

# Finding Long Directed Cycles Is Hard Even When DFVS Is Small or Girth Is Large

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## Abstract

We study the parameterized complexity of two classic problems on directed graphs: HAMILTONIAN CYCLE and its generalization LONGEST CYCLE. Since 2008, it is known that HAMILTONIAN CYCLE is  $W[1]$ -hard when parameterized by *directed treewidth* [Lampis et al., ISSAC'08]. By now, the question of whether it is FPT parameterized by the *directed feedback vertex set* (DFVS) number has become a longstanding open problem. In particular, the DFVS number is the largest natural directed width measure studied in the literature. In this paper, we provide a negative answer to the question, showing that even for the DFVS number, the problem remains  $W[1]$ -hard. As a consequence, we also obtain that LONGEST CYCLE is  $W[1]$ -hard on directed graphs when parameterized multiplicatively above girth, in contrast to the undirected case. This resolves an open question posed by Fomin et al. [ACM ToCT'21] and Gutin and Mnich [arXiv:2207.12278]. Our hardness results apply to the path versions of the problems as well. On the positive side, we show that LONGEST PATH parameterized multiplicatively above girth belongs to the class XP.

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## 1 Introduction

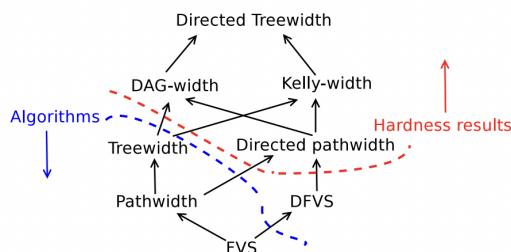
**Hamiltonian Cycle (Path) Parameterized by DFVS.** In HAMILTONIAN CYCLE (PATH), we are given a (directed or undirected) graph  $G = (V, E)$ , and the objective is to determine whether  $G$  contains a simple cycle (path) of length  $n = |V(G)|$ , called a *Hamiltonian cycle (path)*. HAMILTONIAN CYCLE<sup>1</sup> has been widely studied, from various algorithmic (see, e.g., [7, 9–12, 15, 16, 28, 29, 37]) and structural (see, e.g., the survey [50]) points of view. This problem is among the first problems known to be NP-complete [60], and it remains NP-complete on various restricted graph classes (see, e.g., [2, 20, 46]). Nevertheless, a longest path can be found easily in polynomial-time on the large class of directed acyclic graphs (DAGs) using dynamic programming.<sup>2</sup> Thus, it is natural to ask – can we solve HAMILTONIAN CYCLE efficiently on wider classes of directed graphs that resemble DAGs to some extent?

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<sup>1</sup> Throughout the following paragraphs, we refer only to the cycle variant of the problem, but the same statements also hold for the path variant.

<sup>2</sup> For example, see the lecture notes at <https://people.csail.mit.edu/virgi/6.s078/lecture17.pdf>.





■ **Figure 1** Figure 3 in [59]. Caption (verbatim): “*The ecology of digraph widths. Undirected measures refer to the underlying undirected graph. Arrows denote generalizations (for example small treewidth implies small kelly-width). The dashed lines indicate rough borders of known tractability and intractability for most studied problems (including HAMILTONIAN CYCLE).*”

In the undirected realm, the class of acyclic graphs (i.e., forests) can be generalized to graphs of bounded treewidth [27]. The celebrated Courcelle’s theorem [24] states that a wide range of problems, including HAMILTONIAN CYCLE, are *fixed-parameter tractable* (FPT)<sup>3</sup> when parameterized by treewidth. Historically, the notion of *directed treewidth* (Definition 4) was introduced by Johnson et al. [58] as a generalization of the undirected notion contrived to solving linkage problems such as HAMILTONIAN CYCLE; see also [70]. Since then, directed treewidth has been intensively studied (see, e.g., [1, 47, 48, 52, 61, 76]). Over the years, various other width measures for directed graphs have been proposed and studied as well, where the most prominent ones are the DFVS number (which is the minimum number of vertices to remove to make the graph acyclic), *directed pathwidth* (defined similarly to directed treewidth, where we replace trees by paths), *DAG-width* [69] and *Kelly-width* [55]. Notably, DFVS is the largest among them (see Fig. 1). Johnson et al. [58] and Hunter et al. [55] proved that HAMILTONIAN CYCLE (as well as LONGEST CYCLE, defined later) is *slice-wise polynomial* (XP) parameterized by directed treewidth. Later, Lampis et al. [64] (originally in 2008 [63]) showed that, here, this is the “right” form of the time complexity: they proved that HAMILTONIAN CYCLE parameterized by directed pathwidth is W[1]-hard (in fact, even W[2]-hard), implying that the problem is unlikely to be FPT.

Thus, since 2008, it has remained open whether HAMILTONIAN CYCLE parameterized by DFVS is FPT or W[1]-hard (see Fig. 1). As a follow-up to their result, Kaouri et al. [59] wrote: “*Treewidth occupies a sweet spot in the map of width parameters: restrictive enough to be efficient and general enough to be useful. What is the right analogue which occupies a similar sweet spot (if it exists) for digraphs? One direction is to keep searching for the right width that generalizes DAGs, that is searching the area around (and probably above) DFVS. ... The other possibility is that widths which generalize DAGs, such as DFVS and all the currently known widths, may not necessarily be the right path to follow. ... The search is on!*”

Our main contribution is, essentially, the finish line for this search for HAMILTONIAN CYCLE. We prove that, unfortunately, already for DFVS number itself, the problem is W[1]-hard. Formally, HAMILTONIAN CYCLE (PATH) BY DFVS is defined as follows: Given a directed graph  $G = (V, E)$  and a subset  $X \subseteq V(G)$  such that  $G - X$  is acyclic, determine whether  $G$  contains a Hamiltonian cycle (path). Here, the parameter is  $|X|$ . Note that  $X$  need not be given as input – the computation of a minimum-size vertex set whose removal makes a given directed graph acyclic is in FPT [23], so, given a directed graph  $G$ , we can simply run the corresponding algorithm to attain  $X$  whose size is the DFVS number of  $G$ .

<sup>3</sup> A problem is FPT (resp. XP) if it is solvable in time  $f(k) \cdot n^{O(1)}$  (resp.  $n^{f(k)}$ ) for some computable function  $f$  of the parameter  $k$ , where  $n$  is the size of the input.

► **Theorem 1.** HAMILTONIAN CYCLE BY DFVS is  $W[1]$ -hard.

We also obtain the same hardness result for HAMILTONIAN PATH BY DFVS. Notice that HAMILTONIAN CYCLE (PATH) BY DFVS is in XP, since, as mentioned earlier, it is already known to be in XP even when parameterized by the smaller parameter directed treewidth [58]. So, the classification of this problem is resolved.

Lastly, we remark that the choice of the larger directed feedback arc set (DFAS) number is also futile, as it also yields  $W[1]$ -hardness. Indeed, there is a simple reduction that shows this: Replace each vertex  $v$  in  $S$  by two vertices,  $v_{\text{in}}$  and  $v_{\text{out}}$ , so that all vertices that were in-neighbors (out-neighbors) of  $v$  become in-neighbors (out-neighbors) of  $v_{\text{in}}$  ( $v_{\text{out}}$ ), and add the arc  $(v_{\text{in}}, v_{\text{out}})$ . It is immediate that the original instance is a Yes-instance of HAMILTONIAN CYCLE if and only if the new instance is, and that the DFVS number of the original graph equals the DFAS number of the new graph.

**Longest Cycle (Path) Above Girth.** In the LONGEST CYCLE (PATH) problem, also known as  $k$ -LONG CYCLE ( $k$ -PATH), we are given a (directed or undirected) graph  $G$  and a non-negative integer  $k$ , and the objective is to determine whether  $G$  contains a simple cycle (path) of length at least  $k$ . Clearly, LONGEST CYCLE is a generalization of HAMILTONIAN CYCLE as the latter is the special case of the former when  $k = n$ . Over the past four decades, LONGEST CYCLE and LONGEST PATH have been intensively studied from the viewpoint of parameterized complexity. There has been a long race to achieve the best known running time on general (directed and undirected) graphs for the parameter is  $k$  [5, 14, 17, 19, 22, 40, 44, 54, 62, 67, 72, 74, 77–79], where the current winners for directed graphs have time complexity  $4^k \cdot n^{O(1)}$  [79] for LONGEST CYCLE and  $2^k \cdot n^{O(1)}$  [77] for LONGEST PATH. Moreover, the parameterized complexity of the problems was analyzed with respect to structural parameterizations (see, e.g., [7, 28, 29, 33, 37, 45, 49]), on special graph classes (see, e.g., [32, 41, 42, 66, 68, 80]), and when counting replaces decision [3, 4, 6, 13, 18, 25, 26, 34, 65, 75]. In fact, LONGEST PATH is widely considered to be one of the most well studied in the field, being, perhaps, second only to VERTEX COVER [27].

We consider the “multiplicative above guarantee” parameterization for LONGEST PATH where the guarantee is girth<sup>4</sup>, called LONGEST CYCLE (PATH) ABOVE GIRTH, which is defined as follows (for directed graphs): Given a directed graph  $G = (V, E)$  with girth  $g$  and a positive integer  $k$ , determine whether  $G$  contains a cycle (path) of length at least  $g \cdot k$ . Here, the parameter is  $k$ . This parameterization of LONGEST CYCLE (PATH) was considered by Fomin et al. [36], who proved that on undirected graphs the problem is in FPT. They posed the following open question: “For problems on directed graphs, parameterization multiplicatively above girth (which is now the length of a shortest directed cycle) may also make sense. For example, what is the parameterized complexity of DIRECTED LONG CYCLE under this parameterization?” This question was also posed by Gutin et al. [51] as Open Question 9. As our second contribution, we resolve this question – in sharp contrast to the undirected case, the answer is negative:

► **Theorem 2.** LONGEST CYCLE ABOVE GIRTH is  $W[1]$ -hard.

Again, we can transfer the hardness also to the longest path setting. On the positive side, we give a deterministic XP algorithm for the path variant of the problem.

► **Theorem 3.** LONGEST PATH ABOVE GIRTH is in XP.

<sup>4</sup> The girth of a graph is the length of its shortest cycle, and it is easily computable in polynomial time.

Therefore, the classification of this problem is resolved as well. Notice that this theorem also easily implies that when we ask for a path of length at least  $g + k$ , the problem is in FPT (in fact, solvable in time  $2^{O(k)} \cdot n^{O(1)}$ ) as follows. If  $k \geq g$ , then we can simply run some known  $2^{O(k)} \cdot n^{O(1)}$ -time algorithm for LONGEST PATH parameterized by the solution size. Otherwise, when  $k < g$ , we create  $n$  instances of LONGEST PATH ABOVE GIRTH with parameter 2 (thus, solvable in polynomial time by our theorem), one for each vertex  $v$ : Replace  $v$  by a path of length  $g - k$  whose start vertex has all in-neighbors of  $v$  as its in-neighbors and whose end vertex has all out-neighbors of  $v$  as its out-neighbors, and take the disjoint union of the resulting graph and a cycle of length  $g$ ; it is easy to see that we should return Yes if and only if the answer to at least one of these  $n$  instances is Yes.

Other above/below guarantee versions of LONGEST CYCLE (PATH) were also considered in the literature; see [8, 35, 38, 39, 51, 53, 56, 57]. Remarkably, while the parameterized complexity of all of these versions is quite well understood for undirected graphs, the resolution of the parameterized complexity of most of these versions has been (sometimes repeatedly) asked as an open question for directed graphs, where only little is known. It is conceivable that our contributions will shed light on these open questions as well in future works.

## 1.1 Techniques

**Hamiltonian Cycle By DFVS.** Before describing the ideas behind our reduction, we provide some background about the related DISJOINT PATHS problem. Here, we are given a graph with  $k$  vertex pairs  $(s_i, t_i)$  and the goal is to determine whether there exists  $k$  vertex-disjoint<sup>5</sup> paths such that the  $i$ -th path connects  $s_i$  to  $t_i$ . While DISJOINT PATHS is famously known to be FPT (parameterized by  $k$ ) on undirected graphs [71], the problem becomes NP-hard on directed graphs already for  $k = 2$  [43]. What is interesting for us is that when the input is restricted to be a DAG, then DISJOINT PATHS can be solved by an XP algorithm [43], but it is unlikely to admit an FPT algorithm as the problem is W[1]-hard [73].

The latter result suggests a starting point for extending the hardness to our problem. A simple idea to design a reduction from DISJOINT PATHS on DAGs is to just insert edges  $t_1 \rightarrow s_2, t_2 \rightarrow s_3, \dots, t_k \rightarrow s_1$ . Then the set  $\{s_1, \dots, s_k\}$  becomes a DFVS and the existence of  $k$  disjoint  $(s_i, t_i)$ -paths implies the existence of a long cycle. There is a catch, though. This construction might turn a No-instance into an instance with a long cycle because the cycle might traverse the vertices  $s_i, t_i$  in a different order, corresponding to a family of disjoint paths which does not form a solution to the original instance. To circumvent this, we open the black box and give a reduction directly from the basic W[1]-complete problem – MULTICOLORED CLIQUE – while adapting some ideas from the hardness proof for DISJOINT PATHS on DAGs by Slivkins [73].

The construction by Slivkins uses two kinds of gadgets. First, for each  $i = 1, \dots, k$  one needs a *choice gadget* comprising two long directed paths with some arcs from the first path to the second one. The solution path corresponding to the  $i$ -th choice gadget must choose one of these arcs to “change the lane”; the location of this change encodes the choice of the  $i$ -th vertex in the clique. Next, for each pair  $(i, j)$ , where  $1 \leq i < j \leq k$ , one needs a *verification gadget* to check whether the  $i$ -th and  $j$ -th chosen vertices are adjacent. The corresponding solution path must traverse the  $i$ -th and  $j$ -th choice gadgets around the locations of their transitions. The arcs between the choice gadgets are placed in such a way that both these locations can be visited only when the corresponding edge exists in the original graph.

<sup>5</sup> Another studied variant involves finding edge-disjoint paths. While these two problems behave differently in some settings, this is not the case in our context.

Our reduction relies on a similar mechanism of choice gadgets, formed by  $k$  directed paths, corresponding to  $k$  colors in the MULTICOLORED CLIQUE instance. The  $i$ -th path is divided into blocks corresponding to the vertices of color  $i$ . We enforce that a Hamiltonian cycle must omit exactly one block when following such a path, thus encoding a choice of  $k$  vertices. The arcs between blocks from different paths encode the adjacency matrix of the original graph. We would like to guarantee that any Hamiltonian cycle  $C$  visits each omitted block  $k - 1$  times, utilizing the arcs mentioned above. To ensure this, we need to allow  $C$  to “make a step backward” on the directed path during such a visit. On the other hand, we cannot afford to create too many disjoint cycles because DFVS should have size bounded in terms of  $k$ . As a remedy, we attach  $k - 1$  long cycles to each choice gadget, which are connected to every block in a certain way (see Figure 3 on page 8). Using the fact that every vertex in the gadget must be visited by  $C$  at some point, we prove that each long cycle is entered exactly once and each choice gadget is entered exactly  $k$  times in total, imposing a rigid structure on a solution. As a result, we get rid of the flaw occurring in the naive construction and obtain an equivalent instance with DFVS number  $\mathcal{O}(k^2)$ .

**Longest Cycle (Path) Above Girth.** Having established our main hardness result, it is easy to extend it to LONGEST CYCLE (PATH) ABOVE GIRTH. We replace each vertex  $v$  in the DFVS by two vertices  $v_{in}$  and  $v_{out}$ , splitting the in- and out-going arcs of  $v$  between them, and insert a long path from  $v_{in}$  to  $v_{out}$ . This path is set sufficiently long to make the girth  $g$  of the graph comparable to its size. Consequently, the original graph  $G$  is Hamiltonian if and only if the second one has a cycle of length  $g \cdot (\text{dfvs}(G) + 1)$ . So, this problem is also W[1]-hard.

To design an XP algorithm for LONGEST PATH ABOVE GIRTH, we follow the win-win approach employed by the FPT algorithm for the undirected case [36], which relies on a certain version of the Grid Minor Theorem. In general, the theorem states that either a graph contains a  $k \times k$ -grid as a minor, or its treewidth is bounded by  $f(k)$ , for some (in fact, polynomial) function  $f$ . In the first scenario, a sufficiently long path always exists, whereas in the second scenario, the problem is solved via dynamic programming on the tree decomposition.

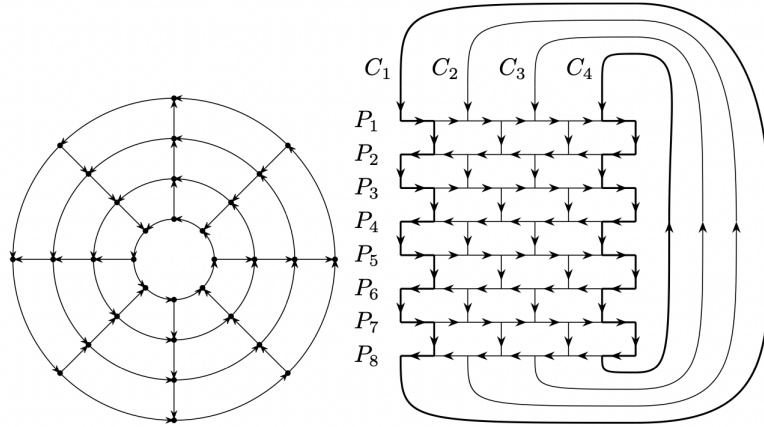
We take advantage of the directed counterpart of the Grid Minor Theorem: either a digraph contains a subgraph isomorphic to a subdivision of the order- $\mathcal{O}(k)$  cylindrical wall or its directed treewidth is bounded by  $f(k)$ , for some function  $f$  [61]. We prove that in the first scenario again a sufficiently long path always exists (so we obtain a Yes-instance), while in the second scenario we can utilize the known algorithm to compute the longest path in XP time with the help of the directed tree decomposition. Finally, let us remark that this argument does not extend to LONGEST CYCLE ABOVE GIRTH because we cannot guarantee the existence of a sufficiently long cycle in a subdivision of a cylindrical wall. We leave it open whether this problem belongs to XP as well.

## 2 Preliminaries

**General Notation.** For an integer  $r$ , let  $[r] = \{1, \dots, r\}$ , and for integers  $r_1, r_2$ , let  $[r_1, r_2] = \{r_1, r_1 + 1, \dots, r_2\}$ . We refer to [27] for standard definitions in Parameterized Complexity.

**Directed Graphs.** We use standard graph theoretic terminology from Diestel’s book [31]. We work with simple directed graphs (digraphs) where the edges are given by a set  $E$  of ordered pairs of vertices. For an edge  $e = (u, v)$  in a digraph  $G$ , we say that  $u$  is an *in-neighbor* of  $v$  and  $v$  is an *out-neighbor* of  $u$ . We refer to  $e$  as an *incoming edge* of  $v$  and an *outgoing edge* of  $u$ . For a vertex set  $S \subseteq V(G)$  we define  $\partial^{out}(S) = \{(u, v) \in E(G) : u \in S\}$ ,





■ **Figure 2** A cylindrical grid and a cylindrical wall of order 4. The figure is taken from [61].

$\partial^{in}(S) = \{(u, v) \in E(G) : v \in S\}$ , and  $\partial(S) = \partial^{out}(S) \cup \partial^{in}(S)$ . For two vertex sets  $A, B \subseteq V(G)$  we write  $E(A, B)$  for  $\partial(A) \cap \partial(B)$ , that is, the set of edges with one endpoint in  $A$  and the other in  $B$ . We use this notation only when the digraph  $G$  is clear from the context. When  $C$  is a subgraph of  $G$  we abbreviate  $\partial(C) = \partial(V(C))$  and likewise for the remaining notation. A digraph  $G$  is *isomorphic* to a digraph  $H$  if there is a bijection  $f : V(G) \rightarrow V(H)$  such that for any  $u, v \in V(G)$  we have  $(u, v) \in E(G)$  if and only if  $(f(u), f(v)) \in E(H)$ . The *length* of a path (or a cycle)  $P$  is the number of edges in  $P$ . When the first and last vertices of a path  $P$  are  $s$  and  $t$ , respectively, we call it an  $(s, t)$ -*path*. A cycle in  $G$  is called *Hamiltonian* if it visits all the vertices of  $G$ . The *girth* of a digraph  $G$  is the shortest length of a cycle in  $G$ . For a rooted tree  $T$  and a node  $t \in V(T)$ ,  $T_t$  denotes the subtree of  $T$  rooted at  $t$ . By orienting each edge in a rooted tree from a parent to its child we obtain an *arborescence*.

**Directed Treewidth.** We move on to the directed counterparts of treewidth and grids.

► **Definition 4** (Directed Tree Decomposition [61]). A directed tree decomposition of a directed graph  $G$  is a triple  $(T, \beta, \gamma)$ , where  $T$  is an arborescence, and  $\beta : V(T) \rightarrow 2^{V(G)}$  and  $\gamma : E(T) \rightarrow 2^{V(G)}$  are functions such that

1.  $\{\beta(t) : t \in V(T)\}$  is a partition of  $V(G)$  into (possibly empty) sets, and
2. if  $e = (s, t) \in E(T)$ ,  $A = \bigcup\{\beta(t') : t' \in V(T_t)\}$  and  $B = V(G) \setminus A$ , then there is no closed (directed) walk in  $G - \gamma(e)$  containing a vertex in  $A$  and a vertex in  $B$ .

For  $t \in V(T)$ , define  $\Gamma(t) := \beta(t) \cup \bigcup\{\gamma(e) : e \text{ is incident with } t\}$ . Moreover, define  $\beta(T) := \bigcup\{\beta(t') : t' \in V(T)\}$ .

The width of  $(T, \beta, \gamma)$  is the least integer  $w$  such that  $|\Gamma(t)| \leq w + 1$  for all  $t \in V(T)$ . The directed treewidth of  $G$  is the least integer  $w$  such that  $G$  has a directed tree decomposition of width  $w$ .

► **Definition 5** (Subdivision). For a directed graph  $G$ , the edge subdivision of  $(u, v) \in E(G)$  is the operation that removes  $(u, v)$  from  $G$  and inserts two edges  $(u, w)$  and  $(w, v)$  with the new vertex  $w$ . A graph derived from  $G$  by a sequence of edge subdivisions is a subdivision of  $G$ .

We now define special graphs called cylindrical grids and cylindrical walls (see Fig. 2).

► **Definition 6** (Cylindrical Grid and Cylindrical Wall [21]). A cylindrical grid of order  $k$ , for some  $k \geq 1$ , is the directed graph  $G_k$  consisting of  $k$  pairwise vertex disjoint directed cycles  $C_1, \dots, C_k$ , together with  $2k$  pairwise vertex disjoint directed paths  $P_1, \dots, P_{2k}$  such that:

- for  $i \in [k]$ ,  $V(C_i) = \{v_{i,1}, v_{i,2}, \dots, v_{i,2k}\}$  and  $E(C_i) = \{(v_{i,j}, v_{i,j+1}) \mid j \in [2k-1]\} \cup \{(v_{i,2k}, v_{i,1})\}$
- for  $i \in \{1, 3, 5, \dots, 2k-1\}$ ,  $E(P_i) = \{(v_{1,i}, v_{2,i}), (v_{2,i}, v_{3,i}), \dots, (v_{k-1,i}, v_{k,i})\}$ , and
- for  $i \in \{2, 4, 6, \dots, 2k\}$ ,  $E(P_i) = \{(v_{k,i}, v_{k-1,i}), (v_{k-1,i}, v_{k-2,i}), \dots, (v_{2,i}, v_{1,i})\}$ .

A cylindrical wall of order  $k$  is the directed graph  $W_k$  obtained from the cylindrical grid  $G_k$  by splitting each vertex of degree 4 as follows: we replace  $v$  by two new vertices  $v_{in}, v_{out}$  and an edge  $(v_{in}, v_{out})$  so that every edge  $(u, v) \in E(G_k)$  is replaced by an edge  $(u, v_{in})$  and every edge  $(v, u) \in E(G_k)$  is replaced by an edge  $(v_{out}, u)$ .

Note that in the cylindrical grid  $G_k$ , the path  $P_i$  is oriented from the first cycle to the last one if  $i$  is odd, and from the last cycle to the first if  $i$  is even (see Figure 2). We have the following theorem for directed graphs, which will be helpful in designing our algorithm in Section 4 using a win/win approach.

► **Theorem 7** (Directed Grid Theorem [61]). *There is a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that for every fixed  $k \in \mathbb{N}$ , when given a directed graph  $G$ , in polynomial time we can compute either:*

1. a subgraph of  $G$  that is isomorphic to a subdivision<sup>6</sup> of  $W_k$  or
2. a directed tree decomposition of  $G$  of width at most  $f(k)$ .

### 3 Hardness of HAMILTONIAN CYCLE BY DFVS

This section is devoted to the proof of Theorem 1. It is based on a parameterized reduction from MULTICOLORED CLIQUE, defined as follows.

MULTICOLORED CLIQUE

**Input:** A graph  $G = (V, E)$ , an integer  $k$ , and a partition  $(V^1, V^2, \dots, V^k)$  of  $V$ .

**Parameter:**  $k$

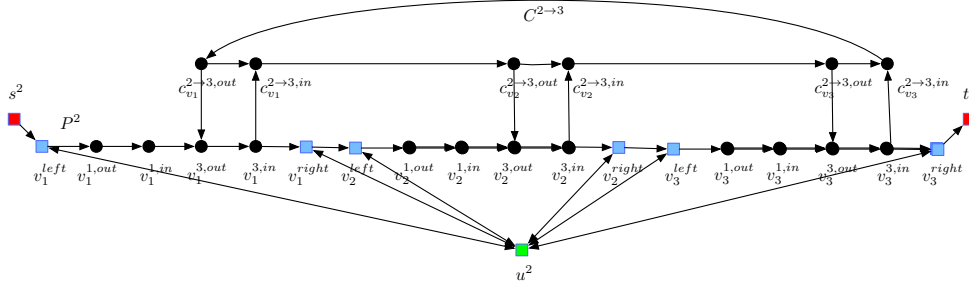
**Question:** Is there a clique of size  $k$  with a vertex from each  $V^i$ ,  $i \in [k]$ ?

This problem is well-known to be W[1]-hard [27, Theorem 13.7]. For an instance  $(G, (V^1, V^2, \dots, V^k))$  of MULTICOLORED CLIQUE, we construct an instance  $(G', X)$  of HAMILTONIAN CYCLE BY DFVS as follows.

**Construction of  $G'$ .** For  $i \in [k]$  we construct a directed path  $P^i$  corresponding to  $V^i$  as follows. Let us fix an arbitrary ordering  $<_i$  of the vertices in  $V^i$ , and accordingly, denote  $V^i = \{v_1, v_2, \dots, v_{|V^i|}\}$ . To each vertex  $v \in V^i$  we associate a directed path  $P_v$  on  $2k$  vertices. We let  $v^{left}$  and  $v^{right}$  denote the first and last vertices of  $P_v$ , respectively. We refer to the  $2(k-1)$  internal vertices of  $P_v$  as  $v^{1,out}, v^{1,in}, v^{2,out}, \dots, v^{i-1,out}, v^{i-1,in}, v^{i+1,out}, v^{i+1,in}, \dots, v^{k,out}, v^{k,in}$  (note that the index  $i$  is avoided). The directed path  $P^i$  is the concatenation of these paths:  $P_{v_1} \rightarrow P_{v_2} \rightarrow \dots \rightarrow P_{v_{|V^i|}}$ .

For every  $i \in [k]$  we create a “universal” vertex  $u^i$  and insert edges  $(u_i, v^{left}), (u_i, v^{right}), (v^{left}, u_i), (v^{right}, u_i)$  for all  $v \in V^i$ . Next, we create  $k-1$  cycles  $C^{i \rightarrow j}$  for  $j \in [k] \setminus \{i\}$ , each of length  $2 \cdot |V^i|$ . The vertices of  $C^{i \rightarrow j}$  are  $c_{v_1}^{i \rightarrow j, out}, c_{v_1}^{i \rightarrow j, in}, c_{v_2}^{i \rightarrow j, out}, c_{v_2}^{i \rightarrow j, in}, \dots, c_{v_{|V^i|}}^{i \rightarrow j, out}, c_{v_{|V^i|}}^{i \rightarrow j, in}$ . We insert edges from  $c_v^{i \rightarrow j, out}$  to  $v^{j, out}$  and from  $v^{j, in}$  to  $c_v^{i \rightarrow j, in}$  for all  $v \in V^i$ . See Figure 3 for an illustration.

<sup>6</sup> The theorem in [61] states that the cylindrical wall of order  $k$  is obtained as a topological minor of  $G$ . For any topological minor  $H$  of  $G$ , there exists a subdivision of  $H$  isomorphic to a subgraph of  $G$ .



■ **Figure 3** Fragment of the graph  $G'$ : the path  $P^2$  (comprising subpaths  $P_{v_1}, P_{v_2}, P_{v_3}$ ) and the cycle  $C^{2 \rightarrow 3}$ .

We add two sets of  $k$  “terminal” vertices:  $S = \{s^1, s^2, \dots, s^k\}$  and  $T = \{t^1, t^2, \dots, t^k\}$ . For  $i \in [k]$  we insert edges from  $s^i, i \in [k]$ , to  $v_1^{left}$  (being the first vertex of the path  $P^i$ ). We also add edges from  $v_{|V^i|}^{right}$  (being the last vertex of the path  $P^i$ ) to  $t^i$ .

We also create two additional sets of  $\binom{k}{2}$  terminal vertices each:  $\widehat{S} = \{\widehat{s}^{i \rightarrow j} : 1 \leq 1 < j \leq k\}$  and  $\widehat{T} = \{\widehat{t}^{i \rightarrow j} : 1 \leq 1 < j \leq k\}$ . For  $i, j \in [k], i < j$ , and  $v \in V^i$ , we insert an edge from  $\widehat{s}^{i \rightarrow j}$  to  $v^{j, in}$  (which belongs to  $P_v$  and hence to  $P^i$ ) and for  $v \in V^j$ , we insert an edge from  $v^{i, out}$  to  $\widehat{t}^{i \rightarrow j}$ . Furthermore, we add edges from every  $t \in T \cup \widehat{T}$  to every  $s \in S \cup \widehat{S}$ .

Finally we encode the adjacency matrix of  $G$ : for each edge  $uv$  in  $E(G)$  with  $u \in V^i$  and  $v \in V^j$  with  $i < j$ , we insert an edge from  $u^{j, out}$  to  $v^{i, in}$ . See Figure 4 for an example.

This concludes the construction of  $G'$ . Clearly, the graph  $G'$  can be computed in polynomial time when given  $(G, (V^1, \dots, V^k))$ . We begin the analysis of  $G'$  by showing that it has a DFVS of size  $\mathcal{O}(k^2)$ .

► **Lemma 8.** *There exists a subset  $X \subseteq V(G')$  such that  $G' - X$  is a DAG and  $|X| = k(k-1) + 2k + \binom{k}{2}$ .*

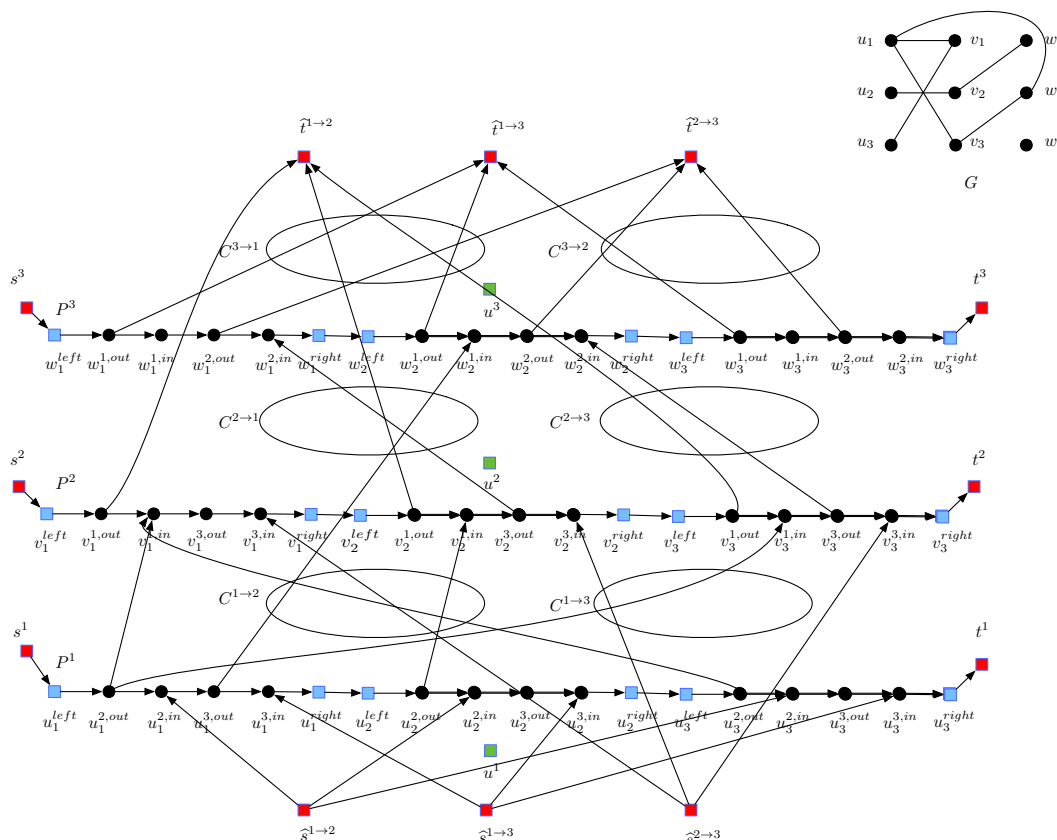
**Proof.** Let  $Y$  be the set of vertices  $\{c_{v^i}^{i \rightarrow j, out} : i, j \in [k], i \neq j\}$  where  $v^i$  stands for the first vertex in  $V^i$ . We set  $X = Y \cup T \cup \widehat{T} \cup \{u^1, \dots, u^k\}$  and claim that  $G' - X$  is a directed acyclic graph. First observe that for each  $i \in [k]$  the graph  $L^i := P^i \cup \bigcup_{j \neq i} C^{i \rightarrow j} - X$  is acyclic because it can be drawn with each edge being either vertical or horizontal facing right (see Figure 3). The remaining edges in  $G' - X$  either start at  $S \cup \widehat{S}$  (these vertices have no incoming edges) or go from  $L^i$  to  $L^j$  for  $i < j$ . Thus  $G' - X$  is a DAG. Finally we check that  $|X| = |Y| + |T| + |\widehat{T}| + k = k(k-1) + 2k + \binom{k}{2}$ . ◀

**Correctness.** We first show that if  $(G, (V^1, V^2, \dots, V^k))$  is a Yes-instance then there exists a Hamiltonian cycle in  $G'$ . In the following lemmas, we refer to the directed feedback vertex set  $X$  from Lemma 8.

► **Lemma 9.** *If  $(G, (V^1, V^2, \dots, V^k))$  is a Yes-instance of MULTICOLORED CLIQUE then  $(G', X)$  is a Yes-instance of HAMILTONIAN CYCLE BY DFVS.*

**Proof.** Let  $\{v_1, \dots, v_k\}$  be a clique in  $G$  with  $v_i \in V^i$  for  $i \in [k]$ . For  $i \in [k]$  we define the path  $Z^i$  that starts at  $s^i$ , follows  $P^i$  until  $v_i^{left}$ , visits  $u^i$ , and then follows  $P^i$  from  $v_i^{right}$  to  $t^i$ . Next, for  $i < j$  we construct the path  $Q^{i \rightarrow j}$  as  $\widehat{s}^{i \rightarrow j} \rightarrow v_j^{j, in} \rightarrow C^{i \rightarrow j} \rightarrow v_i^{i, out} \rightarrow v_j^{j, in} \rightarrow C^{j \rightarrow i} \rightarrow v_j^{j, out} \rightarrow \widehat{t}^{i \rightarrow j}$ . Note that one can traverse the entire cycle  $C^{i \rightarrow j}$  after entering it from  $v_j^{j, in}$  and leave towards  $v_i^{i, out}$ . The edge  $(v_i^{j, out}, v_j^{i, in})$  is present in  $G'$  due to the encoding of the adjacency matrix of  $G$ . The union of all these paths covers the entire vertex set of  $G'$ . It suffices to observe that these paths can be combined into a single cycle using the edges from  $T \cup \widehat{T}$  to  $S \cup \widehat{S}$ . ◀





■ **Figure 4** Graph  $G$  (top right) with a clique  $\{u_1, v_3, w_2\}$  and the graph  $G'$  encoding  $G$ . For the sake of legibility, several groups of edges are omitted in this picture: edges incident to  $u^i$  or the cycles  $C^{i \rightarrow j}$  and edges going from  $T \cup \hat{T}$  to  $S \cup \hat{S}$ .

Proving the converse of Lemma 9 is more challenging. We will reveal the structure of a potential Hamiltonian cycle  $H$  in  $G'$  gradually. First, we show that if  $H$  enters the cycle  $C^{i \rightarrow j}$  through an incoming edge corresponding to a certain vertex  $v \in V^i$ , then it must also leave it through an outgoing edge related to  $v$ , and vice versa. A priori, there might be multiple vertices  $v \in V^i$  for which this happens.

For  $v \in V^i$  we denote the edge from  $c_v^{i \rightarrow j, out}$  to  $v^{j, out}$  by  $e_v^{i \rightarrow j, out}$ . Similarly, let the edge from  $v^{j, in}$  to  $c_v^{i \rightarrow j, in}$  be  $e_v^{i \rightarrow j, in}$ .

► **Lemma 10.** *Let  $H$  be a Hamiltonian cycle in  $G'$  and  $1 \leq i < j \leq k$ . Then  $E(H) \cap \partial(C^{i \rightarrow j})$  is of the form  $\bigcup_{v \in U} \{e_v^{i \rightarrow j, out}, e_v^{i \rightarrow j, in}\}$  where  $U$  is some subset of  $V^i$ .*

**Proof.** Note that  $\partial(C^{i \rightarrow j}) = \{e_{v_1}^{i \rightarrow j, out}, e_{v_1}^{i \rightarrow j, in}, e_{v_2}^{i \rightarrow j, out}, e_{v_2}^{i \rightarrow j, in}, \dots, e_{v_{|V^i|}}^{i \rightarrow j, out}, e_{v_{|V^i|}}^{i \rightarrow j, in}\}$ . First suppose that for some  $v \in V^i$ , we have  $e_v^{i \rightarrow j, in} \in E(H)$ . Since  $H$  is a Hamiltonian cycle in  $G'$ , it has to visit the vertex  $c_v^{i \rightarrow j, out}$ . Thus  $H$  contains an outgoing edge of  $c_v^{i \rightarrow j, out}$ . The only two outgoing edges of  $c_v^{i \rightarrow j, out}$  are  $e_v^{i \rightarrow j, out}$  and  $(c_v^{i \rightarrow j, out}, c_v^{i \rightarrow j, in})$ . Since  $e_v^{i \rightarrow j, in} \in E(H)$  is an incoming edge of  $c_v^{i \rightarrow j, in}$ , the edge  $(c_v^{i \rightarrow j, out}, c_v^{i \rightarrow j, in})$  cannot belong to  $E(H)$ . The only other candidate for the outgoing edge of  $c_v^{i \rightarrow j, out}$  is  $e_v^{i \rightarrow j, out}$ , so  $e_v^{i \rightarrow j, out} \in E(H)$ .

The proof of implication in the other direction is analogous. Hence for each  $v \in V^i$  we have  $e_v^{i \rightarrow j, out} \in E(H) \Leftrightarrow e_v^{i \rightarrow j, in} \in E(H)$ . ◀

As the next step, we show that when  $v \in V^i$  and a Hamiltonian cycle enters some cycle  $C^{i \rightarrow j}$  adjacent to  $P^i$  through a vertex from  $P_v$ , then it must enter all the  $k-1$  cycles adjacent to  $P^i$  through  $P_v$ . Again, we cannot yet exclude a scenario that this would happen for multiple vertices  $v \in V^i$ .

► **Lemma 11.** *For any Hamiltonian cycle  $H$  in  $G'$  and  $i \in [k]$ , there exists a subset  $U \subseteq V^i$  for which we have  $E(H) \cap E(P^i, \bigcup_{j \neq i} C^{i \rightarrow j}) = \bigcup_{v \in U} E(P_v, \bigcup_{j \neq i} C^{i \rightarrow j})$ .*

**Proof.** Targeting a contradiction, suppose there exists a vertex  $v \in V^i$  such that at least one edge of  $E(P_v, \bigcup_{j \neq i} C^{i \rightarrow j})$  is in  $E(H)$  and at least one edge of  $E(P_v, \bigcup_{j \neq i} C^{i \rightarrow j})$  is not in  $E(H)$ . From Lemma 10 we know that  $E(H) \cap E(P_v, \bigcup_{j \neq i} C^{i \rightarrow j})$  is a union of pairs of the form  $\{e_v^{i \rightarrow j, out}, e_v^{i \rightarrow j, in}\}$  for some  $j \in [k] \setminus \{i\}$ . Hence there indices  $p, j \in [k] \setminus \{i\}$  so that  $(v^{p, in}, v^{j, out}) \in E(P^i)$ , one of the sets  $E(B_v, C^{i \rightarrow p}), E(B_v, C^{i \rightarrow j})$  has empty intersection with  $E(H)$  and the other one is contained in  $E(H)$ . We can assume w.l.o.g. that  $j = p+1$  and the first of these sets is empty.

We arrive at the following scenario:  $p \in [k] \setminus \{i\}$ ,  $e_v^{i \rightarrow p, out}, e_v^{i \rightarrow p, in} \notin E(H)$  and  $e_v^{i \rightarrow p+1, out}, e_v^{i \rightarrow p+1, in} \in E(H)$ . Since  $H$  is a Hamiltonian cycle, it has to visit the vertex  $v^{p, in}$ . Thus, it has to traverse an outgoing edge of  $v^{p, in}$ . The only outgoing edges of  $v^{p, in}$  are the edges  $e_v^{i \rightarrow p, in}$  and  $(v^{p, in}, v^{p+1, out})$ . By the assumption,  $e_v^{i \rightarrow p, in} \notin E(H)$  so  $(v^{p, in}, v^{p+1, out}) \in E(H)$ . On the other hand, since  $e_v^{i \rightarrow p+1, out} \in E(H)$  is an incoming edge of  $v^{p+1, out}$ , we have  $(v^{p, in}, v^{p+1, out}) \notin E(H)$  as it is also an incoming edge to  $v^{p+1, out}$ . This yields a contradiction. ◀

We want to argue now that in fact the set  $U$  in Lemma 11 is a singleton, that is, for fixed  $i \in [k]$  each cycle  $C^{i \rightarrow j}$  is being entered exactly once and from the same subpath  $P_v$  of  $P^i$ . To this end, we take advantage of the universal vertices  $u^i$ .

► **Lemma 12.** *For any Hamiltonian cycle  $H$  in  $G'$  and  $i \in [k]$  there exists  $v \in V^i$  such that  $E(H) \cap E(P^i, \bigcup_{j \neq i} C^{i \rightarrow j}) = E(P_v, \bigcup_{j \neq i} C^{i \rightarrow j})$ .*

**Proof.** Let  $U$  be the set from Lemma 11. Targeting a contradiction, suppose that there exist two distinct  $v, w \in U$ . That is,  $E(H) \cap E(P^i, \bigcup_{j \neq i} C^{i \rightarrow j}) \supseteq E(P_v, \bigcup_{j \neq i} C^{i \rightarrow j}) \cup E(P_w, \bigcup_{j \neq i} C^{i \rightarrow j})$ .

The Hamiltonian cycle  $H$  must contain an outgoing edge of  $v^{left}$ , which is the first vertex on the path  $P_v$ . The only out-neighbors of  $v^{left}$  are  $u^i$  and  $v^{p, out}$  where  $p = 1$  when  $i \neq 1$  or  $p = 2$  when  $i = 1$ . Recall that the  $e_v^{i \rightarrow p, out}$  is present in  $E(B_v^i, \bigcup_{j \neq i} C^{i \rightarrow j})$  and thereby in  $E(H)$  by the assumption. Also, since the edge  $e_v^{i \rightarrow p, out}$  is an incoming edge of  $v^{p, out}$ , the edge  $(v^{left}, v^{p, out})$  cannot be in  $E(H)$ . Thus the outgoing edge of  $v^{left}$  in  $H$  is  $(v^{left}, u^i)$ .

Now consider the vertex  $w^{left}$ , which is the first one of the path  $P_w$ . By the same argument as above, we have  $(w^{left}, u^i) \in E(H)$ . This implies that  $u^i$  has two in-neighbors in  $H$ , a contradiction.

It is also impossible that  $U = \emptyset$  because then  $E(H) \cap \partial(C^{i \rightarrow j}) = \emptyset$  for each  $j \neq i$  implying that  $H$  is disconnected. Consequently,  $U$  contains exactly one element. ◀

We can summarize the arguments given so far as follows: for a Hamiltonian cycle  $H$  in  $G'$  and  $i \in [k]$ , there exists  $v_i \in V^i$ , such that, for all  $j \neq i$ , it holds that  $E(H) \cap \partial(C^{i \rightarrow j}) = E(P_{v_i}, C^{i \rightarrow j}) = \{e_{v_i}^{i \rightarrow j, out}, e_{v_i}^{i \rightarrow j, in}\}$ . We make note of a simple implication of this fact.

► **Lemma 13.** *Let  $H$  be a Hamiltonian cycle in  $G'$  and  $i, j \in [k]$ . Let  $v \in V^i$  satisfy  $E(H) \cap \partial(C^{i \rightarrow j}) = \{e_v^{i \rightarrow j, out}, e_v^{i \rightarrow j, in}\}$ . Then  $e = (v^{j, out}, v^{j, in})$  does not belong to  $E(H)$ .*

**Proof.** Since  $E(H) \cap \partial(C^{i \rightarrow j})$  comprises exactly two edges, we infer that  $H$  contains the path  $Q = v^{j,in} \rightarrow c_v^{i \rightarrow j,in} \rightarrow \dots \rightarrow c_v^{i \rightarrow j,out} \rightarrow v^{j,out}$  with all internal vertices from  $V(C^{i \rightarrow j})$ . If  $H$  traversed the edge  $e = (v^{j,out}, v^{j,in})$ , then it would contain the cycle  $C$  formed by  $Q$  and  $e$ . This would imply  $H = C$ , which contradicts that  $H$  is Hamiltonian.  $\blacktriangleleft$

We are going to show that  $H$  can include only one edge that goes from  $V(P^i)$  to  $V(P^j)$ ; it will follow that this edge must be  $(v_i^{j,out}, v_j^{i,out})$ .

► **Lemma 14.** *Let  $H$  be a Hamiltonian cycle in  $G'$ . For each pair  $i, j \in [k]$  with  $i < j$  we have  $|E(H) \cap E(P^i, P^j)| \leq 1$ .*

**Proof.** Suppose that  $|E(H) \cap E(P^i, P^j)| \geq 2$ . Then there exist  $u, w \in V^j$  and  $e_u \in \partial^{in}(u^{i,in}) \cap \partial^{out}(P^i)$ ,  $e_w \in \partial^{in}(w^{i,in}) \cap \partial^{out}(P^i)$  such that  $e_u, e_w \in E(H)$ . This implies that the edges  $(u^{i,out}, u^{i,in})$ ,  $(w^{i,out}, w^{i,in}) \in E(P^j)$  cannot be used by  $H$ . Since  $i < j$ , the vertex  $u^{i,out}$  has only one out-neighbor different than  $u^{i,in}$ : the terminal  $\hat{t}^{i \rightarrow j}$ . Hence  $(u^{i,out}, \hat{t}^{i \rightarrow j}) \in E(H)$ . But the same argument applies to  $w^{i,out}$ . As a consequence, two incoming edges of  $\hat{t}^{i \rightarrow j}$  are being used by  $H$  and so we arrive at a contradiction.  $\blacktriangleleft$

Finally, we prove the second implication in the correctness proof of the reduction.

► **Lemma 15.** *If  $(G', X)$  is a Yes-instance of HAMILTONIAN CYCLE BY DFVS then  $(G, (V^1, V^2, \dots, V^k))$  is a Yes-instance of MULTICOLORED CLIQUE.*

**Proof.** Let  $H$  be a Hamiltonian cycle in  $G'$  and  $v_i \in V^i$  be the vertex given by Lemma 12 for  $i \in [k]$ . Fix a pair of indices  $i < j$ . By Lemma 13 we know that the edge  $(v_i^{j,out}, v_i^{j,in})$  does not belong to  $E(H)$ . The remaining outgoing edges of  $v_i^{j,out}$  belong to  $E(P^i, P^j)$  and  $H$  must utilize one of them. By the same argument,  $H$  must traverse one of the incoming edges of  $v_j^{i,in}$  that belongs to  $E(P^i, P^j)$ . Due to Lemma 14, the Hamiltonian cycle  $H$  can use at most one edge from  $E(P^i, P^j)$ . Consequently, we obtain that  $(v_i^{j,out}, v_j^{i,in}) \in E(H)$ . In particular, this means that  $(v_i^{j,out}, v_j^{i,in})$  is present in  $E(G')$  and, by the construction of  $G'$ , implies that  $v_i v_j \in E(G)$ . Therefore,  $\{v_1, v_2, \dots, v_k\}$  forms a clique in  $G$ .  $\blacktriangleleft$

Lemmas 9 and 15 constitute that the instances  $(G, (V^1, V^2, \dots, V^k))$  and  $(G', X)$  are equivalent while Lemma 8 ensures that  $|X| = \mathcal{O}(k^2)$ . We have thus obtained a parameterized reduction from MULTICOLORED CLIQUE to HAMILTONIAN CYCLE BY DFVS, proving Theorem 1.

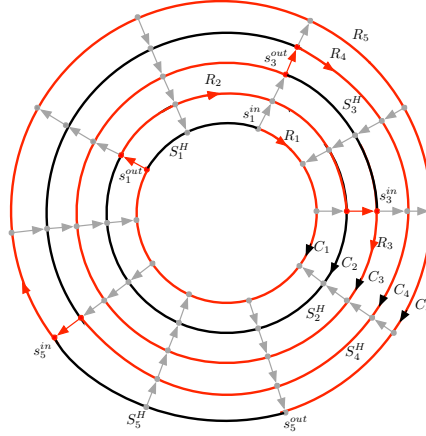
Due to space constraints, the W[1]-hardness proofs for LONGEST CYCLE ABOVE GIRTH and the path variants of both problems are provided in the full version.

#### 4 XP Algorithm for LONGEST PATH ABOVE GIRTH

Johnson et al. [58] proved that HAMILTONIAN PATH is in XP parameterized by directed treewidth. This result was later extended by de Oliveira Oliveira [30] to capture a wider range of problems expressible in Monadic Second Order (MSO) logic with counting constraints. As a special case of this theorem, we have the following.

► **Theorem 16** ([30]). *LONGEST PATH on directed graphs is in XP when parameterized by directed treewidth.*

We will use the Directed Grid Theorem (Theorem 7), which either returns a cylindrical wall of order  $\mathcal{O}(k)$  in  $G$  or concludes that the directed treewidth of  $G$  is bounded in terms of  $k$ . In the former case, we prove that a path of length  $g \cdot k$  always exists in  $G$ . In the latter case, we use Theorem 16 to solve the problem optimally by an XP algorithm.



■ **Figure 5** The path  $R$  (colored in red) in a subdivision of  $W_5$ , described in Lemma 20.

Throughout this section we abbreviate  $W = W_{2k+1}$  (the cylindrical wall of order  $2k + 1$ ). In order to establish correctness of the algorithm, we need to argue that any subdivision  $H$  of  $W$  admits a path  $k$  times longer than the girth of  $H$ .

We denote the  $2k + 1$  vertex disjoint cycles in  $W$  by  $C_1^W, C_2^W, \dots, C_{2k+1}^W$  counting from the innermost one. For a subdivision  $H$  of  $W$  we denote the counterpart of  $C_i^W$  in  $H$  as  $C_i^H$ . We refer to the girth of  $H$  as  $g$ . Note that  $g$  lower bounds the length of each cycle  $C_i^H$ .

► **Definition 17.** A subpath in  $C_i^W$  is called a segment of  $C_i^W$  if its endpoints have out-neighbors in  $C_{i+1}^W$  and none of its internal vertices has out-neighbors in  $C_{i+1}^W$ . A subpath in  $C_i^H$  is a segment of  $C_i^H$  if it is a subdivision of a segment in  $C_i^W$ .

We make note of a few properties of segments.

► **Observation 18.** For every  $i \in [2k + 1]$  the following hold.

1. Each cycle  $C_i^W$  is a cyclic concatenation of  $k + 1$  segments of length 4.
2. Each cycle  $C_i^H$  is a cyclic concatenation of  $k + 1$  segments.
3. If  $i > 1$  then each segment of  $C_i^W$  has a unique internal vertex with an in-neighbor in  $C_{i-1}^W$ .

We show that in every cycle  $C_i^H$  we can choose a segment  $S$  so that the path obtained by traversing all vertices in  $C_i^H$  that are not internal vertices of  $S$  is almost as long as  $C_i^H$ .

► **Lemma 19.** For a segment  $S$  of  $C_i^H$ , let  $\widehat{S}$  be the subpath of  $C_i^H$  that starts at the end of  $S$  and ends at the start of  $S$ . Then there exists a segment  $S_i^H$  of  $C_i^H$  such that  $\widehat{S}_i^H$  has length at least  $g - \frac{g}{k+1}$ .

**Proof.** By Observation 18 we know that  $C_i^H$  can be partitioned into  $k + 1$  segments. By a counting argument, there exist a segment  $S_i^H$  of size at most  $\frac{|E(C_i^H)|}{k+1}$ . Thus, for the path  $\widehat{S}_i^H$  complementing  $S_i^H$ , we have  $|E(\widehat{S}_i^H)| \geq |E(C_i^H)| \cdot (1 - \frac{1}{k+1}) \geq g \cdot (1 - \frac{1}{k+1})$ . ◀

Let  $s_i^{out}$  denote the first vertex on  $S_i^H$ . If  $i > 1$ , we define  $s_i^{in}$  to be the unique internal vertex of  $S_i^H$  corresponding to a vertex in  $C_i^W$  with an in-neighbor in  $C_{i-1}^W$ . If  $i = 1$ , we define  $s_i^{in}$  to be the last vertex on  $S_i^H$ .

► **Lemma 20.** Let  $H$  be a subdivision of  $W_{2k+1}$  and  $g$  be the girth of  $H$ . Then there exists a path of length at least  $g \cdot k$  in  $H$ .

**Proof.** For  $i \in [2k + 1]$  we define path  $R_i$  in  $C_i$  as follows. If  $i$  is odd, we set  $R_i$  to be the subpath of  $C_i$  from  $s_i^{in}$  to  $s_i^{out}$ . If  $i$  is even, the path  $R_i$  begins at the unique vertex in  $C_i^H$  that can be reached via a subdivided edge from  $s_{i-1}^{out}$  and  $R_i$  ends at the unique vertex in  $C_i^H$  from which  $s_{i+1}^{in}$  is reachable by via a subdivided edge. Note that the first and last indices in  $[2k + 1]$  are odd, so the paths  $R_i$  are well-defined. We can now concatenate  $R_1, R_2, \dots, R_{2k+1}$  using the subdivided edges between the cycles to obtain a path  $R$  in  $H$ ; see Figure 5.

We argue that  $R$  is sufficiently long. There are  $k + 1$  odd indices in  $[2k + 1]$ , and, for each of them,  $R_i$  contains the path  $\widehat{S}_i^H$  (from Lemma 19) of length at least  $g - \frac{g}{k+1}$ . Since these form vertex disjoint subpaths of  $R$ , we conclude that  $E(R) \geq (g - \frac{g}{k+1})(k + 1) = g(k + 1) - g = g \cdot k$ . ◀

We are ready to summarize the entire algorithm.

► **Theorem 3.** LONGEST PATH ABOVE GIRTH *is in XP.*

**Proof.** We first execute the algorithm from Theorem 7, which returns either a directed tree decomposition of width  $f(k)$ , for some computable function  $f$ , or a subgraph  $H$  of  $G$  that is a subdivision of  $W_{2k+1}$ . Note that the girth of  $H$  is at least the girth of  $G$ . In the first case, we apply the algorithm from Theorem 16 to find the longest path in  $G$  in polynomial time for fixed  $k$ . In the latter case, Lemma 20 asserts that  $H$  contains a path of length at least  $k$  times the girth of  $H$ , so we can report that  $(G, k)$  a Yes-instance. ◀

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## References

- 1 Isolde Adler. Directed tree-width examples. *Journal of Combinatorial Theory, Series B*, 97(5):718–725, 2007.
- 2 Takanori Akiyama, Takao Nishizeki, and Nobuji Saito. NP-completeness of the hamiltonian cycle problem for bipartite graphs. *Journal of Information processing*, 3(2):73–76, 1980.
- 3 Noga Alon and Shai Gutner. Balanced hashing, color coding and approximate counting. In *Parameterized and Exact Computation, 4th International Workshop, IWPEC 2009, Copenhagen, Denmark, September 10-11, 2009, Revised Selected Papers*, pages 1–16, 2009.
- 4 Noga Alon and Shai Gutner. Balanced families of perfect hash functions and their applications. *ACM Trans. Algorithms*, 6(3):54:1–54:12, 2010.
- 5 Noga Alon, Raphael Yuster, and Uri Zwick. Color-coding. *J. ACM*, 42(4):844–856, 1995.
- 6 Vikraman Arvind and Venkatesh Raman. Approximation algorithms for some parameterized counting problems. In *Algorithms and Computation: 13th International Symposium, ISAAC 2002 Vancouver, BC, Canada, November 21–23, 2002 Proceedings 13*, pages 453–464. Springer, 2002.
- 7 Benjamin Bergougnoux, Mamadou Moustapha Kanté, and O-joung Kwon. An optimal xp algorithm for hamiltonian cycle on graphs of bounded clique-width. *Algorithmica*, 82(6):1654–1674, 2020.
- 8 Ivona Bezáková, Radu Curticapean, Holger Dell, and Fedor V. Fomin. Finding detours is fixed-parameter tractable. *SIAM J. Discret. Math.*, 33(4):2326–2345, 2019.
- 9 Andreas Björklund. Determinant sums for undirected hamiltonicity. *SIAM J. Comput.*, 43(1):280–299, 2014. doi:10.1137/110839229.
- 10 Andreas Björklund. Exploiting sparsity for bipartite hamiltonicity. In *29th International Symposium on Algorithms and Computation, ISAAC 2018, December 16-19, 2018, Jiaoxi, Yilan, Taiwan*, pages 3:1–3:11, 2018.
- 11 Andreas Björklund. An asymptotically fast polynomial space algorithm for hamiltonicity detection in sparse directed graphs. In *38th International Symposium on Theoretical Aspects of Computer Science, STACS 2021, March 16-19, 2021, Saarbrücken, Germany (Virtual Conference)*, pages 15:1–15:12, 2021.

- 12 Andreas Björklund and Thore Husfeldt. The parity of directed hamiltonian cycles. In *54th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2013, 26-29 October, 2013, Berkeley, CA, USA*, pages 727–735, 2013.
- 13 Andreas Björklund, Thore Husfeldt, Petteri Kaski, and Mikko Koivisto. Counting paths and packings in halves. In *Algorithms-ESA 2009: 17th Annual European Symposium, Copenhagen, Denmark, September 7-9, 2009. Proceedings 17*, pages 578–586. Springer, 2009.
- 14 Andreas Björklund, Thore Husfeldt, Petteri Kaski, and Mikko Koivisto. Narrow sieves for parameterized paths and packings. *J. Comput. Syst. Sci.*, 87:119–139, 2017.
- 15 Andreas Björklund, Petteri Kaski, and Ioannis Koutis. Directed hamiltonicity and out-branchings via generalized laplacians. In *44th International Colloquium on Automata, Languages, and Programming, ICALP 2017, July 10-14, 2017, Warsaw, Poland*, pages 91:1–91:14, 2017.
- 16 Andreas Björklund and Ryan Williams. Computing permanents and counting hamiltonian cycles by listing dissimilar vectors. In *46th International Colloquium on Automata, Languages, and Programming, ICALP 2019, July 9-12, 2019, Patras, Greece*, pages 25:1–25:14, 2019.
- 17 Hans L. Bodlaender. On linear time minor tests with depth-first search. *Journal of Algorithms*, 14(1):1–23, 1993.
- 18 Cornelius Brand, Holger Dell, and Thore Husfeldt. Extensor-coding. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing (STOC)*, pages 151–164, 2018.
- 19 Cornelius Brand and Kevin Pratt. Parameterized applications of symbolic differentiation of (totally) multilinear polynomials. In *48th International Colloquium on Automata, Languages, and Programming, ICALP 2021, July 12-16, 2021, Glasgow, Scotland (Virtual Conference)*, pages 38:1–38:19, 2021.
- 20 Michael Buro. Simple amazons endgames and their connection to hamilton circuits in cubic subgrid graphs. In *Computers and Games: Second International Conference, CG 2000 Hamamatsu, Japan, October 26–28, 2000 Revised Papers 2*, pages 250–261. Springer, 2001.
- 21 Victor Campos, Raul Lopes, Ana Karolinna Maia, and Ignasi Sau. Adapting the directed grid theorem into an fpt algorithm. *Electronic Notes in Theoretical Computer Science*, 346:229–240, 2019.
- 22 Jianer Chen, Joachim Kneis, Songjian Lu, Daniel Mölle, Stefan Richter, Peter Rossmanith, Sing-Hoi Sze, and Fenghui Zhang. Randomized divide-and-conquer: Improved path, matching, and packing algorithms. *SIAM Journal on Computing*, 38(6):2526–2547, 2009.
- 23 Jianer Chen, Yang Liu, Songjian Lu, Barry O’Sullivan, and Igor Razgon. A fixed-parameter algorithm for the directed feedback vertex set problem. *J. ACM*, 55(5):21:1–21:19, 2008.
- 24 Bruno Courcelle and Joost Engelfriet. *Graph Structure and Monadic Second-Order Logic - A Language-Theoretic Approach*, volume 138 of *Encyclopedia of mathematics and its applications*. Cambridge University Press, 2012. doi:10.1017/CB09780511977619.
- 25 Radu Curticapean, Holger Dell, and Dániel Marx. Homomorphisms are a good basis for counting small subgraphs. In *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing (STOC)*, pages 210–223, 2017.
- 26 Radu Curticapean and Dániel Marx. Complexity of counting subgraphs: Only the boundedness of the vertex-cover number counts. In *2014 IEEE 55th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 130–139. IEEE, 2014.
- 27 Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer, 2015.
- 28 Marek Cygan, Stefan Kratsch, and Jesper Nederlof. Fast hamiltonicity checking via bases of perfect matchings. *Journal of the ACM (JACM)*, 65(3):1–46, 2018.
- 29 Marek Cygan, Jesper Nederlof, Marcin Pilipczuk, Michał Pilipczuk, Johan MM Van Rooij, and Jakub Onufry Wojtaszczyk. Solving connectivity problems parameterized by treewidth in single exponential time. *ACM Transactions on Algorithms (TALG)*, 18(2):1–31, 2022.



- 30 Mateus de Oliveira Oliveira. An algorithmic metatheorem for directed treewidth. *Discrete Applied Mathematics*, 204:49–76, 2016. doi:10.1016/j.dam.2015.10.020.
- 31 Reinhard Diestel. *Graph Theory, 4th Edition*, volume 173 of *Graduate texts in mathematics*. Springer, 2012.
- 32 Frederic Dorn, Fedor V Fomin, and Dimitrios M Thilikos. Catalan structures and dynamic programming in h-minor-free graphs. *Journal of Computer and System Sciences*, 78(5):1606–1622, 2012.
- 33 Martin Doucha and Jan Kratochvíl. Cluster vertex deletion: A parameterization between vertex cover and clique-width. In *MFCS*, volume 2012, pages 348–359. Springer, 2012.
- 34 Jörg Flum and Martin Grohe. The parameterized complexity of counting problems. *SIAM Journal on Computing*, 33(4):892–922, 2004.
- 35 Fedor V. Fomin, Petr A. Golovach, William Lochet, Danil Sagunov, Kirill Simonov, and Saket Saurabh. Detours in directed graphs. In *39th International Symposium on Theoretical Aspects of Computer Science, STACS 2022, March 15-18, 2022, Marseille, France (Virtual Conference)*, pages 29:1–29:16, 2022.
- 36 Fedor V. Fomin, Petr A. Golovach, Daniel Lokshtanov, Fahad Panolan, Saket Saurabh, and Meirav Zehavi. Multiplicative parameterization above a guarantee. *ACM Trans. Comput. Theory*, 13(3):18:1–18:16, 2021.
- 37 Fedor V. Fomin, Petr A. Golovach, Daniel Lokshtanov, Saket Saurabh, and Meirav Zehavi. Clique-width III: hamiltonian cycle and the odd case of graph coloring. *ACM Trans. Algorithms*, 15(1):9:1–9:27, 2019.
- 38 Fedor V. Fomin, Petr A. Golovach, Danil Sagunov, and Kirill Simonov. Algorithmic extensions of dirac’s theorem. In *Proceedings of the 2022 ACM-SIAM Symposium on Discrete Algorithms, SODA 2022, Virtual Conference / Alexandria, VA, USA, January 9 - 12, 2022*, pages 406–416, 2022.
- 39 Fedor V. Fomin, Petr A. Golovach, Danil Sagunov, and Kirill Simonov. Longest cycle above erdős-gallai bound. In *30th Annual European Symposium on Algorithms, ESA 2022, September 5-9, 2022, Berlin/Potsdam, Germany*, pages 55:1–55:15, 2022.
- 40 Fedor V. Fomin, Daniel Lokshtanov, Fahad Panolan, Saket Saurabh, and Meirav Zehavi. Long directed  $(s, t)$ -path: FPT algorithm. *Inf. Process. Lett.*, 140:8–12, 2018.
- 41 Fedor V. Fomin, Daniel Lokshtanov, Fahad Panolan, Saket Saurabh, and Meirav Zehavi. Decomposition of map graphs with applications. In *46th International Colloquium on Automata, Languages, and Programming, ICALP 2019, July 9-12, 2019, Patras, Greece*, pages 60:1–60:15, 2019.
- 42 Fedor V. Fomin, Daniel Lokshtanov, Fahad Panolan, Saket Saurabh, and Meirav Zehavi. Finding, hitting and packing cycles in subexponential time on unit disk graphs. *Discret. Comput. Geom.*, 62(4):879–911, 2019.
- 43 Steven Fortune, John Hopcroft, and James Wyllie. The directed subgraph homeomorphism problem. *Theoretical Computer Science*, 10(2):111–121, 1980. doi:10.1016/0304-3975(80)90009-2.
- 44 Harold N Gabow and Shuxin Nie. Finding a long directed cycle. *ACM Transactions on Algorithms (TALG)*, 4(1):1–21, 2008.
- 45 Robert Ganian. Improving vertex cover as a graph parameter. *Discrete Mathematics & Theoretical Computer Science*, 17, 2015.
- 46 Michael R Garey, David S Johnson, and Larry Stockmeyer. Some simplified np-complete problems. In *Proceedings of the sixth annual ACM symposium on Theory of computing (STOC)*, pages 47–63, 1974.
- 47 Archontia C. Giannopoulou, Ken-ichi Kawarabayashi, Stephan Kreutzer, and O-joung Kwon. The directed flat wall theorem. In *Proceedings of the 2020 ACM-SIAM Symposium on Discrete Algorithms, SODA 2020, Salt Lake City, UT, USA, January 5-8, 2020*, pages 239–258, 2020.
- 48 Archontia C. Giannopoulou, Ken-ichi Kawarabayashi, Stephan Kreutzer, and O-joung Kwon. Directed tangle tree-decompositions and applications. In *Proceedings of the 2022 ACM-SIAM Symposium on Discrete Algorithms, SODA 2022, Virtual Conference / Alexandria, VA, USA, January 9 - 12, 2022*, pages 377–405, 2022.

- 49 Petr A. Golovach, R. Krithika, Abhishek Sahu, Saket Saurabh, and Meirav Zehavi. Graph hamiltonicity parameterized by proper interval deletion set. In *LATIN 2020: Theoretical Informatics - 14th Latin American Symposium, São Paulo, Brazil, January 5-8, 2021, Proceedings*, pages 104–115, 2020.
- 50 Ronald J Gould. Advances on the hamiltonian problem—a survey. *Graphs and Combinatorics*, 19(1):7–52, 2003.
- 51 Gregory Gutin and Matthias Mnich. A survey on graph problems parameterized above and below guaranteed values. *arXiv preprint arXiv:2207.12278*, 2022.
- 52 Meike Hatzel, Ken-ichi Kawarabayashi, and Stephan Kreutzer. Polynomial planar directed grid theorem. In *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2019, San Diego, California, USA, January 6-9, 2019*, pages 1465–1484, 2019.
- 53 Meike Hatzel, Konrad Majewski, Michal Pilipczuk, and Marek Sokolowski. Simpler and faster algorithms for detours in planar digraphs. In *2023 Symposium on Simplicity in Algorithms, SOSA 2023, Florence, Italy, January 23-25, 2023*, pages 156–165, 2023.
- 54 Falk Hüffner, Sebastian Wernicke, and Thomas Zichner. Algorithm engineering for color-coding with applications to signaling pathway detection. *Algorithmica*, 52(2):114–132, 2008.
- 55 Paul Hunter and Stephan Kreutzer. Digraph measures: Kelly decompositions, games, and orderings. *Theor. Comput. Sci.*, 399(3):206–219, 2008.
- 56 Ashwin Jacob, Michal Włodarczyk, and Meirav Zehavi. Long directed detours: Reduction to 2-disjoint paths. *CoRR*, abs/2301.06105, 2023.
- 57 Bart M. P. Jansen, László Kozma, and Jesper Nederlof. Hamiltonicity below dirac’s condition. In *Graph-Theoretic Concepts in Computer Science - 45th International Workshop, WG 2019, Vall de Núria, Spain, June 19-21, 2019, Revised Papers*, pages 27–39, 2019.
- 58 Thor Johnson, Neil Robertson, Paul D. Seymour, and Robin Thomas. Directed tree-width. *J. Comb. Theory, Ser. B*, 82(1):138–154, 2001.
- 59 Georgia Kaouri, Michael Lampis, and Valia Mitsou. New directions in directed treewidth. *Parameterized Complexity News: Newsletter of the Parameterized Complexity Community*, September:4–5, 2009.
- 60 Richard M. Karp. Reducibility among combinatorial problems. In *Proceedings of a symposium on the Complexity of Computer Computations, held March 20-22, 1972, at the IBM Thomas J. Watson Research Center, Yorktown Heights, New York, USA*, pages 85–103, 1972.
- 61 Ken-ichi Kawarabayashi and Stephan Kreutzer. The directed grid theorem. In *Proceedings of the forty-seventh annual ACM symposium on Theory of Computing*, pages 655–664, 2015.
- 62 Ioannis Koutis. Faster algebraic algorithms for path and packing problems. In *Automata, Languages and Programming, 35th International Colloquium, ICALP 2008, Reykjavik, Iceland, July 7-11, 2008, Proceedings, Part I: Track A: Algorithms, Automata, Complexity, and Games*, pages 575–586, 2008.
- 63 Michael Lampis, Georgia Kaouri, and Valia Mitsou. On the algorithmic effectiveness of digraph decompositions and complexity measures. In *Algorithms and Computation, 19th International Symposium, ISAAC 2008, Gold Coast, Australia, December 15-17, 2008. Proceedings*, pages 220–231, 2008.
- 64 Michael Lampis, Georgia Kaouri, and Valia Mitsou. On the algorithmic effectiveness of digraph decompositions and complexity measures. *Discret. Optim.*, 8(1):129–138, 2011.
- 65 Daniel Lokshtanov, Andreas Björklund, Saket Saurabh, and Meirav Zehavi. Approximate counting of  $k$ -paths: Simpler, deterministic, and in polynomial space. *ACM Trans. Algorithms*, 17(3):26:1–26:44, 2021.
- 66 Daniel Lokshtanov, Matthias Mnich, and Saket Saurabh. Planar  $k$ -path in subexponential time and polynomial space. In *WG*, pages 262–270. Springer, 2011.
- 67 Burkhard Monien. How to find long paths efficiently. In *North-Holland Mathematics Studies*, volume 109, pages 239–254. Elsevier, 1985.

- 68 Jesper Nederlof. Detecting and counting small patterns in planar graphs in subexponential parameterized time. In *Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing, STOC 2020, Chicago, IL, USA, June 22-26, 2020*, pages 1293–1306, 2020.
- 69 Jan Obdržálek. Dag-width: connectivity measure for directed graphs. In *Proceedings of the Seventeenth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2006, Miami, Florida, USA, January 22-26, 2006*, pages 814–821, 2006.
- 70 Bruce A. Reed. Introducing directed tree width. *Electron. Notes Discret. Math.*, 3:222–229, 1999.
- 71 Neil Robertson and Paul D Seymour. Graph minors. XIII. The disjoint paths problem. *Journal of combinatorial theory, Series B*, 63(1):65–110, 1995. doi:10.1006/jctb.1995.1006.
- 72 Hadas Shachnai and Meirav Zehavi. Representative families: A unified tradeoff-based approach. *J. Comput. Syst. Sci.*, 82(3):488–502, 2016.
- 73 Aleksanders Slivkins. Parameterized tractability of edge-disjoint paths on directed acyclic graphs. *SIAM J. Discret. Math.*, 24(1):146–157, 2010. doi:10.1137/070697781.
- 74 Dekel Tsur. Faster deterministic parameterized algorithm for  $k$ -path. *Theor. Comput. Sci.*, 790:96–104, 2019.
- 75 Virginia Vassilevska and Ryan Williams. Finding, minimizing, and counting weighted subgraphs. In *Proceedings of the forty-first annual ACM symposium on Theory of computing (STOC)*, pages 455–464, 2009.
- 76 Sebastian Wiederrecht. A note on directed treewidth. *CoRR*, abs/1910.01826, 2019. arXiv:1910.01826.
- 77 Ryan Williams. Finding paths of length  $k$  in  $\mathcal{O}^*(2^k)$  time. *Inf. Process. Lett.*, 109(6):315–318, 2009.
- 78 Meirav Zehavi. Mixing color coding-related techniques. In *Algorithms - ESA 2015 - 23rd Annual European Symposium, Patras, Greece, September 14-16, 2015, Proceedings*, pages 1037–1049, 2015.
- 79 Meirav Zehavi. A randomized algorithm for long directed cycle. *Inf. Process. Lett.*, 116(6):419–422, 2016.
- 80 Meirav Zehavi, Fedor V. Fomin, Daniel Lokshtanov, Fahad Panolan, and Saket Saurabh. Eth-tight algorithms for long path and cycle on unit disk graphs. *J. Comput. Geom.*, 12(2):126–148, 2021.