $v_s^i v_t^j$ and $v_t^i v_s^j$, together, are both in $E(G)$ or both not in $E(G)$. Hence, by looking at a sum in I, we will be able to tell whether it is corresponding to a pair of symmetry edges between V^i and V^j, or not.

Next, we define $s = n^{10}$, $q = ks + \binom{k}{2}s$, $a_0 = q+1$, and $a_{n+1} = a_n - 1$. We construct H as in Figure 3. First, for each color class V^i of G, add three factors R_i, S_i, T_i to H, where:

- R_i is an edgeless graph of size $|\beta| - s$.
- S_i is a $(a_0, I \cup \{a_{n+1}\}, c)$-degree deletion graph, where we put the costs $c(a_j) = s - \frac{1}{2}a_j$ for $a_j \in I$ and $c(a_{n+1}) = a_0$.
- T_i is an edgeless graph of size s.

We then make S_i adjacent with R_i and T_i. Secondly, for each pair of color classes V^i, V^j with $i < j$, we add another two factors U_{ij}, R_{ij}, where:

- R_{ij} is an edgeless graph of size $|\beta| - 2s$.
- U_{ij} is built as follows. Suppose integer set I_{ij} consists of all $a+b$ such that a, $b \in I$ and symmetry edges $f_i^{-1}(b)f_j^{-1}(a), f_i^{-1}(a)f_j^{-1}(b) \in E(G)$. Let $\ell_{ij} = \min(I_{ij})$.
  Then U_{ij} is a $(a_0, I_{ij} \cup \{\ell_{ij} - 1\}, c_{ij})$-degree deletion graph, where we put the cost $c_{ij}(a+b) = s - \frac{1}{2(k-1)}(a+b)$, and we put $c_{ij}(\ell_{ij} - 1) = a_0$.

We then make U_{ij} adjacent with R_{ij}, and adjacent with T_i and T_j. To avoid cumbersome notation, we define $U_{ij} = U_{ji}$, $R_{ij} = R_{ji}$, $c_{ij} = c_{ji}$, and $I_{ij} = I_{ji}$. This completes the construction of H. It is easy to see that H can be constructed in polynomial time and the number of factors in H is a polynomial function in k.

All that remains is to argue that the objective instance is equivalent to the original one. The following lemma gives the forward direction. We only provide a sketch here.

▶ **Lemma 11.** *Suppose that G contains a multicolored clique C of size k. Then there is $X \subseteq V(H)$ of size at most $q = ks + \binom{k}{2}s$ such that $\Delta(H - X) \leq |\beta|$.*

Proof sketch. For $i \in [k]$, let $f_i^{-1}(\hat{a}_i)$ be the vertex of V^i that belongs to the multicolored clique C, where $\hat{a}_i \in I$ is the number associated with the vertex. For any $i \neq j$, we know that vertices $f_i^{-1}(\hat{a}_i)$ and $f_j^{-1}(\hat{a}_j)$ are in C, which means that $f_i^{-1}(\hat{a}_i)f_j^{-1}(\hat{a}_j) \in E(G)$. This implies that $\hat{a}_i + \hat{a}_j$ is in the I_{ij} list that was used to construct U_{ij}. We then show how to construct the vertex set X for H. The intersection of each R_i and X is empty. For each T_i, add $\hat{a}_i$ vertices to X. For each S_i, add $c(\hat{a}_i) = s - \frac{1}{2}\hat{a}_i$ vertices to X. This can be done so that $S_i - X$ has maximum degree $\hat{a}_i$ according to Definition 4. The intersection of each R_{ij} and X is empty. For each U_{ij}, we can add $c_{ij}(\hat{a}_i + \hat{a}_j) = s - \frac{\hat{a}_i+\hat{a}_j}{2(k-1)}$ vertices to X so that $U_{ij} - X$ has maximum degree $\hat{a}_i + \hat{a}_j$ according to Definition 4. It is not hard to verify that X with exactly q vertices satisfies that $\Delta(H - X) \leq |\beta|$. ◀

The converse direction is much more difficult.

▶ **Lemma 12.** *Suppose that there is $X \subseteq V(H)$ with $|X| \leq q$ such that $H - X$ has maximum degree at most $|\beta|$. Then G contains a multicolored clique of size k.*

Proof sketch. Let $X \subseteq V(H)$ be of size at most q such that $H - X$ has maximum degree $|\beta|$ or less. To ease notation slightly, for a factor M of H, we will write $X(M) := X \cap V(M)$ and $\chi(M) := |X(M)|$. The proof is divided into a series of claims. ◀

▷ Claim 13. For each $i \in [k]$, we may assume that $\chi(R_i) = 0$. Moreover for each distinct $i, j \in [k]$, we may assume that $\chi(R_{ij}) = 0$.

The rough idea is that a vertex of $X(R_i)$ can always be replaced by a vertex of $T_i \setminus X$ if the latter is non-empty. If $V(T_i) \setminus X$ is empty, then after an unavoidable at least $c(a_1)$ vertices deletion from S_i, no deletion in R_i is needed since each remaining vertex of S_i in $H - X$ has maximum degree at most $|\beta| - s + a_1 < |\beta|$. The idea for $\chi(R_{ij}) = 0$ goes the same way.

▷ Claim 14. For any $i \in [k]$, we may assume $\Delta(S_i - X) \in \{a_1, \ldots, a_n\}$, and that $\chi(S_i) = c(\Delta(S_i - X))$.

The rough idea is that Case 2 of Definition 4 states that we can delete $c(a_j)$ vertices from S_i to make $\Delta(S_i - X) = a_j$, where $0 \leq j \leq n + 1$. In fact, this is the smallest maximum degree we can achieve in S_i by deleting between $c(a_j)$ and $c(a_{j+1}) - 1$ vertices, because of the stepwise behavior of arbitrary deletion tables. Moreover, we already know that $\chi(R_i) = 0$ based on Claim 13. So, if $\Delta(S_i - X) = a_0$ there is a vertex in $S_i - X$ with degree $|\beta| - s + a_0 > |\beta|$, and if $\Delta(S_i - X) = a_n - 1$ or less then $c(a_n - 1) = a_0 > q$ vertices have to be deleted from S_i.

▷ Claim 15. For any distinct $i, j \in [k]$, we may assume that $\Delta(U_{ij} - X) \in I_{ij}$, and that $\chi(U_{ij}) = c_{ij}(\Delta(U_{ij} - X))$.

The idea for proving this claim is similar to that of Claim 14. Our next step is to argue that the degree chosen by U_{ij} must be the sum of the degrees chosen by S_i and S_j. More specifically, for $i \in [k]$, we say that S_i *chose* $a_j \in I$ if $\Delta(S_i - X) = a_j$ and $\chi(S_i) = c(a_j)$. Likewise, for distinct $i, j \in [k]$, we say that U_{ij} *chose* $a, b \in I$ if $\Delta(U_{ij} - X) = a + b$ and $\chi(U_{ij}) = c_{ij}(a+b)$. Note that by Claim 14, each S_i chooses one a_j and by Claim 15, each U_{ij} chooses one pair a, b such that symmetry edges $f_i^{-1}(b)f_j^{-1}(a), f_i^{-1}(a)f_j^{-1}(b) \in E(G)$. The point to make is that if S_i and S_j chose a and b, respectively, then U_{ij} must have chosen a, b.

▷ Claim 16. For each $i \neq j$, if S_i chose $a \in I$ and S_j chose $b \in I$, then U_{ij} chose a, b.

We sketch it as follows. For each $i \in [k]$, we will denote by $\hat{a}_i$ the element of I that S_i chose. We divide the U_{ij}'s into three groups:

$$U^{<} = \{U_{ij} : U_{ij} \text{ chose } a', b' \text{ such that } a' + b' < \hat{a}_i + \hat{a}_j\}$$
$$U^{=} = \{U_{ij} : U_{ij} \text{ chose } a', b' \text{ such that } a' + b' = \hat{a}_i + \hat{a}_j\}$$
$$U^{>} = \{U_{ij} : U_{ij} \text{ chose } a', b' \text{ such that } a' + b' > \hat{a}_i + \hat{a}_j\}$$

To prove the claim, it suffices to show that $U^{<}$ and $U^{>}$ are empty (this is because $U_{ij} \in U^{=}$ is only possible if U_{ij} chose $\hat{a}_i, \hat{a}_j$, since all the sum pairs are distinct). The rough idea is as follows. If each U_{ij} chose the correct $\hat{a}_i, \hat{a}_j$, then each of them will incur a deletion cost of $c_{ij}(\hat{a}_i + \hat{a}_j) = s - \frac{\hat{a}_i + \hat{a}_j}{2(k-1)}$ and end up cancelling the deletion costs of the S_i and T_i factors. If $U^{<}$ is non-empty, it incurs extra deletion cost with respect to $c_{ij}(\hat{a}_i + \hat{a}_j)$ with no real benefit. The complicated case is when $U^{>}$ is non-empty. In this case, $U_{ij} - X$ has higher degree than if it had chosen $\hat{a}_i, \hat{a}_j$ and incurs less deletions than $c_{ij}(\hat{a}_i + \hat{a}_j)$. However, this needs to be compensated with extra deletions in T_i and T_j. By using a charging argument, we can show that the sum of extra deletions required for all the $U^{>}$ members outweighs the deletions saved in the U_{ij}'s of $U^{>}$.

We can now construct a multicolored clique. Define $C = \{f_i^{-1}(\hat{a}_i) : i \in [k] \text{ and } S_i \text{ chose } \hat{a}_i\}$. We claim that C is a clique. By Claim 14, each S_i chooses some $\hat{a}_i$ and thus $|C| = k$. Now let $f_i^{-1}(\hat{a}_i), f_j^{-1}(\hat{a}_j)$ be two vertices of C, where $i < j$. Then $\hat{a}_i, \hat{a}_j$ were chosen by S_i and S_j, respectively, and by Claim 15 we know that U_{ij} chose $\hat{a}_i + \hat{a}_j$. By the construction of the U_{ij} solution table, this is only possible if symmetry edges $f_i^{-1}(\hat{a}_i)f_j^{-1}(\hat{a}_j), f_i^{-1}(\hat{a}_j)f_j^{-1}(\hat{a}_i) \in E(G)$. Therefore, $f_i^{-1}(\hat{a}_i)f_j^{-1}(\hat{a}_j) \in E(G)$ and C is a clique.

6 FPT algorithms for succinct solution tables

In this section, we study the Bounded Degree Deletion problem (i.e. $\alpha = 1, \beta < 0$) on graph classes that admit succinct solution tables. Let $G = (V, E)$ be a graph. Assume $\mathbb{S}_x \subseteq 2^V$ consists of all subsets of V with size $x \in [0, |V|]$. $X \in \mathbb{S}_x$ is called an *x-deletion set* of G if $\Delta(G - X) = \min\{\Delta(G - Y) : Y \in \mathbb{S}_x\}$. An *$x$-deletion* of G is the process of deleting all vertices of an x-deletion set from G. Clearly, the degree deletion function $f(x)$ of G is the maximum degree of G after an x-deletion. A *piecewise linear function* $g : \mathbb{R} \to \mathbb{R}$ is a continuous function defined on a sequence of intervals, such that the function is linearly restricted to each of the intervals (each such linear function is called a *sub-function* of g). In addition, a *constant piecewise linear function* is a piecewise linear function that consists of a constant number of linear sub-functions.

▶ **Definition 17.** *Let $\mathcal{G}$ be a polynomial-time recognizable graph class. For $G \in \mathcal{G}$, suppose that $f^G(x)$ is the degree deletion function of $G = (V, E)$, where $x \in [0, |V|]$. We say that $\mathcal{G}$ admits a succinct solution table for Bounded Degree Deletion if, for every $G \in \mathcal{G}$, there exists a function $g^G : \mathbb{R} \to \mathbb{R}$ that satisfies at least one of the following conditions:*
1. *g^G is a constant piecewise linear function such that $f^G(x) = g^G(x)$ for every integer $x \in [0, |V(G)|]$,*
2. *g^G is a piecewise convex linear function such that $f^G(x) = \lceil g^G(x) \rceil$ for every integer $x \in [0, |V(G)|]$.*

Moreover, g^G can be described and constructed in polynomial time with respect to $|V(G)|$.

A graph class admits a succinct solution table of type t for Bounded Degree Deletion if we refer to the condition t, where $t \in \{1, 2\}$. Type 1 implies the solution table can be divided into a constant number of blocks, each of which can be encoded into a small number of bits (note that for any fixed graph class, the upper bound on the constant should be fixed as well). In type 2, the solution table could be convex, but could also be *non-convex* with a larger number of blocks, but we can reduce this special non-convex function to a convex function using ceilings (this occurs with `cluster`-*mc*). As we show, these definitions capture the intuition of solution tables that can be merged in FPT time.

▶ **Theorem 18.** *Let $\mathcal{G}$ be a graph class that admits a succinct solution table for* Bounded Degree Deletion*. Assume a minimum $\mathcal{G}$-modular partition is given. Then,* Bounded Degree Deletion *is FPT parameterized by $\mathcal{G}$-modular cardinality.*

Application. Let graph class BDT include all graphs with bounded degree and treewidth. We prove BDT and `cluster` admit succinct solution tables of type 1 and type 2, respectively. Thus, Bounded Degree Deletion is FPT in parameter *BDT-mc* or `cluster`-*mc*. Clearly, *BDT-mc* is bounded by a function of vertex cover number and is incomparable with *mw* (*linear forest* has unbounded *mw*). Moreover, `cluster`-*mc* is a parameter with size at most *nd* and at least *mw*.

▶ **Theorem 19.** *BDT admits a succinct solution table for* Bounded Degree Deletion*. Therefore, the* Bounded Degree Deletion *is FPT in parameter BDT-mc.*

Proof sketch. For every $G \in$ BDT, the degree deletion function of G can be represented by an efficiently computable step function with a constant number of steps. ◀

▶ **Lemma 20.** *Assume cluster graph H contains b complete graphs K_a, where K_a is the maximum size complete graph in H. Let $q \in [b, |V(H)|]$. Suppose R is obtained from H by deleting exactly one vertex from every K_a of H. Then, H has a q-deletion such that the maximum degree of the remaining graph is h if and only if R has a $(q-b)$-deletion such that the maximum degree of the remaining graph is h.*

▶ **Theorem 21.** *Cluster admits a succinct solution table for* Bounded Degree Deletion*. Therefore, the* Bounded Degree Deletion *is FPT in parameter* `cluster`*-mc.*

Proof sketch. Let G be a cluster graph and $f^G(x)$ be the degree deletion function of G. We can construct a piecewise convex linear function g^G based on the structure of G, and prove that $f^G(x) = \lceil g^G(x) \rceil$ by using the properties of the two functions and Lemma 20. ◀

Open problems. We conclude with some interesting problems.

- Can we characterize graph classes are easily mergeable? For instance, is the class of H-free graphs easily mergeable, for any fixed graph H?
- Is Bounded Degree Deletion fixed-parameter tractable in parameter ($K_{1,t}$-free)-modular cardinality, where $t \geq 3$ is either fixed or a parameter?
- Is k-Domination FPT in parameter `cluster`-modular cardinality, where β is related to the input size?
- Is α-Domination FPT in parameter `cluster`-modular cardinality?
- The Red-Blue Capacitated Dominating Set problem is W[1]-hard in *cw* [17]. It is not hard to prove it to be FPT in *mw* using succinct solution tables. Does the same hold for the Red-Blue Exact Saturated Capacitated Dominating Set?
- Are Edge Dominating Set, Max-cut, and Partition Into Triangles FPT in parameter `cograph`-modular cardinality?

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