

On the Finite Variable-Occurrence Fragment of the Calculus of Relations with Bounded Dot-Dagger Alternation

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Abstract

We introduce the *k*-variable-occurrence fragment, which is the set of terms having at most *k* occurrences of variables. We give a sufficient condition for the decidability of the equational theory of the *k*-variable-occurrence fragment using the finiteness of a monoid. As a case study, we prove that for Tarski's calculus of relations with bounded dot-dagger alternation (an analogy of quantifier alternation in first-order logic), the equational theory of the *k*-variable-occurrence fragment is decidable for each *k*.

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1 Introduction

Since *the satisfiability problem of first-order logic* is undecidable [8, 30] in general, (un-)decidable classes of first-order logic are widely studied [7]; for example, the undecidability holds even for the *Kahr–Moore–Wang (KMW) class*¹ $[\forall\exists\forall, (0, \omega)]$ [14], but it is decidable for the *Bernays–Schönfinkel–Ramsey (BSR) class* $[\exists^*\forall^*, \text{all}] = [3, 27]$. The calculus of relations (CoR) [28], revived by Tarski, is an algebraic system on binary relations; its expressive power is equivalent to that of the three-variable fragment of first-order logic with equality [28, 29], w.r.t. binary relations. The equational theory of CoR is undecidable [28, 29]² in general, which follows from the undecidability of the KMW class, but, for example, it is decidable for the *(existential) positive fragment* [2, 26] and the *existential fragment* [22] of CoR, which follows from the decidability of the BSR class. On the undecidability of CoR, the undecidability holds even for the 1-variable fragment [16] and even for the 1-variable fragment only with union, composition, and complement [19], where the *k*-variable fragment denotes the set of terms having at most *k* variables.

¹ Recall the notation for *prefix-vocabulary classes* [7, Def. 1.3.1]. E.g., $[\forall\exists\forall, (0, \omega)]$ denotes the set of prenex sentences φ of first-order logic without equality, function symbols, nor constants such that the quantifier prefix of φ is $\forall\exists\forall$; φ has ω (countably infinitely many) binary relation symbols and φ does not have 1- nor *i*-ary relation symbols for $i \geq 3$.

² In [29], the undecidability of the equational theory is shown for more general classes of relation algebras.



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Then, from the undecidability result for the 1-variable fragment of CoR [16, 19] above, the following natural question arises – Is it decidable for the k -variable-occurrence fragment of CoR? Here, the k -variable-occurrence fragment denotes the set of terms having at most k occurrences of variables. For example, when a, b are variables and l is a constant, the term $(a \cdot b) \cdot (l \cdot (a \cdot b))$ has 4 occurrences of variables and 1 occurrence of constants; thus, this term is in the 4-variable-occurrence fragment (cf. the term is in the 2-variable fragment since the variables a, b occur). While one may seem that this restriction immediately implies the decidability, the equational theory of the k -variable-occurrence fragment on some (single) algebra is undecidable in general even when $k = 0$ (Remark 2).

Our contribution is to prove that the equational theory of the k -variable-occurrence fragment is decidable for CoR *with bounded dot-dagger alternation*, where the dot-dagger alternation [20, 21] is an analogy of the quantifier alternation in first-order logic. Note that the equational theory of the k -variable fragment is undecidable in general for CoR [16, 19] (even with bounded dot-dagger alternation (Prop. 24)).

Our strategy is to prove that the number of terms in the k -variable-occurrence fragment is finite up to the semantic equivalence relation. To this end, (1) we decompose terms as much as possible; and then (2) we show that each decomposed part is finite up to the semantic equivalence relation by collecting valid equations. By the preprocessing of (1), one can see that for (2), essentially, it suffices to prove the finiteness of some monoid (using the method of Sect. 2). Its finiteness is not clear, as it is undecidable whether a (finitely presented) monoid is finite in general; but, fortunately, we can prove the finiteness (Thm. 25) by finding valid equations (Fig. 1).

The rest of this paper is structured as follows. Sect. 2 introduces the k -variable-occurrence fragment for general algebras and gives a framework to prove the decidability from the finiteness of a monoid. Sect. 3 recalls the syntax and semantics of CoR and the dot-dagger alternation hierarchy. In Sect. 4, based on Sect. 2, we prove that the equational theory of CoR with bounded dot-dagger alternation is decidable. Sect. 5 concludes this paper.

We write \mathbb{N} for the set of all non-negative integers. For $l, r \in \mathbb{N}$, we write $[l, r]$ for the set $\{i \in \mathbb{N} \mid l \leq i \leq r\}$. For a set A , we write $\#A$ for the cardinality of A and $\wp(A)$ for the power set of A . For a set A and an equivalence relation \sim on A , we write A/\sim for the quotient set of A by \sim and $[a]_{\sim}$ for the equivalence class of an element $a \in A$ on \sim .

2 On the k -variable-occurrence fragment

We fix Σ as a non-empty *finite* set of variables. We fix S as a *finite* algebraic signature; S is a map from a finite domain (of functions) to \mathbb{N} . For each $\langle f, n \rangle \in S$, we write $f^{(n)}$; it is the function symbol f with arity n . We also let $S^{(n)} \triangleq \{f^{(m)} \in S \mid m = n\}$. The set \mathbf{T} of S -terms over Σ is defined as the minimal set closed under the following two rules: $a \in \Sigma \implies a \in \mathbf{T}$; $(f^{(n)} \in S \text{ and } t_1, \dots, t_n \in \mathbf{T}) \implies f(t_1, \dots, t_n) \in \mathbf{T}$.

An S -algebra A is a tuple $\langle |A|, \{f^A\}_{f^{(n)} \in S} \rangle$, where $|A|$ is a non-empty finite set and $f^A: |A|^n \rightarrow |A|$ is an n -ary map for each $f^{(n)} \in S$. A *valuation* $\mathbf{v}: \Sigma \rightarrow |A|$ on an S -algebra A is a map; we write $\hat{\mathbf{v}}: \mathbf{T} \rightarrow |A|$ for the unique homomorphism extending \mathbf{v} . For a class \mathcal{C} of S -algebras, the equivalence relation $\sim_{\mathcal{C}}$ on \mathbf{T} is defined by: $t \sim_{\mathcal{C}} s \iff \hat{\mathbf{v}}(t) = \hat{\mathbf{v}}(s)$ for all valuations \mathbf{v} on all algebras in \mathcal{C} . For a set $T \subseteq \mathbf{T}$, the equational theory of T over \mathcal{C} is the set $\{\langle t, s \rangle \in T^2 \mid t \sim_{\mathcal{C}} s\}$.

► **Definition 1** (*k*-variable-occurrence fragment). For an *S*-term t , let $\text{vo}(t)$ be the number of occurrences of variables in t : $\text{vo}(t) \triangleq \begin{cases} 1 & (t \in \Sigma) \\ \sum_{i=1}^n \text{vo}(t_i) & (t = f(t_1, \dots, t_n)) \end{cases}$. For each set T of *S*-terms, the *k*-variable-occurrence fragment $T_{(\text{vo} \leq k)}$ is the set $\{t \in T \mid \text{vo}(t) \leq k\}$. (Similarly, let $T_{(\text{vo} = k)} \triangleq \{t \in T \mid \text{vo}(t) = k\}$.) Clearly, $T = \bigcup_{k \in \mathbb{N}} T_{(\text{vo} \leq k)}$.

► **Remark 2.** The equational theory of the *k*-variable-occurrence fragment is undecidable in general, even when $k = 0$. It follows from the reduction from the word problem for monoids. Let $M = \langle |M|, \circ^M, \mathbf{1}^M \rangle$ be a (finitely) presented monoid with finite generators $C = \{c_1, \dots, c_l\}$ such that the word problem for M is undecidable (by Markov [17] and Post [25]). We define $S \triangleq \{c_1^{(1)}, \dots, c_l^{(1)}\} \cup \{\mathbf{1}^{(0)}\}$ and the *S*-algebra $A \triangleq \langle |A|, \{f^A\}_{f^{(n)} \in S} \rangle$ by: $|A| = |M|$; $c_i^A(x) = c_i \circ^M x$ for $i \in [1, l]$; $\mathbf{1}^A = \mathbf{1}^M$. By definition, for all two words $a_1 \dots a_n, b_1 \dots b_m$ over C : they are equivalent in M iff $a_1(a_2(\dots a_n(\mathbf{1}))) \sim_{\{A\}} b_1(b_2(\dots b_m(\mathbf{1})))$.

In the rest of this section, we fix \mathcal{C} as a class of *S*-algebras.

2.1 On the finiteness of *k*-variable-occurrence fragment: from 1 to *k*

How can we show the decidability of the equational theory of the *k*-variable-occurrence fragment? We consider proving it from the finiteness up to the semantic equivalence relation:

► **Proposition 3** (Cor. of [5, 15] for the complexity). Let $T \subseteq \mathbf{T}$ be a subterm-closed³ set. If the set $T/\sim_{\mathcal{C}}$ is finite, the equational theory of T over \mathcal{C} is decidable. Moreover, it is decidable in DLOGTIME-uniform NC¹ if the input is given as a well-bracketed string.

Proof Sketch. Because \mathcal{C} is fixed and $T/\sim_{\mathcal{C}}$ is finite, for each $t \in T$, one can calculate the index of the equivalence class of t on $\sim_{\mathcal{C}}$ by using the (finite and possibly partial) Cayley table of each operator; thus, the equational theory is decidable. Moreover, according to this algorithm, if the input is given as a well-bracketed string, one can also construct a parenthesis context-free grammar such that for all $t, s \in T$, the well-bracketed string encoding the equation $t = s$ is in the language iff $t \sim_{\mathcal{C}} s$. Hence, the complexity is shown because every language recognized by a parenthesis context-free grammar is in ALOGTIME [5, 6] (ALOGTIME is equivalent to DLOGTIME-uniform NC¹ [18]). ◀

For the *k*-variable-occurrence fragment, the finiteness of $T_{(\text{vo} \leq 1)}$ (with Prop. 3) can imply the decidability of the equational theory of $T_{(\text{vo} \leq k)}$ (Lem. 6) by the following decomposition lemma. Here, we write $t[s/a]$ for the term t in which each a has been replaced with s .

► **Lemma 4.** Let $T \subseteq \mathbf{T}$ be a subterm-closed set. Let $k \geq 2, a \in \Sigma$. Then, for all $t \in T_{(\text{vo} = k)}$, there are $t_0 \in T_{(\text{vo} \leq 1)}, f^{(n)} \in S, t_1, \dots, t_n \in T_{(\text{vo} \leq k-1)}$ such that $t = t_0[f(t_1, \dots, t_n)/a]$.

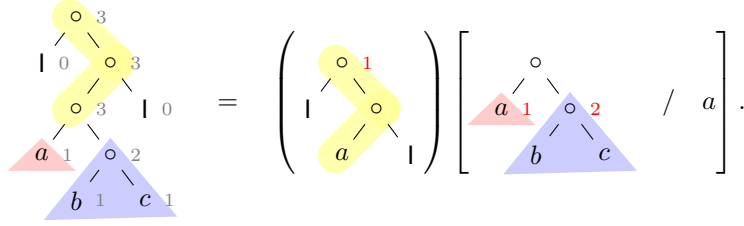
Proof. By induction on t . Since $k \geq 2$, there are $g^{(m)} \in S, s_1, \dots, s_m \in T$ s.t. $t = g(s_1, \dots, s_m)$ and $\sum_{i=1}^m \text{vo}(s_i) = k$. Case $\text{vo}(s_i) \leq k-1$ for all i : By letting $t_0 \triangleq a$, we have $t = t_0[g(s_1, \dots, s_m)/a]$. Otherwise: Let i be s.t. $\text{vo}(s_i) = k$. Since $s_i \in T_{(\text{vo} = k)}$, let $u \in T_{(\text{vo} \leq 1)}, f^{(n)} \in S, t_1, \dots, t_n \in T_{(\text{vo} \leq k-1)}$ be the ones obtained by IH w.r.t. s_i , so that $s_i = u[f(t_1, \dots, t_n)/a]$. By letting $t_0 \triangleq g(s_1, \dots, s_{i-1}, u, s_{i+1}, \dots, s_m)$, we have:

³ A set $T \subseteq \mathbf{T}$ is subterm-closed if for every $t \in T$, if s a subterm of t , then $s \in T$.

$$\begin{aligned}
 t_0[f(t_1, \dots, t_n)/a] &= g(s_1, \dots, s_{i-1}, u, s_{i+1}, \dots, s_m)[f(t_1, \dots, t_n)/a] \\
 &= g(s_1, \dots, s_{i-1}, u[f(t_1, \dots, t_n)/a], s_{i+1}, \dots, s_m) \quad (\text{vo}(s_j) = 0 \text{ if } j \neq i) \\
 &= g(s_1, \dots, s_{i-1}, s_i, s_{i+1}, \dots, s_m) = t. \quad (s_i = u[f(t_1, \dots, t_n)/a])
 \end{aligned}$$

Hence, this completes the proof. \blacktriangleleft

► **Example 5** (of Lem. 4). If $S = \{\circ^{(2)}, \text{l}^{(0)}\}$ and $a, b, c \in \Sigma$, the term $t = \text{l} \circ ((a \circ (b \circ c)) \circ \text{l}) \in \mathbf{T}_{(\text{vo} \leq 3)}$ has the following decomposition: $t = (\text{l} \circ (a \circ \text{l})) [a \circ (b \circ c) / a]$. Then $\text{l} \circ (a \circ \text{l}) \in \mathbf{T}_{(\text{vo} \leq 1)}$ and $a, (b \circ c) \in \mathbf{T}_{(\text{vo} \leq 2)}$. The following is an illustration of the decomposition, where the number written in each subterm s denotes $\text{vo}(s)$:



Using this decomposition iteratively, we have the following:

► **Lemma 6.** *Let $T \subseteq \mathbf{T}$ be a subterm-closed set. Assume that $T_{(\text{vo} \leq 1)} / \sim_c$ is finite. Then, for each $k \in \mathbb{N}$, the set $T_{(\text{vo} \leq k)} / \sim_c$ is finite.*

Proof. It suffices to prove: for all $k \geq 2$, $T_{(\text{vo} = k)} / \sim_c$ is finite. By induction on k . We have:

$$\begin{aligned}
 &\#(T_{(\text{vo} = k)} / \sim_c) \\
 &\leq \#(\{t_0[f(t_1, \dots, t_n)/a] \mid f^{(n)} \in S, t_1, \dots, t_n \in T_{(\text{vo} \leq k-1)}, t_0 \in T_{(\text{vo} \leq 1)}\} / \sim_c) \quad (\text{Lem. 4}) \\
 &\leq \sum_{f^{(n)} \in S} \#(\{t_0[f(t_1, \dots, t_n)/a] \mid t_1, \dots, t_n \in T_{(\text{vo} \leq k-1)}, t_0 \in T_{(\text{vo} \leq 1)}\} / \sim_c).
 \end{aligned}$$

Then the set $\{t_0[f(t_1, \dots, t_n)/a] \mid t_1, \dots, t_n \in T_{(\text{vo} \leq k-1)}, t_0 \in T_{(\text{vo} \leq 1)}\} / \sim_c$ is finite because $T_{(\text{vo} \leq k-1)} / \sim_c$ is finite (by IH) and \sim_c satisfies the congruence law. Thus, the last term above is finite since S is finite. Hence $T_{(\text{vo} = k)} / \sim_c$ is finite. \blacktriangleleft

2.2 The monoid of the 1-variable-occurrence fragment

Thanks to Lem. 6, we can focus on the 1-variable-occurrence fragment. For the 1-variable-occurrence fragment, it suffices to consider a monoid. For a set A of characters, we write A^* for the set of all words (i.e., finite sequences) over the alphabet A . We write wv for the concatenation of words w and v and write ε for the empty word. We write $\|w\|$ for the length of a word w .

► **Definition 7.** *Let $\dot{\Sigma}$ be the (possibly infinite) set of characters defined by:*

$$\dot{\Sigma} \triangleq \bigcup_{f^{(n)} \in S, i \in [1, n]} \{f(t_1, \dots, t_{i-1}, _, t_{i+1}, \dots, t_n) \mid \forall j \in [1, n] \setminus \{i\}, t_j \in \mathbf{T}_{(\text{vo} \leq 0)}\}.$$

($_$ denotes “blank”.) For a word $w \in \dot{\Sigma}^*$ and a term $t \in \mathbf{T}$, let $w[t]$ be the term defined by:

$$w[t] \triangleq \begin{cases} f(t_1, \dots, t_{i-1}, w'[t], t_{i+1}, \dots, t_n) & (w = f(t_1, \dots, t_{i-1}, _, t_{i+1}, \dots, t_n)w') \\ t & (w = \varepsilon) \end{cases}.$$

► **Example 8** (of Def. 7). If $S = \{\circ^{(2)}, \mathbf{l}^{(0)}\}$, then $\dot{\Sigma} = \{(\mathbf{l} \circ _), ((\mathbf{l} \circ \mathbf{l}) \circ _), \dots (_ \circ \mathbf{l}), \dots\}$. For example, if $w = ((\mathbf{l} \circ \mathbf{l}) \circ _)(_ \circ \mathbf{l})(\mathbf{l} \circ _)$, we have:

$$\begin{aligned} w[a] &= (((\mathbf{l} \circ \mathbf{l}) \circ _)(_ \circ \mathbf{l})(\mathbf{l} \circ _)[a]) = (\mathbf{l} \circ \mathbf{l}) \circ ((_ \circ \mathbf{l})(\mathbf{l} \circ _)[a]) \\ &= (\mathbf{l} \circ \mathbf{l}) \circ (((\mathbf{l} \circ _)[a]) \circ \mathbf{l}) \\ &= (\mathbf{l} \circ \mathbf{l}) \circ ((\mathbf{l} \circ \varepsilon[a]) \circ \mathbf{l}) = (\mathbf{l} \circ \mathbf{l}) \circ ((\mathbf{l} \circ a) \circ \mathbf{l}). \end{aligned}$$

► **Proposition 9.** For all $t \in \mathbf{T}_{(\mathbf{v}_0 \leq 1)}$, there are $w \in \dot{\Sigma}^*$ and $s \in S^{(0)} \cup \Sigma$ s.t. $t = w[s]$.

Proof. By easy induction on t . ◀

► **Definition 10.** Let $\sim_{\mathcal{C}}$ be the equivalence relation on $\dot{\Sigma}^*$ defined by:

$$w \sim_{\mathcal{C}} v \iff w[a] \sim_{\mathcal{C}} v[a] \text{ where } a \in \Sigma \text{ is any variable.}$$

► **Lemma 11.** If $\dot{\Sigma}^*/\sim_{\mathcal{C}}$ is finite, then $\mathbf{T}_{(\mathbf{v}_0 \leq 1)}/\sim_{\mathcal{C}}$ is finite.

Proof. By Prop. 9 (and that the set $S^{(0)} \cup \Sigma$ is finite). ◀

Moreover, if $\mathbf{T}_{(\mathbf{v}_0 \leq 0)}/\sim_{\mathcal{C}}$ is finite, it suffices to consider a finite subset of $\dot{\Sigma}$, as follows:

► **Lemma 12.** Assume that $\mathbf{T}_{(\mathbf{v}_0 \leq 0)}/\sim_{\mathcal{C}}$ is finite. Let $T_0 = \{t_1, \dots, t_n\} \subseteq \mathbf{T}_{(\mathbf{v}_0 \leq 0)}$ be such that $\mathbf{T}_{(\mathbf{v}_0 \leq 0)}/\sim_{\mathcal{C}} = \{[t_1]_{\sim_{\mathcal{C}}}, \dots, [t_n]_{\sim_{\mathcal{C}}}\}$. Let $\dot{\Sigma}_0 \subseteq \dot{\Sigma}$ be the finite set defined by:

$$\dot{\Sigma}_0 \triangleq \bigcup_{f^{(n)} \in S, i \in [1, n]} \{f(t_1, \dots, t_{i-1}, _, t_{i+1}, \dots, t_n) \mid \forall j \in [1, n] \setminus \{i\}, t_j \in T_0\}.$$

Then $\dot{\Sigma}_0^*/\sim_{\mathcal{C}}$ is finite $\implies \dot{\Sigma}^*/\sim_{\mathcal{C}}$ is finite.

Proof. For every $a \in \dot{\Sigma}$, there is $b \in \dot{\Sigma}_0$ s.t. $a \sim_{\mathcal{C}} b$. By the congruence law of $\sim_{\mathcal{C}}$, for every $w \in \dot{\Sigma}^*$, there is some $v \in \dot{\Sigma}_0^*$ s.t. $w \sim_{\mathcal{C}} v$. Since $\dot{\Sigma}_0^*/\sim_{\mathcal{C}}$ is finite, this completes the proof. ◀

► **Example 13** (of Lem. 12). If $S = \{\circ^{(2)}, \mathbf{l}^{(0)}\}$ (so, $\dot{\Sigma} = \{(\mathbf{l} \circ _), ((\mathbf{l} \circ \mathbf{l}) \circ _), \dots (_ \circ \mathbf{l}), \dots\}$) and \mathcal{C} is the class of all monoids, we have: $\mathbf{T}_{(\mathbf{v}_0 \leq 0)}/\sim_{\mathcal{C}} = \{[\mathbf{l}]_{\sim_{\mathcal{C}}}\}$. Thus the set $\dot{\Sigma}_0 = \{(\mathbf{l} \circ _), (_ \circ \mathbf{l})\}$ is sufficient for considering the finiteness of $\dot{\Sigma}^*/\sim_{\mathcal{C}}$.

Thus, to prove the finiteness of $\mathbf{T}_{(\mathbf{v}_0 \leq k)}/\sim_{\mathcal{C}}$, it suffices to prove that both $\mathbf{T}_{(\mathbf{v}_0 \leq 0)}/\sim_{\mathcal{C}}$ and $\dot{\Sigma}_0^*/\sim_{\mathcal{C}}$ are finite:

► **Lemma 14.** If $\mathbf{T}_{(\mathbf{v}_0 \leq 0)}/\sim_{\mathcal{C}}$ and $\dot{\Sigma}_0^*/\sim_{\mathcal{C}}$ are finite, then for each $k \in \mathbb{N}$, the set $\mathbf{T}_{(\mathbf{v}_0 \leq k)}/\sim_{\mathcal{C}}$ is finite (hence, the equational theory of $\mathbf{T}_{(\mathbf{v}_0 \leq k)}$ over \mathcal{C} is decidable).

Proof. We have: $\mathbf{T}_{(\mathbf{v}_0 \leq 0)}/\sim_{\mathcal{C}}$ and $\dot{\Sigma}_0^*/\sim_{\mathcal{C}}$ are finite $\implies \dot{\Sigma}^*/\sim_{\mathcal{C}}$ is finite (by Lem. 12) $\implies \mathbf{T}_{(\mathbf{v}_0 \leq 1)}/\sim_{\mathcal{C}}$ is finite (by Lem. 11) $\implies \mathbf{T}_{(\mathbf{v}_0 \leq k)}/\sim_{\mathcal{C}}$ is finite (by Lem. 6). (The decidability is obtained from Prop. 3.) ◀

2.3 Finiteness from finding equations

For languages L, K , we write LK for the concatenation of L and K : $LK = \{wv \mid w \in L, v \in K\}$. For words w_i ($i \in I$), we write $\bigcup_{i \in I} w_i$ for the language $\{w_i \mid i \in I\}$.

For the finiteness of $\dot{\Sigma}_0^*/\sim_{\mathcal{C}}$, we consider finding equations $\{\langle w_i, v_i \rangle \mid i \in I\}$ and then applying the following:

► **Lemma 15.** *Let $\dot{\Sigma}_0 \subseteq \dot{\Sigma}$ be a finite set. Let $\langle \cdot \rangle \subseteq (\dot{\Sigma}_0^*)^2$ be a well-founded relation s.t.*

- $\langle \cdot \rangle$ satisfies the congruence law (i.e., $v < v' \implies wv' < wv$);
- $\langle \cdot \rangle$ has no infinite antichains.⁴

Then, the following are equivalent:

1. There is a finite set $\{\langle w_i, v_i \rangle \mid i \in I\} \subseteq \langle \cdot \rangle \cap (\sim_{\mathcal{C}})$ such that the language $\dot{\Sigma}_0^*(\bigcup_{i \in I} v_i)\dot{\Sigma}_0^*$ over the alphabet $\dot{\Sigma}_0$ is cofinite.⁵
2. $\dot{\Sigma}_0^*/\sim_{\mathcal{C}}$ is finite.

Proof. $1 \implies 2$: By induction on the well-founded relation $\langle \cdot \rangle$, we prove: For every $w \in \dot{\Sigma}_0^*$, there is some $v \in \dot{\Sigma}_0^* \setminus (\dot{\Sigma}_0^*(\bigcup_{i \in I} v_i)\dot{\Sigma}_0^*)$ such that $w \sim_{\mathcal{C}} v$. If $w \in \dot{\Sigma}_0^* \setminus (\dot{\Sigma}_0^*(\bigcup_{i \in I} v_i)\dot{\Sigma}_0^*)$, by letting $v = w$. Otherwise, since $w \in \dot{\Sigma}_0^*(\bigcup_{i \in I} v_i)\dot{\Sigma}_0^*$, there are $i \in I$ and $w', w'' \in \dot{\Sigma}_0^*$ such that $w = w'v_iw''$. By $w'w_iw'' < w'v_iw''$ (the congruence law of $\langle \cdot \rangle$) and IH, there is $v \in \dot{\Sigma}_0^* \setminus (\dot{\Sigma}_0^*(\bigcup_{i \in I} v_i)\dot{\Sigma}_0^*)$ s.t. $w'w_iw'' \sim_{\mathcal{C}} v$. We also have $w'v_iw'' \sim_{\mathcal{C}} w'w_iw''$ (by $v_i \sim_{\mathcal{C}} w_i$ with the congruence law of $\sim_{\mathcal{C}}$). Thus $w'v_iw'' \sim_{\mathcal{C}} v$ (by transitivity of $\sim_{\mathcal{C}}$).

$2 \implies 1$: Let $W \triangleq \bigcup_{X \in \dot{\Sigma}_0^*/\sim_{\mathcal{C}}} \{w \in X \mid w \text{ is minimal w.r.t. } \langle \cdot \rangle \cap X^2\}$. Let $\text{Subw}(W)$ be the subword closure of W (i.e., the minimal set $W' \supseteq W$ s.t. $w'ww'' \in W' \implies w \in W'$). Let $V \triangleq (\text{Subw}(W)\dot{\Sigma}_0) \setminus \text{Subw}(W)$. Then $(\dot{\Sigma}_0^*V\dot{\Sigma}_0^*) = \dot{\Sigma}_0^* \setminus \text{Subw}(W)$ holds, as follows. For \subseteq : Let $w \in \dot{\Sigma}_0^*$, $v \in V$, $w' \in \dot{\Sigma}_0^*$. If we assume $wv'w' \in \text{Subw}(W)$, then $v \in \text{Subw}(W)$, but this contradicts $v \in V$; thus, $wv'w' \notin \text{Subw}(W)$. For \supseteq : By induction on the length of $w \in \dot{\Sigma}_0^* \setminus \text{Subw}(W)$. If $w \in V$, clear. If $w \notin V$, let $w = w'a$ (note that $w \neq \varepsilon$, because $\varepsilon \in \text{Subw}(W)$ always by that W is not empty). Then, $w' \in \dot{\Sigma}_0^* \setminus \text{Subw}(W)$ (if not, since $w' \in \text{Subw}(W)$ and $w \notin \text{Subw}(W)$, $w \in V$, reaching a contradiction). Thus by IH, $w' \in \dot{\Sigma}_0^*V\dot{\Sigma}_0^*$; thus $w \in \dot{\Sigma}_0^*V\dot{\Sigma}_0^*$. Hence, we have $\dot{\Sigma}_0^* \setminus (\dot{\Sigma}_0^*V\dot{\Sigma}_0^*) = \text{Subw}(W)$. Now, the set W is finite because $\dot{\Sigma}_0^*/\sim_{\mathcal{C}}$ is finite and for each $X \in \dot{\Sigma}_0^*/\sim_{\mathcal{C}}$, the number of minimal elements w is finite (because $\langle \cdot \rangle$ has no infinite antichains); thus $\text{Subw}(W)$ is finite; thus V is finite. Let $V = \{v_1, \dots, v_n\}$. For every $i \in [1, n]$, there is $w_i \in W \cap [v_i]_{\sim_{\mathcal{C}}}$ s.t. $w_i < v_i$, because v_i is not minimal w.r.t. $\langle \cdot \rangle \cap [v_i]_{\sim_{\mathcal{C}}}^2$. Hence, $\{\langle w_i, v_i \rangle \mid i \in [1, n]\}$ is the desired set. ◀

The *shortlex order* (aka *length-lexicographical order*) is an example of $\langle \cdot \rangle$ in Lem. 15 (because it is a well-ordering [4, Def. 2.2.3] and it satisfies the congruence law).

► **Example 16** (toy example of Lem. 15 ($1 \implies 2$)). Let $S = \{\circ^{(2)}, \text{I}^{(0)}, -^{(1)}\}$ and \mathcal{C} be the class of groups. Let $\dot{\Sigma}_0 = \{\text{I} \circ _, (_ \circ \text{I}), _-\}$. Then, we have the following 3 equations:

$$\varepsilon \sim_{\mathcal{C}} (\text{I} \circ _) \qquad \varepsilon \sim_{\mathcal{C}} (_ \circ \text{I}) \qquad \varepsilon \sim_{\mathcal{C}} (_ -)(_-).$$

(This is because $a \sim_{\mathcal{C}} (\text{I} \circ a)$, $a \sim_{\mathcal{C}} (a \circ \text{I})$, and $a \sim_{\mathcal{C}} (a^-)^-$ hold, respectively.) Then, the language $(\dot{\Sigma}_0^*(\bigcup_{i \in I} v_i)\dot{\Sigma}_0^*) = \dot{\Sigma}_0^* \setminus \{\varepsilon, _-\}$ is cofinite. Thus, from the 3 equations, we have that $\dot{\Sigma}_0^*/\sim_{\mathcal{C}}$ is finite. (Use a shortlex order as $\langle \cdot \rangle$ in Lem. 15.)

⁴ This assumption is used only in the direction of $2 \implies 1$.

⁵ A language L over an alphabet A is cofinite if its complemented language $A^* \setminus L$ is finite.

While it is undecidable whether a given (finitely presented) monoid $\dot{\Sigma}_0^*/\sim_{\mathcal{C}}$ is finite [17] (see also [4, Thm. 7.3.7 with Def. 7.3.2(b)]) in general (cf. 2 of Lem. 15), it is decidable (in linear time) whether the language of a given regular expression of the form $\dot{\Sigma}_0^*(\bigcup_{i \in I} v_i)\dot{\Sigma}_0^*$ is cofinite (cf. 1 of Lem. 15):

► **Proposition 17.** *The following is decidable in linear time (more precisely, $\mathcal{O}(n)$ time on a random-access machine for n the number of symbols in the given regular expression): Given a regular expression of the form $\dot{\Sigma}_0^*(\bigcup_{i \in I} v_i)\dot{\Sigma}_0^*$ over the alphabet $\dot{\Sigma}_0$, is its language cofinite?*

Proof Sketch. By the Aho-Corasick algorithm, we can construct a deterministic finite automaton (DFA) from a given regular expression of the form $\dot{\Sigma}_0^*(\bigcup_{i \in I} v_i)\dot{\Sigma}_0^*$ in linear time [1, Sect. 8]. By taking the complemented language of the DFA, it suffices to show that the following problem is in $\mathcal{O}(n)$ time: given a DFA with n states, is its language finite? Then we can give the following algorithm: From the graph induced by the DFA, remove all the states not reachable from the starting state and remove all the states not reachable to any accepting states by using the depth-first search; check whether there exists some cycle in the graph by the depth-first search. ◀

Thus, thanks to $1 \implies 2$ of Lem. 15, we can focus on finding a finite set of equations. While it is undecidable in general whether there exists such a set, we give a possibly non-terminating pseudo-code in Algorithm 1, which can help to find equations (e.g., Fig. 1).⁶

■ **Algorithm 1** Possibly non-terminating pseudo-code for ensuring the finiteness of $\dot{\Sigma}_0^*/\sim_{\mathcal{C}}$.

Require: Given $S, \Sigma, \sim_{\mathcal{C}}, \dot{\Sigma}_0$. ▷ $\dot{\Sigma}_0$ is a finite alphabet of Lem. 12.

Ensure: Is $\dot{\Sigma}_0^*/\sim_{\mathcal{C}}$ finite?

- 1: $\Gamma \leftarrow \emptyset$ ▷ We use I, w_i, v_i to denote $\Gamma = \{\langle w_i, v_i \rangle \mid i \in I\}$.
- 2: **while** the language $\dot{\Sigma}_0^*(\bigcup_{i \in I} v_i)\dot{\Sigma}_0^*$ over $\dot{\Sigma}_0$ is not cofinite **do**
- 3: $\langle w, v \rangle \leftarrow$ a fresh pair in $\dot{\Sigma}_0^* \times \dot{\Sigma}_0^*$ ▷ $<$ is a binary relation in (Lem. 15).
- 4: **if** $w < v$ and $w \sim_{\mathcal{C}} v$ **then** $\Gamma \leftarrow \Gamma \cup \{\langle w, v \rangle\}$
- 5: **end if**
- 6: **end while**
- 7: **return** True

► **Remark 18.** If “given w, v , does $w \sim_{\mathcal{C}} v$ hold” is decidable, then Algorithm 1 is a semi-algorithm (i.e., if $\dot{\Sigma}_0^*/\sim_{\mathcal{C}}$ is finite, it is terminated and returns True; otherwise, not terminated). This is because $2 \implies 1$ of Lem. 15 also holds.

3 The calculus of relations with bounded dot-dagger alternation

In the remaining part of this paper, as a case study of the k -variable-occurrence fragment presented in Sect. 2, we consider the calculus of relations with bounded dot-dagger alternation. In this section, we recall the definitions of the calculus of relations (CoR) and the dot-dagger alternation hierarchy.

⁶ Usually, to calculate $\sim_{\mathcal{C}}$ is a bottleneck. For relaxing this problem, for example, hashing words by using some algebras in \mathcal{C} is practically useful for reducing the number of $\sim_{\mathcal{C}}$ calls (since if the hash of two words w, v are different, then we immediately have that w, v are not equivalent w.r.t. $\sim_{\mathcal{C}}$).

3.1 CoR: syntax and semantics

We fix Σ as a non-empty *finite* set of variables. Consider the finite algebraic signature $S^{\text{CoR}} \triangleq \{\perp^{(0)}, \top^{(0)}, -^{(1)}, \cup^{(2)}, \cap^{(2)}, \text{I}^{(0)}, \text{D}^{(0)}, \cdot^{(2)}, \dagger^{(2)}\} \cup \{\pi^{(1)} \mid \pi \text{ is a map in } [1, 2]^{[1, 2]}\}$ (we consider algebras of binary relations and each π is used for a projection of binary relations). The set \mathbf{T}^{CoR} of CoR terms is defined as follows:

$$\mathbf{T}^{\text{CoR}} \ni t, s, u ::= a \mid \perp \mid \top \mid s \cup u \mid s \cap u \mid s^- \\ \mid \text{I} \mid \text{D} \mid s \cdot u \mid s \dagger u \mid s^\pi \quad (a \in \Sigma, \pi \in [1, 2]^{[1, 2]}).$$

Additionally, for a term t , we use t^\sim to denote the term $t^\sim \triangleq t^{\{1 \rightarrow 2, 2 \rightarrow 1\}}$. Here, we use the infix notation for binary operators, the superscript notation for unary operators, and parenthesis in ambiguous situations, as usual.

For binary relations R, S on a set W , the *identity relation* I_W on W , the *difference relation* D_W on W , the *(relational) composition (relative product)* $R \cdot S$, the *dagger (relative sum)* $R \dagger S$, and the *projection* R^π are defined by:

$$\begin{aligned} \text{I}_W &\triangleq \{\langle x, y \rangle \in W^2 \mid x = y\} && \text{(identity)} \\ \text{D}_W &\triangleq \{\langle x, y \rangle \in W^2 \mid x \neq y\} && \text{(difference)} \\ R \cdot S &\triangleq \{\langle x, y \rangle \in W^2 \mid \exists z \in W, \langle x, z \rangle \in R \wedge \langle z, y \rangle \in S\} && \text{(relative product)} \\ R \dagger S &\triangleq \{\langle x, y \rangle \in W^2 \mid \forall z \in W, \langle x, z \rangle \in R \vee \langle z, y \rangle \in S\} && \text{(relative sum)} \\ R^\pi &\triangleq \{\langle x_1, x_2 \rangle \in W^2 \mid \langle x_{\pi(1)}, x_{\pi(2)} \rangle \in R\} && \text{(projection)}. \end{aligned}$$

A structure \mathfrak{A} is a tuple $\langle |\mathfrak{A}|, \{a^\mathfrak{A}\}_{a \in \Sigma} \rangle$, where $|\mathfrak{A}|$ is a non-empty set and $a^\mathfrak{A} \subseteq |\mathfrak{A}|^2$ is a binary relation for each $a \in \Sigma$. For a structure \mathfrak{A} , the *binary relation map* $\llbracket _ \rrbracket_{\mathfrak{A}} : \mathbf{T}^{\text{CoR}} \rightarrow \wp(|\mathfrak{A}|^2)$ is the unique homomorphism extending $\llbracket a \rrbracket_{\mathfrak{A}} = a^\mathfrak{A}$ w.r.t. the set-theoretic operators and the aforementioned binary relation operators; i.e., $\llbracket t \rrbracket_{\mathfrak{A}}$ is defined as follows:

$$\begin{aligned} \llbracket a \rrbracket_{\mathfrak{A}} &\triangleq a^\mathfrak{A} \quad (a \in \Sigma) & \llbracket \perp \rrbracket_{\mathfrak{A}} &\triangleq \emptyset & \llbracket \top \rrbracket_{\mathfrak{A}} &\triangleq |\mathfrak{A}|^2 & \llbracket \text{I} \rrbracket_{\mathfrak{A}} &\triangleq \text{I}_{|\mathfrak{A}|} & \llbracket \text{D} \rrbracket_{\mathfrak{A}} &\triangleq \text{D}_{|\mathfrak{A}|} \\ \llbracket t \cup s \rrbracket_{\mathfrak{A}} &\triangleq \llbracket t \rrbracket_{\mathfrak{A}} \cup \llbracket s \rrbracket_{\mathfrak{A}} & \llbracket t \cap s \rrbracket_{\mathfrak{A}} &\triangleq \llbracket t \rrbracket_{\mathfrak{A}} \cap \llbracket s \rrbracket_{\mathfrak{A}} & \llbracket t^- \rrbracket_{\mathfrak{A}} &\triangleq |\mathfrak{A}|^2 \setminus \llbracket t \rrbracket_{\mathfrak{A}} \\ \llbracket t \cdot s \rrbracket_{\mathfrak{A}} &\triangleq \llbracket t \rrbracket_{\mathfrak{A}} \cdot \llbracket s \rrbracket_{\mathfrak{A}} & \llbracket t \dagger s \rrbracket_{\mathfrak{A}} &\triangleq \llbracket t \rrbracket_{\mathfrak{A}} \dagger \llbracket s \rrbracket_{\mathfrak{A}} & \llbracket t^\pi \rrbracket_{\mathfrak{A}} &\triangleq \llbracket t \rrbracket_{\mathfrak{A}}^\pi. \end{aligned}$$

It is well-known that w.r.t. binary relations, CoR has the same expressive power as the three-variable fragment of first-order logic with equality:⁷

► **Proposition 19** ([28, 29, 11]). *W.r.t. binary relations, the expressive power of \mathbf{T}^{CoR} is equivalent to that of the three-variable fragment of first-order logic with equality.*

Let REL be the class of all structures. Let $\text{REL}_{\geq m}$ (resp. $\text{REL}_{\leq m}$) be the class of structures \mathfrak{A} of $\#|\mathfrak{A}| \geq m$ (resp. $\#|\mathfrak{A}| \leq m$). For $\mathcal{C} \subseteq \text{REL}$, the equivalence relation $\sim_{\mathcal{C}}$ on \mathbf{T}^{CoR} is defined by: $t \sim_{\mathcal{C}} s \iff \llbracket t \rrbracket_{\mathfrak{A}} = \llbracket s \rrbracket_{\mathfrak{A}}$ for every $\mathfrak{A} \in \mathcal{C}$. For $T \subseteq \mathbf{T}^{\text{CoR}}$, the equational theory of T over \mathcal{C} is the set $\{\langle t, s \rangle \in T^2 \mid t \sim_{\mathcal{C}} s\}$. We mainly consider \sim_{REL} : the equational theory over REL . The following are some instances w.r.t. \sim_{REL} :

$$\begin{aligned} a \cdot (b \cdot c) &\sim_{\text{REL}} (a \cdot b) \cdot c & a \cap (b \cup c) &\sim_{\text{REL}} (a \cap b) \cup (a \cap c) & (a^\sim)^\sim &\sim_{\text{REL}} a \\ a \cdot a^\sim &\not\sim_{\text{REL}} a^\sim \cdot a & a \cdot (b \dagger c) &\not\sim_{\text{REL}} (a \cdot b) \dagger c & a^{\{1 \rightarrow 1, 2 \rightarrow 1\}} &\sim_{\text{REL}} (a \cap \text{I}) \cdot \top. \end{aligned}$$

⁷ Namely, for every formula φ with two distinct free variables z_1, z_2 in the three-variable fragment of first-order logic with equality, there is $t \in \mathbf{T}^{\text{CoR}}$ such that for all \mathfrak{A} , $\llbracket \lambda z_1 z_2. \varphi \rrbracket_{\mathfrak{A}} = \llbracket t \rrbracket_{\mathfrak{A}}$. Conversely, for every $t \in \mathbf{T}^{\text{CoR}}$, there is φ such that for all \mathfrak{A} , $\llbracket t \rrbracket_{\mathfrak{A}} = \llbracket \lambda z_1 z_2. \varphi \rrbracket_{\mathfrak{A}}$. Here, $\llbracket \lambda z_1 z_2. \varphi \rrbracket_{\mathfrak{A}} \triangleq \{\langle x, y \rangle \in |\mathfrak{A}|^2 \mid \varphi \text{ is true on } \mathfrak{A} \text{ if } z_1, z_2 \text{ are mapped to } x, y, \text{ respectively}\}$.

The following propositions hold because for each $m \in \mathbb{N}$, the number of structures \mathfrak{A} of $\#|\mathfrak{A}| \leq m$ is finite up to isomorphism and each structure is finite.

► **Proposition 20.** *For each $m \in \mathbb{N}$, $\mathbf{T}^{\text{CoR}}/\sim_{\text{REL}_{\leq m}}$ is finite.*

► **Proposition 21.** *Let $T \subseteq \mathbf{T}^{\text{CoR}}$ be a subterm-closed set and $m \geq 1$. Then, T/\sim_{REL} is finite $\iff T/\sim_{\text{REL}_{\geq m}}$ is finite. Additionally, the equational theory of T over REL is decidable \iff the equational theory of T over $\text{REL}_{\geq m}$ is decidable.*

Proof. Because $t \sim_{\text{REL}} s \iff t \sim_{\text{REL}_{\leq m-1}} s \wedge t \sim_{\text{REL}_{\geq m}} s$. By Prop. 20 with Prop. 3, $T/\sim_{\text{REL}_{\leq m-1}}$ is finite and the equational theory of T over $\text{REL}_{\leq m-1}$ is decidable. ◀

3.2 The dot-dagger alternation hierarchy

► **Definition 22** (the dot-dagger alternation hierarchy [21]). *The sets, $\{\Sigma_n^{\text{CoR}}, \Pi_n^{\text{CoR}}\}_{n \in \mathbb{N}}$, are the minimal sets satisfying the following:*

- $\Sigma_0^{\text{CoR}} = \Pi_0^{\text{CoR}} = \{t \in \mathbf{T}^{\text{CoR}} \mid t \text{ does not contain } \cdot \text{ nor } \dagger\}$;
- For $n \geq 0$, $\Sigma_n^{\text{CoR}} \cup \Pi_n^{\text{CoR}} \subseteq \Sigma_{n+1}^{\text{CoR}} \cap \Pi_{n+1}^{\text{CoR}}$;
- For $n \geq 1$, if $s, u \in \Sigma_n^{\text{CoR}}$, then $s \cup u, s \cap u, s \cdot u, s^\pi \in \Sigma_n^{\text{CoR}}$ and $s \dagger u \in \Pi_{n+1}^{\text{CoR}}$;
- For $n \geq 1$, if $s, u \in \Pi_n^{\text{CoR}}$, then $s \cup u, s \cap u, s \dagger u, s^\pi \in \Pi_n^{\text{CoR}}$ and $s \cdot u \in \Sigma_{n+1}^{\text{CoR}}$.

For example, $a \cdot b \in \Sigma_1^{\text{CoR}}$ and $a \cdot (b \dagger c) \in \Sigma_2^{\text{CoR}}$ (the term $a \cdot b$ means that for some z , $a(x, z)$ and $b(z, y)$). The term $a \cdot (b \dagger c)$ means that for some z , for every w , $a(x, z)$ and $(b(z, w)$ or $c(w, y))$. Here, x and y indicate the source and the target, respectively, and each $a'(x', y')$ denotes that there is an a' -labelled edge from x' to y'). The dot-dagger alternation hierarchy is an analogy of the quantifier alternation hierarchy in first-order logic (by viewing \cdot as \exists and \dagger as \forall). This provides a fine-grained analogy of Prop. 19 w.r.t. the number of quantifier alternations, as follows:

► **Proposition 23** ([21, Cor. 3.14]; cf. Prop. 19). *W.r.t. binary relations, the expressive power of Σ_n^{CoR} (resp. Π_n^{CoR}) is equivalent to that of the level Σ_n (resp. Π_n) in the quantifier alternation hierarchy of the three-variable fragment of first-order logic with equality.*

Because there are recursive translations for Prop. 23 [21], the following (un-)decidability results follow from those in first-order logic.

► **Proposition 24.** *The equational theory of Σ_n^{CoR} (resp. Π_n^{CoR}) is decidable if $n \leq 1$ and is undecidable if $n \geq 2$.*

Proof Sketch. When Σ is a countably infinite set, they follow from the BSR class $[\exists^* \forall^*, \text{all}] =$ [3, 27] and the reduction class $[\forall \exists \wedge \forall^3, (\omega, 1)]$ [7, Cor. 3.1.19]. We can strengthen this result even if $\#\Sigma = 1$ by using a variant of the translation in [19, Lem. 11] for encoding countably infinitely many variables by one variable. (See [23] for more details.) ◀

4 On the k -variable-occurrence fragment of Σ_n^{CoR}

We now consider $(\Sigma_n^{\text{CoR}})_{(\text{vo} \leq k)}$: the k -variable-occurrence fragment of the level Σ_n^{CoR} in the dot-dagger alternation hierarchy. Clearly, $\Sigma_n^{\text{CoR}} = \bigcup_{k \in \mathbb{N}} (\Sigma_n^{\text{CoR}})_{(\text{vo} \leq k)}$. While the equational theory of Σ_n^{CoR} is undecidable in general (Prop. 24), we show that the equational theory of $(\Sigma_n^{\text{CoR}})_{(\text{vo} \leq k)}$ is decidable (Cor. 26). Our goal in this section is to show the following:

► **Theorem 25.** For each $n, k \in \mathbb{N}$, $(\Sigma_n^{\text{CoR}})_{(\text{vo} \leq k)} / \sim_{\text{REL}}$ is finite.

Combining with Prop. 3 yields the following decidability and complexity upper bound. The complexity lower bound is because the equational theory can encode the *boolean sentence value problem* [5] (even if $n = k = 0$), as a given boolean sentence φ is true iff $t \sim_{\text{REL}} \top$, where t is the term obtained from φ by replacing $\wedge, \vee, \top, \text{F}$ with \cap, \cup, \top, \perp , respectively.

► **Corollary 26.** For $n, k \in \mathbb{N}$, the equational theory of $(\Sigma_n^{\text{CoR}})_{(\text{vo} \leq k)}$ over REL is decidable. Moreover, it is complete for DLOGTIME-uniform NC^1 under DLOGTIME reductions if the input is given as a well-bracketed string.

To prove Thm. 25, we consider the finiteness of $\mathbf{T}_{(\text{vo} \leq 0)}^{\text{CoR}} / \sim_{\text{REL}}$ in Sect. 4.1 and the finiteness of a monoid for $(\Sigma_n^{\text{CoR}})_{(\text{vo} \leq k)} / \sim_{\text{REL}}$ in Sect. 4.2, respectively (cf. Lem. 14).

4.1 On the finiteness of $\mathbf{T}_{(\text{vo} \leq 0)}^{\text{CoR}}$

For the finiteness of $\mathbf{T}_{(\text{vo} \leq 0)}^{\text{CoR}} / \sim_{\text{REL}}$, by Prop. 21, it suffices to show the following:

► **Lemma 27.** $\mathbf{T}_{(\text{vo} \leq 0)}^{\text{CoR}} / \sim_{\text{REL}_{\geq 3}} = \{[\perp]_{\sim_{\text{REL}_{\geq 3}}}, [\top]_{\sim_{\text{REL}_{\geq 3}}}, [I]_{\sim_{\text{REL}_{\geq 3}}}, [D]_{\sim_{\text{REL}_{\geq 3}}}\}$.

Proof. W.r.t. $\sim_{\text{REL}_{\geq 3}}$, we prove that the four elements are closed under each operator. For the operators $\cap, -, \cdot, \smile$, this is shown by the following Cayley tables:

\cap	\top	\perp	I	D
\top	\top	\perp	I	D
\perp	\perp	\perp	\perp	\perp
I	I	\perp	I	\perp
D	D	\perp	\perp	D

$-$	\diagdown
\top	\perp
\perp	\top
I	D
D	I

\cdot	\top	\perp	I	D
\top	\top	\perp	\top	\top
\perp	\perp	\perp	\perp	\perp
I	\top	\perp	I	D
D	\top	\perp	D	\top

\smile	\diagdown
\top	\top
\perp	\perp
I	I
D	D

Note that $D \cdot D \sim_{\text{REL}_{\geq 3}} \top$ holds thanks to “ ≥ 3 ”. When $\#(|\mathfrak{A}|) \geq 3$, we have: $\langle x, y \rangle \in \llbracket D \cdot D \rrbracket_{\mathfrak{A}}$ iff $(\exists z \in |\mathfrak{A}|, z \neq x \wedge z \neq y) \text{ iff } |\mathfrak{A}| \setminus \{x, y\} \neq \emptyset \text{ iff True}$ (cf. Remark 28). (Similarly for $D \cdot \top \sim_{\text{REL}_{\geq 2}} \top$.) For the other operators (\cup, \dagger, π), they can be expressed by using $\cap, -, \cdot, \smile$ as follows: $t \cup s \sim_{\text{REL}} (t^- \cap s^-)^-$, $t \dagger s \sim_{\text{REL}} (t^- \cdot s^-)^-$, $t^{\{1 \rightarrow 1, 2 \rightarrow 2\}} \sim_{\text{REL}} t$, $t^{\{1 \rightarrow 1, 2 \rightarrow 1\}} \sim_{\text{REL}} (t \cap I) \cdot \top$, $t^{\{1 \rightarrow 2, 2 \rightarrow 2\}} \sim_{\text{REL}} \top \cdot (t \cap I)$, and $t^{\{1 \rightarrow 2, 2 \rightarrow 1\}} = t^\smile$. Hence, this completes the proof. ◀

► **Remark 28.** $D \cdot D \not\sim_{\text{REL}} \top$, whereas $D \cdot D \sim_{\text{REL}_{\geq 3}} \top$. For example when $\#|\mathfrak{A}| = 1$, since $\llbracket D \rrbracket_{\mathfrak{A}} = \llbracket \perp \rrbracket_{\mathfrak{A}}$, we have $\llbracket D \cdot D \rrbracket_{\mathfrak{A}} = \emptyset \neq |\mathfrak{A}| = \llbracket \top \rrbracket_{\mathfrak{A}}$. ($D \cdot D$ is not equivalent to neither one of the four constants w.r.t. \sim_{REL} ; thus, there are many constants w.r.t. \sim_{REL} .)

4.2 Monoid for $(\Sigma_n^{\text{CoR}})_{(\text{vo} \leq k)}$

Next, we decompose terms, and then we reduce the finiteness of $(\Sigma_n^{\text{CoR}})_{(\text{vo} \leq k)} / \sim_{\text{REL}}$ to that of a monoid (cf. Sect. 2.2).

► **Lemma 29.** For each $n, k \in \mathbb{N}$, if $(\Sigma_n^{\text{CoR}})_{(\text{vo} \leq 1)} / \sim_{\text{REL}}$ is finite, $(\Sigma_n^{\text{CoR}})_{(\text{vo} \leq k)} / \sim_{\text{REL}}$ is finite.

Proof. By specializing T with $(\Sigma_n^{\text{CoR}})_{(\text{vo} \leq k)}$ and \mathcal{C} with REL, in Prop. 3 and Lem. 6. ◀

► **Lemma 30.** For each $n, k \in \mathbb{N}$, $(\Sigma_n^{\text{CoR}})_{(\text{vo} \leq k)} / \sim_{\text{REL}}$ is finite iff $(\Pi_n^{\text{CoR}})_{(\text{vo} \leq k)} / \sim_{\text{REL}}$ is finite.

Proof. \Leftarrow : For every term t in $(\Pi_n^{\text{CoR}})_{(\text{vo} \leq k)}$, there is some s in $(\Sigma_n^{\text{CoR}})_{(\text{vo} \leq k)}$ such that $t \sim_{\text{REL}} s^-$. Such s can be obtained from the term t^- by taking the complement normal form using the following equations:

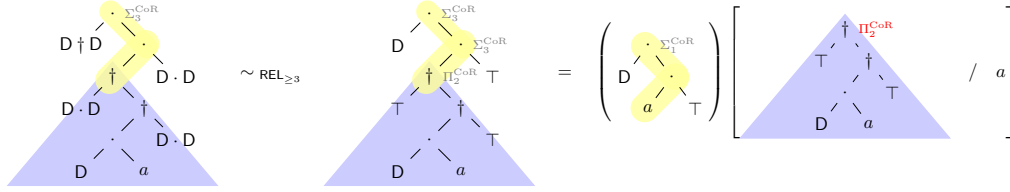
$$\begin{array}{ccccccc} \top^- \sim_{\text{REL}} \perp & & \perp^- \sim_{\text{REL}} \top & & \top^- \sim_{\text{REL}} \text{D} & & \text{D}^- \sim_{\text{REL}} \top \\ (s \cup u)^- \sim_{\text{REL}} s^- \cap u^- & & (s \cap u)^- \sim_{\text{REL}} s^- \cup u^- & & (s^-)^- \sim_{\text{REL}} s & & (s^\pi)^- \sim_{\text{REL}} (s^-)^\pi. \end{array}$$

\Rightarrow : As with \Leftarrow . ◀

► **Lemma 31** (cf. Lem. 4). Let $a \in \Sigma$. For all $n \geq 2$, $t \in (\Sigma_n^{\text{CoR}})_{(\text{vo} \leq 1)}$, there are $t_0 \in (\Sigma_1^{\text{CoR}})_{(\text{vo} \leq 1)}$ and $t_1 \in (\Pi_{n-1}^{\text{CoR}})_{(\text{vo} \leq 1)}$ such that $t \sim_{\text{REL}_{\geq 3}} t_0[t_1/a]$.

Proof. By induction on the pair of n and t . We distinguish the following cases. Case $t \in (\Sigma_{n-1}^{\text{CoR}})_{(\text{vo} \leq 1)}$: Clear, by IH ($\cdot: \Pi_{n-2}^{\text{CoR}} \subseteq \Pi_{n-1}^{\text{CoR}}$). Case $t \in (\Pi_{n-1}^{\text{CoR}})_{(\text{vo} \leq 1)}$: By letting $t_0 = a$ and $t_1 = t$. Case $t = s \cup u$: By $\text{vo}(t) = 1$, $\text{vo}(s) = 0$ or $\text{vo}(u) = 0$ holds. Sub-case $\text{vo}(s) = 0$: By Lem. 27, let $s' \in \{\perp, \top, \text{I}, \text{D}\}$ be s.t. $s \sim_{\text{REL}_{\geq 3}} s'$. By IH w.r.t. u , let $u_0 \in (\Sigma_1^{\text{CoR}})_{(\text{vo} \leq 1)}$, $u_1 \in (\Pi_{n-1}^{\text{CoR}})_{(\text{vo} \leq 1)}$ be s.t. $u \sim_{\text{REL}_{\geq 3}} u_0[u_1/a]$. By letting $t_0 = s' \cup u_0$ and $t_1 = u_1$, we have $t \sim_{\text{REL}_{\geq 3}} t_0[t_1/a]$. Sub-case $\text{vo}(u) = 0$: As with Sub-case $\text{vo}(s) = 0$. Case $t = s \cap u, s \cdot u, s^\pi$: As with Case $t = s \cup u$. ◀

The following is an illustrative example of the decomposition of Lem. 31:



► **Lemma 32.** For each $n \in \mathbb{N}$, if $(\Sigma_1^{\text{CoR}})_{(\text{vo} \leq 1)} / \sim_{\text{REL}}$ is finite, $(\Sigma_n^{\text{CoR}})_{(\text{vo} \leq 1)} / \sim_{\text{REL}}$ is finite.

Proof. By induction on n . Case $n \leq 1$: By the assumption (note that $(\Sigma_0^{\text{CoR}})_{(\text{vo} \leq 1)} \subseteq (\Sigma_1^{\text{CoR}})_{(\text{vo} \leq 1)}$). Case $n \geq 2$: By the assumption, $(\Sigma_1^{\text{CoR}})_{(\text{vo} \leq 1)} / \sim_{\text{REL}}$ is finite. By IH with Lem. 30, $(\Pi_{n-1}^{\text{CoR}})_{(\text{vo} \leq 1)} / \sim_{\text{REL}}$ is finite. Combining them with Lem. 31 (and Prop. 21 for changing \sim_{REL} and $\sim_{\text{REL}_{\geq 3}}$ mutually) yields that $(\Sigma_n^{\text{CoR}})_{(\text{vo} \leq 1)} / \sim_{\text{REL}}$ is finite. ◀

For $S \subseteq S^{\text{CoR}}$, let $\mathbf{T}^S \subseteq \mathbf{T}^{\text{CoR}}$ be the set of all terms over the signature S . Then we have:

► **Lemma 33.** If $\mathbf{T}_{(\text{vo} \leq 1)}^{\{\cap, \cdot, \text{I}, \text{D}\}} / \sim_{\text{REL}}$ is finite, then $(\Sigma_1^{\text{CoR}})_{(\text{vo} \leq 1)} / \sim_{\text{REL}}$ is finite.

Proof sketch. Note that $t, s \in \Sigma_1^{\text{CoR}} ::= u \mid t \cup s \mid t \cap s \mid t \cdot s \mid t^\pi$ (where $u \in \Sigma_0^{\text{CoR}}$, $\pi \in [1, 2]^{[1, 2]}$) and $u, u' \in \Sigma_0^{\text{CoR}} ::= a \mid u \cup u' \mid u \cap u' \mid u^- \mid \top \mid \perp \mid u^\pi$ (where $a \in \Sigma$, $\pi \in [1, 2]^{[1, 2]}$). By taking the complement ($-$) and projection (π) normal form and replacing \perp with $\text{I} \cap \text{D}$ and \top with $\text{I} \cup \text{D}$, for each $t \in (\Sigma_1^{\text{CoR}})_{(\text{vo} \leq 1)}$ and $a \in \Sigma$, there are $t_0 \in \mathbf{T}_{(\text{vo} \leq 1)}^{\{\cup, \cap, \cdot, \text{I}, \text{D}\}}$ and $t_1 \in \mathbf{T}_{(\text{vo} \leq 1)}^{\{-\} \cup \{\pi \mid \pi \in [1, 2]^{[1, 2]}\}}$ such that $t \sim_{\text{REL}} t_0[t_1/a]$. Moreover, by the distributive law of \cup w.r.t. \cdot and \cap , for each $t \in \mathbf{T}_{(\text{vo} \leq 1)}^{\{\cup, \cap, \cdot, \text{I}, \text{D}\}}$, there are $n \in \mathbb{N}$ and $t_1, \dots, t_n \in \mathbf{T}_{(\text{vo} \leq 1)}^{\{\cap, \cdot, \text{I}, \text{D}\}}$ such that $t \sim_{\text{REL}} t_1 \cup \dots \cup t_n$. Because $\mathbf{T}_{(\text{vo} \leq 1)}^{\{\cap, \cdot, \text{I}, \text{D}\}} / \sim_{\text{REL}}$ is finite (by the assumption) and $\mathbf{T}_{(\text{vo} \leq 1)}^{\{-\} \cup \{\pi \mid \pi \in [1, 2]^{[1, 2]}\}} / \sim_{\text{REL}}$ is clearly finite, $(\Sigma_1^{\text{CoR}})_{(\text{vo} \leq 1)} / \sim_{\text{REL}}$ is finite. Hence, this completes the proof. (See [23] for more details of the proof.) ◀

69:12 On the Finite Variable-Occurrence Fragment of the Calculus of Relations

Combining Lems. 29, 32, and 33 yields that to prove that $(\Sigma_n^{\text{CoR}})_{(\text{vo} \leq k)} / \sim_{\text{REL}}$ is finite, it suffices to prove that $\mathbf{T}_{(\text{vo} \leq 1)}^{\{\cap, \cdot, \text{I}, \text{D}\}} / \sim_{\text{REL}}$ is finite.

Let $\dot{\Sigma}$ be the set of characters of Def. 7 from the signature $\{\cap^{(2)}, \cdot^{(2)}, \text{I}^{(0)}, \text{D}^{(0)}, \smile^{(1)}\}$. That is, $\dot{\Sigma} \triangleq \{(_ \cap t), (t \cap _), (_ \cdot t), (t \cdot _) \mid t \in (\mathbf{T}_{(\text{vo} \leq 0)}^{\{\cap, \cdot, \text{I}, \text{D}, \smile\}}) \cup \{\smile\}$. (While \smile does not occur in $\mathbf{T}_{(\text{vo} \leq 1)}^{\{\cap, \cdot, \text{I}, \text{D}\}}$, we introduce \smile for replacing the primitive character $(\text{D} \cdot _)$ with $\smile (_ \cdot \text{D})$. This is not essential but is useful for reducing the number of equations and for simplifying the notation (Def. 35).) Let $\dot{\sim}_{\text{REL}_{\geq 5}}$ be the equivalence relation on $\dot{\Sigma}^*$ defined by: $w \dot{\sim}_{\text{REL}_{\geq 5}} v \iff w[a] \sim_{\text{REL}_{\geq 5}} v[a]$ where $a \in \Sigma$ is any variable (recall Def. 10).⁸

► **Lemma 34.** *If $\dot{\Sigma}^* / \dot{\sim}_{\text{REL}_{\geq 5}}$ is finite, then $\mathbf{T}_{(\text{vo} \leq 1)}^{\{\cap, \cdot, \text{I}, \text{D}\}} / \sim_{\text{REL}}$ is finite.*

Proof. Since $\dot{\Sigma}^* / \dot{\sim}_{\text{REL}_{\geq 5}}$ is finite, we have that $\mathbf{T}_{(\text{vo} \leq 1)}^{\{\cap, \cdot, \text{I}, \text{D}, \smile\}} / \sim_{\text{REL}_{\geq 5}}$ is finite (Lem. 11); thus, $\mathbf{T}_{(\text{vo} \leq 1)}^{\{\cap, \cdot, \text{I}, \text{D}\}} / \sim_{\text{REL}_{\geq 5}}$ is finite. Hence by Prop. 21, this completes the proof. ◀

We consider the following finite subset $\dot{\Sigma}_0$ of $\dot{\Sigma}$ (cf. Lem. 12):

► **Definition 35.** *Let $\dot{\Sigma}_0 \subseteq \dot{\Sigma}$ be the finite set $\{\cap_{\text{I}}, \cap_{\text{D}}, \cdot_{\text{D}}, \smile\}$, where $\cap_{\text{I}}, \cap_{\text{D}}, \cdot_{\text{D}}$ are abbreviations of $(_ \cap \text{I}), (_ \cap \text{D}), (_ \cdot \text{D})$, respectively.*

► **Lemma 36.** *If $\dot{\Sigma}_0^* / \dot{\sim}_{\text{REL}_{\geq 5}}$ is finite, then $\dot{\Sigma}^* / \dot{\sim}_{\text{REL}_{\geq 5}}$ is finite.*

Proof. It suffices to prove the following: for every $a \in \dot{\Sigma}$, there is $w \in \dot{\Sigma}_0^*$ such that $a \dot{\sim}_{\text{REL}_{\geq 5}} w$. Case $a = (_ \cap t), (_ \cdot t)$: Since $\text{vo}(t) = 0$, by using Lem. 27, they are shown by distinguishing the following four sub-cases, as follows:

	$t \sim_{\text{REL}_{\geq 3}} \perp$	$t \sim_{\text{REL}_{\geq 3}} \top$	$t \sim_{\text{REL}_{\geq 3}} \text{I}$	$t \sim_{\text{REL}_{\geq 3}} \text{D}$
$a = (_ \cap t)$	$\cap_{\text{I}} \cap_{\text{D}}$	ε	\cap_{I}	\cap_{D}
$a = (_ \cdot t)$	$\cap_{\text{I}} \cap_{\text{D}}$	$\cdot_{\text{D}} \cdot_{\text{D}}$	ε	\cdot_{D}

Case $a = (t \cap _), (t \cdot _)$: By $(t \cap _) = \smile (_ \cap t)$ and applying the above case analysis for $(_ \cap t)$, this case can be proved (similarly for $(t \cdot _)$). Case $a = \smile$: Since $\smile \in \dot{\Sigma}_0$. ◀

Thus, our goal is to prove that $\dot{\Sigma}_0^* / \dot{\sim}_{\text{REL}_{\geq 5}}$ is finite.

4.3 On the finiteness of the monoid

For the finiteness of $\dot{\Sigma}_0^* / \dot{\sim}_{\text{REL}_{\geq 5}}$ (cf. Lem. 15), we present the 21 equations in Fig. 1.⁹ For $i \in [1, 21]$, let w_i, v_i be words such that $w_i = v_i$ denotes the i -th equation.

► **Lemma 37** (soundness). *For each $i \in [1, 21]$, $w_i \dot{\sim}_{\text{REL}_{\geq 5}} v_i$.*

Proof Sketch. We prove $w_i[a] \sim_{\text{REL}_{\geq 5}} v_i[a]$, where a is any variable. This equation can be translated to the validity of a first-order sentence via the standard translation [28]. Here, we add the formula $\exists x_1, \dots, x_5, \wedge_{i,j \in [1,5]; i \neq j} x_i \neq x_j$ as an axiom, for forcing $\dot{\sim}_{\text{REL}_{\geq 5}}$. Thanks to this encoding, each of them can also be tested by using ATP/SMT systems. Nevertheless, in

⁸ The condition “ ≥ 5 ” is needed for some equations in Fig. 1.

⁹ The most technical part of the paper is to collect these equations. They are obtained by running a program based on Algorithm 1 using ATP/SMT systems.

$$\begin{array}{llll}
\cap_l = \cap_l \cap_l & (1) & \varepsilon = \smile \smile & (7) & \cdot_D \cdot_D = \cdot_D \cdot_D \cdot_D & (13) \\
\cap_D = \cap_D \cap_D & (2) & \cap_D \cap_l = \cdot_D \cap_l \cap_D & (8) & \cdot_D \cdot_D \cap_D = \cdot_D \cdot_D \cap_l \cdot_D & (14) \\
\cap_l = \cap_l \smile & (3) & \cap_D \cap_l = \cap_l \cdot_D \cap_l & (9) & \cap_D \cdot_D \cdot_D = \cdot_D \cap_l \cdot_D \cdot_D & (15) \\
\cap_l = \smile \cap_l & (4) & \cap_l \cdot_D = \cap_l \cdot_D \cap_D & (10) & \smile \cdot_D \cap_l = \cap_D \smile \cdot_D \cap_l & (16) \\
\cap_l \cap_D = \cap_D \cap_l & (5) & \cdot_D \cap_l = \cap_D \cdot_D \cap_l & (11) & \cdot_D \smile \cdot_D \smile = \smile \cdot_D \smile \cdot_D & (17) \\
\cap_D \smile = \smile \cap_D & (6) & \cdot_D \cdot_D = \cdot_D \cap_D \cdot_D & (12) & \smile \cdot_D \cap_l \cdot_D = \cap_D \smile \cdot_D \cdot_D \cap_D & (18) \\
& & & & \cdot_D \smile \cdot_D \cdot_D \smile = \smile \cdot_D \cdot_D \smile \cdot_D & (19) \\
\cdot_D \cap_D \smile \cdot_D \cap_D \smile \cdot_D \cap_D \smile \cdot_D \cap_D \smile \cdot_D \cap_D \smile \cdot_D \cap_D \smile \cdot_D \cap_D \smile & (20) \\
\cdot_D \cap_l \cdot_D \smile \cdot_D \cap_l \cdot_D \smile \cdot_D \smile \cdot_D \cap_l \cdot_D \smile \cdot_D \cap_l \cdot_D \smile \cdot_D & (21)
\end{array}$$

■ **Figure 1** Equations for the finiteness.

the following, as an example, we give explicit proof for Equation (13). By using the standard translation, Equation (13) is translated into the following formula in first-order logic, where x_0, y_0 are free variables:

$$\begin{aligned}
& (\exists y_1, (\exists y_2, a(x_0, y_2) \wedge y_2 \neq y_1) \wedge y_1 \neq y_0) \\
\leftrightarrow & (\exists y_1, (\exists y_2, (\exists y_3, a(x_0, y_3) \wedge y_3 \neq y_2) \wedge y_2 \neq y_1) \wedge y_1 \neq y_0).
\end{aligned}$$

This formula is valid under $\text{REL}_{\geq 5}$, which can be shown by using the axiom above (notice that under $\text{REL}_{\geq 5}$, y_1 on the left and y_1, y_2 on the right always exist, by taking a vertex not assigned by any variable occurring in each formula; thus, both formulas are equivalent to the formula $\exists y, a(x_0, y)$). Even without the encoding to first-order logic, this equation can also be shown as follows:

$$\begin{aligned}
(\cdot_D \cdot_D)[a] &= (a \cdot D) \cdot D = a \cdot (D \cdot D) && \text{(associativity law)} \\
&= a \cdot \top && (\top \sim_{\text{REL}_{\geq 3}} D \cdot D) \\
&= a \cdot (D \cdot D \cdot D) && (\top \sim_{\text{REL}_{\geq 3}} \top \cdot D \text{ and } \top \sim_{\text{REL}_{\geq 3}} D \cdot D) \\
&= ((a \cdot D) \cdot D) \cdot D = (\cdot_D \cdot_D \cdot_D)[a]. && \text{(associativity law)}
\end{aligned}$$

See [23, 24] for all the equations. ◀

► **Lemma 38.** *The language $\bigcup_{i \in [1, 21]} \dot{\Sigma}_0^* v_i \dot{\Sigma}_0^*$ over $\dot{\Sigma}_0$ is cofinite.*

Proof Sketch. It suffices to prove that for some $n \in \mathbb{N}$, the following hold: there is no word $w \in \dot{\Sigma}_0^* \setminus (\bigcup_{i \in [1, 21]} \dot{\Sigma}_0^* v_i \dot{\Sigma}_0^*)$ such that $\|w\| \geq n$ (since the set $\{w \in \dot{\Sigma}_0^* \mid \|w\| \leq n - 1\}$ is finite). This holds when $n \geq 29$, which can be tested by using Z3 (an ATP/SMT system) [9] and can be checked by drawing its DFA (see [23, 24], for more details). ◀

Thus, we have obtained the following:

► **Lemma 39.** *$\dot{\Sigma}^* / \sim_{\text{REL}_{\geq 5}}$ is finite.*

Proof. By Lems. 37 and 38, we can apply Lem. 15, where $<$ is the shortlex order on $\dot{\Sigma}_0^*$ induced by: $\cap_l < \cap_D < \cdot_D < \smile$. By the form, $w_i < v_i$ is clear for each $i \in [1, 21]$. ◀

Finally, Thm. 25 is obtained as follows:

Proof of Thm. 25. We have: $\dot{\Sigma}_0^*/\sim_{\text{REL}_{\geq 5}}$ is finite (Lem. 39) $\implies \dot{\Sigma}^*/\sim_{\text{REL}_{\geq 5}}$ is finite (Lem. 36) $\implies \mathbf{T}_{(\text{vo} \leq 1)}^{\{\cap, \cdot, !, \text{D}\}}/\sim_{\text{REL}}$ is finite (Lem. 34) $\implies (\Sigma_1^{\text{CoR}})_{(\text{vo} \leq 1)}/\sim_{\text{REL}}$ is finite (Lem. 33) $\implies (\Sigma_n^{\text{CoR}})_{(\text{vo} \leq 1)}/\sim_{\text{REL}}$ is finite (Lem. 32) $\implies (\Sigma_n^{\text{CoR}})_{(\text{vo} \leq k)}/\sim_{\text{REL}}$ is finite (Lem. 29). \blacktriangleleft

\blacktriangleright **Remark 40.** The finite axiomatizability of the equational theory of $(\Sigma_n^{\text{CoR}})_{(\text{vo} \leq k)}$ over REL immediately follows from the finiteness of $(\Sigma_n^{\text{CoR}})_{(\text{vo} \leq k)}/\sim_{\text{REL}}$.

5 Conclusion

We have introduced the *k-variable-occurrence fragment* and presented an approach for showing the decidability of the equational theory from the finiteness. As a case study, we have proved that the equational theory of $(\Sigma_n^{\text{CoR}})_{(\text{vo} \leq k)}$ is decidable, whereas that of Σ_n^{CoR} is undecidable in general. We leave the decidability open for the equational theory of CoR with *full dot-dagger alternation* (i.e., $\mathbf{T}_{(\text{vo} \leq k)}^{\text{CoR}}$, in this paper). Our approach may apply to some other algebras/logics. It would be interesting to consider the finite variable-occurrence fragment for other systems (e.g., CoR with antidomain [13, 10], dynamic logics [12], relation algebras). It would also be interesting to extend our result to *first-order logic with equality* (cf. Prop. 19) – for example, is the *k-atomic-predicate-occurrence fragment* of the *m*-variable fragment of first-order logic with equality decidable?

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