The Tight Spanning Ratio of the Rectangle Delaunay Triangulation

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───── **Abstract** ─────

Spanner construction is a well-studied problem and Delaunay triangulations are among the most popular spanners. Tight bounds are known if the Delaunay triangulation is constructed using an equilateral triangle, a square, or a regular hexagon. However, all other shapes have remained elusive. In this paper we extend the restricted class of spanners for which tight bounds are known. We prove that Delaunay triangulations constructed using rectangles with aspect ratio A have spanning ratio at most $\sqrt{2}\sqrt{1 + A^2 + A\sqrt{A^2 + 1}}$, which matches the known lower bound.

1 Introduction

A geometric graph is a weighted graph in the plane where every vertex v has coordinates (x_v, y_v) and the weight of an edge between any two vertices is the Euclidean distance between its endpoints. A *geometric spanner* is defined to be a class of subgraphs where the shortest path distance between any two vertices is at most the Euclidean distance between these two vertices multiplied by a constant t. The smallest constant t for which this property holds is called the *spanning ratio* or *stretch factor* of the geometric spanner. A comprehensive overview on the topic of geometric spanners can be found in the book by Narasimhan and Smid [11] and the survey by Bose and Smid [5].

One way to construct a geometric spanner is by using a Delaunay triangulation. The Delaunay triangulation is defined as follows: for any two vertices u and v, if there exists a circle with u and v on its boundary and no other vertex in its interior, then the edge between u and v is part of the Delaunay triangulation. Equivalently, this can be defined using three vertices u, v, and w, where the triangle connecting these three vertices is part of the Delaunay triangulation if and only if the unique circle through these three points does not contain any other vertices in its interior. For simplicity, it is usually assumed that no three points are collinear and no four points lie on the boundary of the circle.

The tight spanning ratio of the Delaunay triangulation is not known. Dobkin et al. [8] showed an upper bound of $\pi(1 + \sqrt{5})/2 \approx 5.09$ for the spanning ratio, which Keil and Gutwin [9] improved to $4\pi/3\sqrt{3} \approx 2.42$. Currently, the best upper bound is 1.998, proven by Xia [14]. A lower bound on the spanning ratio was provided by Bose et al. [4], who showed that this is strictly larger than $\frac{\pi}{2}$. This was later improved to 1.59 by Xia and Zhang [15].

Usually, the distance between two points u and v in the plane is defined as $((x_u - x_v)^2 + (y_u - y_v)^2)^{\frac{1}{2}}$. This distance can be generalized to a family of metrics L_p where the distance between u and v is defined as $((x_u - x_v)^p + (y_u - y_v)^p)^{\frac{1}{p}}$. The shape of a "circle" varies in different metrics, leading to different Delaunay triangulations in different metrics. For example, the shape of the "circle" would be a diamond or square in the L_1 and L_∞ metrics.

In 1986, Lee and Lin [10] introduced the notion of generalized Delaunay triangulations. Instead of using a circle to construct the graph, generalized Delaunay triangulations can be constructed using arbitrary geometric shapes. It was proven that any generalized Delaunay triangulation constructed using a convex shape is a spanner [2].

Although generalized Delaunay triangulations using arbitrary convex shapes are known to be spanners, their spanning ratios are less well understood. Tight bounds on the spanning ratio are only known when an equilateral triangle, a square, or a regular hexagon is used in the construction. When using equilateral triangles, Chew [7] showed that the spanning ratio is 2. When using squares, Chew [6] showed an upper bound of $\sqrt{10} \approx 3.16$, and Bonichon et al. [1] showed a matching upper and lower bound of $\sqrt{4 + 2\sqrt{2}} \approx 2.61$. When using regular hexagons, Perković et al. [12] showed a tight bound of 2.

Bose et al. [3] studied generalized Delaunay triangulations using rectangles. For rectangles with aspect ratio A, they showed an upper bound of $\sqrt{2}(2A + 1)$ and a lower bound of $\sqrt{2}\sqrt{1 + A^2 + A\sqrt{A^2 + 1}}$. Inspired by the proof of Bonichon et al. [1], by significantly extending and generalizing their approach we obtain a tight bound of $\sqrt{2}\sqrt{1 + A^2 + A\sqrt{A^2 + 1}}$. This extends the class of shapes for which a tight bound is known for the spanning ratio of generalized Delaunay triangulations. We note that the proof of our result is not a straightforward extension of Bonichon et al. [1], as we cannot simply rotate our lemmas to get them to prove both the horizontal and vertical cases simultaneously.

2 Preliminaries

Let us first formally define the rectangle Delaunay triangulation of a set of points P. Given an arbitrary axis-aligned rectangle R, the rectangle Delaunay triangulation is constructed by considering scaled translates of R (rotations are not allowed). Such scaled translates are also referred to as homothets. Given two vertices u and v in P, the rectangle Delaunay triangulation contains an edge between u and v if and only if there exists a scaled translate of R with u and v on its boundary which contains no vertices of P in the interior. Equivalently, the rectangle Delaunay triangulation contains a triangle $\triangle uvw$ if and only if there exists a scaled translate of R with u, v, and w on its boundary which contains no vertices of P in the interior. We note that different rectangles can give different rectangle Delaunay triangulations.

For our proofs, we assume that P is in general position. Specifically, we assume that no four vertices lie on the boundary of any scaled translate of R and that no two vertices lie on a line parallel to any of the sides of R (i.e., no two vertices lie on a vertical or horizontal line). These assumptions are common for Delaunay graphs and are required to guarantee their planarity.

Throughout this paper, we use A to denote the aspect ratio of the rectangle R used in the construction of the rectangle Delaunay triangulation, i.e., $A = l/s$ where l and s are the length of the long and short side of R respectively. We also use $d_t(u, v)$ to denote the length of the shortest path in the rectangle Delaunay triangulation between u and v, $d_x(u, v)$ to denote the difference in x-coordinate between u and v, $d_y(u, v)$ to denote the difference in y-coordinate between u and v, and $d_2(u, v)$ to denote the Euclidean distance between u and v.

Due to space constraints, we have deferred proofs of several lemmas to the full version of this paper [13].

3 Bounding the Spanning Ratio

To show an upper bound on the spanning ratio between any two vertices u and v, we consider the sequence of triangles $T_1, T_2, ..., T_k$ intersecting with line segment uv. The order of this sequence is determined by the order in which these triangles are encountered when following uv from u to v (as shown in Figure 1). Each triangle except T_1 and T_k intersects the interior of uv twice. Hence, we can define the last line segment of T_i ($1 \leq i < k$) that intersects uv as the line segment involved in the second intersection. We use h_i and l_i to denote the endpoints of the last line segment of T_i, where h_i is the endpoint above uv and l_i is the endpoint below uv. Since all T_i are triangles, we have that for every T_i and T_{i+1}, either $l_i = l_{i+1}$ or $h_i = h_{i+1}$. We also define $h_0 = l_0 = u$, $l_k = h_k = v$.
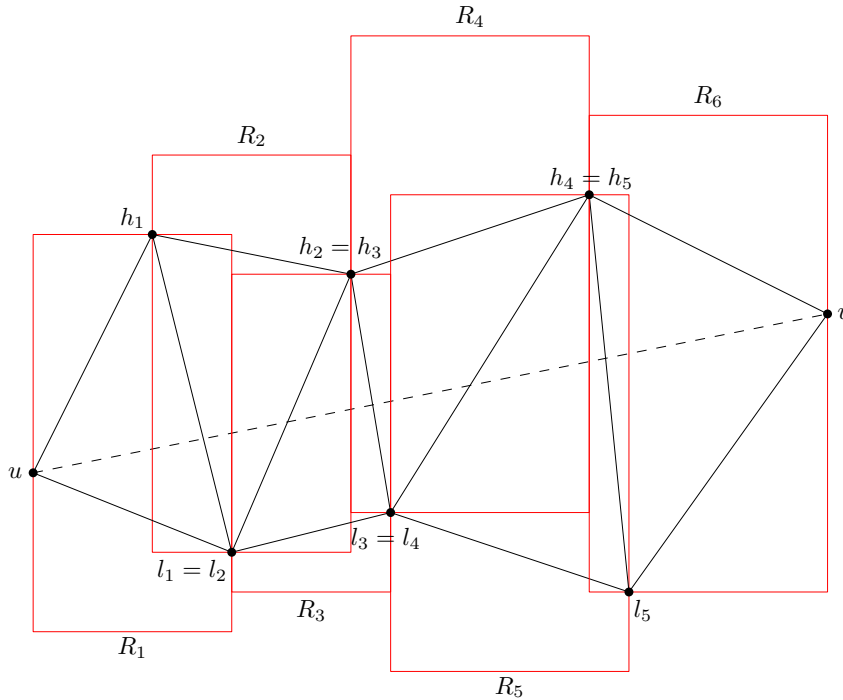
Figure 1 The triangles intersecting uv and their associated rectangles and h_i and l_i.

Each triangle T_i also has an associated rectangle R_i: the scaled translate of R that has the three vertices of T_i on its boundary. For ease of reference, we use W (west), N (north), E (east), and S (south) to refer to the four sides of a rectangle. We also use these sides to classify an edge, for example, if an endpoint of an edge lies on the W side of R_i and the other endpoint lies on the N side of R_i, we call the edge a WN edge. We also define u to be on the E side of R_0 (not associated with any triangle), as this will simplify some of the lemma statements.

Define L to be the length of the vertical side of R divided by the length of the horizontal size of R. Note that L can be either A or $1/A$. For our proofs, it is helpful to distinguish between edges of slope less than the slope of the diagonal of R and those with larger slope.

▶ **Definition 1.** *An edge is gentle if it has a slope within [-L, L]. Otherwise it is steep.*

We let u, v be any two vertices in the rectangle Delaunay triangulation. Fix the (x, y)-coordinate system so that we have $Ld_x(u, v) > d_y(u, v)$. Note that this is without loss of generality, since we can simply switch the x- and y-axes if needed. This implies that if we consider a scaled translate of R with u lying in the lower left corner and passing through v, then v lies on the E side. Without loss of generality, we assume u to be at the origin $(0, 0)$ and v to be at (x, y). We use $R(u, v)$ to denote the rectangle with u and v in opposite corners.

In order to bound the spanning ratio of the rectangle Delaunay triangulation, we first define what it means for a rectangle to have potential. This later helps us to bound the total length of the shortest path between u and v in the rectangle Delaunay triangulation.

▶ **Definition 2.** *The inductive point c of a rectangle R_i is the point with larger x-coordinate out of h_i and l_i. Rectangle R_i is inductive if edge (l_i, h_i) is gentle.*

▶ **Definition 3.** *A rectangle R_i has potential if $d_t(u, h_i) + d_t(u, l_i) + d_{R_i}(h_i, l_i) \leq (2 + 2L)x_i$ where $d_{R_i}(h_i, l_i)$ is the Euclidean distance when moving clockwise from h_i to l_i along the sides of R_i and x_i is the x-coordinate of the E side of R_i.*

We are now ready to prove that rectangles that are not inductive pass on their potential.

▶ **Lemma 4.** *If $R(u, v)$ is empty and (u, v) is not an edge in the rectangle Delaunay triangulation, then R_1 has potential. Furthermore, for any $1 \leq i < k$, if R_i has potential but is not inductive, then R_{i+1} has potential.*

Next, we bound the distance from u to the inductive point of a rectangle with potential when this inductive point lies on the E side of the rectangle.

▶ **Lemma 5.** *If rectangle R_i has potential and its inductive point c $(c = h_i$ or $c = l_i)$ lies on the E side of R_i, then $d_t(u, c) \leq (1 + L)x_c$.*

Now we shift our focus to paths consisting of gentle edges (see Figure 2).

▶ **Definition 6.** *If h_j is on the E side of R_j, the maximal high path ending at h_j is h_j itself; otherwise, it is the path $h_i, h_{i+1}, ..., h_j$ such that h_m is not on the E side of R_m (for $i < m \leq j$) and either $i = 0$ or h_i is on the E side of R_i.*

If l_j is on the E side of R_j, the maximal low path ending at l_j is l_j; otherwise, it is the path $l_i, l_{i+1}, ..., l_j$ such that l_m is not on the E side of R_m (for $i < m \leq j$) and either $i = 0$ or l_i is on the E side of R_i.

Next, we bound the length of these maximal high and maximal low paths.

▶ **Lemma 7.** *If the path $h_i, h_{i+1}, ..., h_j$ is a maximal high path then $d_t(h_i, h_j) \leq (x_{h_j} - x_{h_i}) + (y_{h_j} - y_{h_i})$. Similarly, if the path $l_i, l_{i+1}, ..., l_j$ is a maximal low path then $d_t(l_i, l_j) \leq (x_{l_j} - x_{l_i}) + (y_{l_i} - y_{l_j})$.*

We now use the above lemmas to prove bounds on the path length from u to the inductive point on the first inductive rectangle (if one exists) when $R(u, v)$ does not contain any vertices. Note that in Property 2 of Lemma 8, we differentiate between $L = A$ and $L = 1/A$, which is crucial in proving Theorem 10.

▶ **Lemma 8.** *Let $R(u, v)$ not contain any vertices of P and let (u, v) not be an edge of the rectangle Delaunay triangulation. The following properties hold:*
1. *If no rectangle in $R_1, ..., R_k$ is inductive then*

$$d_t(u, v) \leq (L + \sqrt{L^2 + 1})x + y.$$
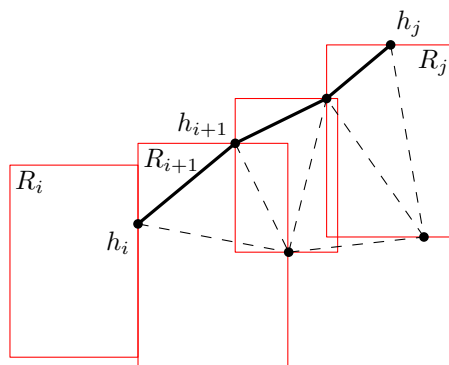
Figure 2 An example of a maximal high path (thick edges). The other edges of the triangles are shown using dashed line segments.

2. *Otherwise, let R_j be the first inductive rectangle in the sequence $R_1, ..., R_k$.*

   a. *If h_j is the inductive point of R_j and $L = A$, then*

   $$d_t(u, h_j) + (y_{h_j} - y) \leq (A + \sqrt{A^2 + 1})x_{h_j}.$$

   b. *If h_j is the inductive point of R_j and $L = \frac{1}{A}$, then*

   $$d_t(u, h_j) + A(y_{h_j} - y) \leq \left(1 + \sqrt{\frac{1}{A^2} + 1}\right) x_{h_j}.$$

   c. *If l_j is the inductive point of R_j and $L = A$, then*

   $$d_t(u, l_j) - y_{l_j} \leq (A + \sqrt{A^2 + 1})x_{l_j}.$$

   d. *If l_j is the inductive point of R_j and $L = \frac{1}{A}$, then*

 $$d_t(u, l_j) - Ay_{l_j} \leq \left(1 + \sqrt{\frac{1}{A^2} + 1}\right) x_{l_j}.$$

Proof. *Property 1:* By Lemma 4, if no rectangle in $R_1, ..., R_k$ is inductive then the last rectangle must have potential since R_1 has potential. Since no two vertices have the same y-coordinate, v must lie on the E side of the last rectangle. Thus, we can use Lemma 5 to conclude that $d_t(u, v) \leq (1 + L)x \leq (L + \sqrt{L^2 + 1})x + y$.

Property 2a: We consider the situation where R_j is the first inductive rectangle in the sequence $R_1, ..., R_k$. Let $l_i, ..., l_{j-1} = l_j$ be the maximal low path ending at l_j, and recall that h_j is the inductive point of R_j. By Lemma 4 we know that R_i has potential, since R_1 has potential and no rectangle before R_i is inductive. Since R_i has potential and l_i is on the E side of R_i, by Lemma 5 we know $d_t(u, l_i) \leq (1 + L)x_{l_i}$. See Figure 3. Since $L = A$, we have

$$d_t(u, h_j) + (y_{h_j} - y) \leq d_t(u, l_i) + d_t(l_i, l_j) + d_2(l_j, h_j) + (y_{h_j} - y)$$
$$\leq (1 + A)x_{l_i} + d_t(l_i, l_j) + d_2(l_j, h_j) + y_{h_j}.$$

Since $l_i, ..., l_{j-1} = l_j$ is a maximal low path, by Lemma 7 we know $d_t(l_i, l_j) \leq (x_{l_j} - x_{l_i}) + (y_{l_i} - y_{l_j})$. Hence, we obtain that:

$$d_t(u, h_j) + (y_{h_j} - y) \leq (1 + A)x_{l_i} + (x_{l_j} - x_{l_i}) + (y_{l_i} - y_{l_j}) + d_2(l_j, h_j) + y_{h_j}$$
$$= Ax_{l_i} + x_{l_j} + (y_{l_i} - y_{l_j}) + d_2(l_j, h_j) + y_{h_j}.$$
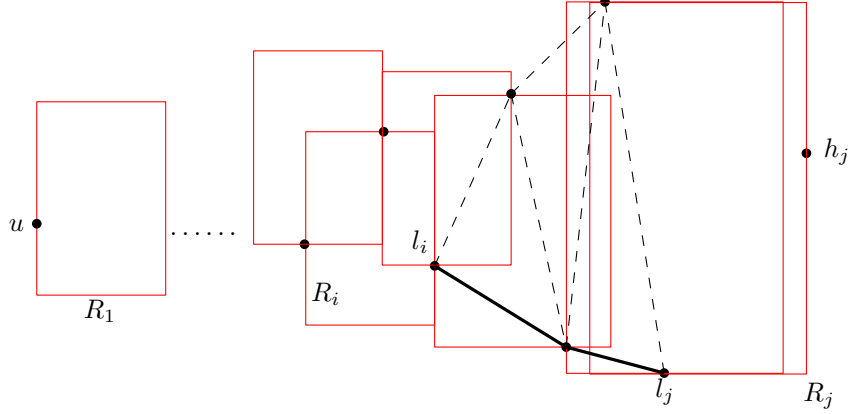
Figure 3 In Property 2a, R_j is the first inductive rectangle, h_j is the inductive point of R_j, and $l_i, ..., l_{j-1} = l_j$ is the maximal low path ending at l_j. R_i has potential and l_i is on the E side of R_i.

Because R_j is inductive, we know that edge (l_j, h_j) is gentle. Therefore, $d_2(l_j, h_j) \leq \sqrt{1 + A^2}(x_{h_j} - x_{l_j})$ and thus:

$$d_t(u, h_j) + (y_{h_j} - y) \leq Ax_{l_i} + x_{l_j} + (y_{l_i} - y_{l_j}) + \sqrt{1 + A^2}(x_{h_j} - x_{l_j}) + y_{h_j}$$
$$\leq Ax_{l_i} + (y_{l_i} - y_{l_j}) + \sqrt{1 + A^2}x_{h_j} + y_{h_j}.$$

Furthermore, again because edge (l_j, h_j) is gentle, we have that $y_{h_j} - y_{l_j} \leq A(x_{h_j} - x_{l_j})$ and therefore:

$$d_t(u, h_j) + (y_{h_j} - y) \leq Ax_{l_i} + y_{l_i} + \sqrt{1 + A^2}x_{h_j} + A(x_{h_j} - x_{l_j})$$
$$\leq y_{l_i} + \sqrt{1 + A^2}x_{h_j} + Ax_{h_j}.$$

Note that when $i \leq j$, then $x_{l_i} \leq x_{l_j}$. Finally, since $R(u, v)$ is empty, l_i must lie below it and thus $y_{l_i} < 0$, which leads to: $d_t(u, h_j) + (y_{h_j} - y) \leq (A + \sqrt{A^2 + 1})x_{h_j}$.

Property 2b: Let R_j be the first inductive rectangle in the sequence $R_1, ..., R_k$. Let $l_i, ..., l_{j-1} = l_j$ be the maximal low path ending at l_j, and recall that h_j is the inductive point of R_j. By Lemma 4, R_i has potential, and by Lemma 5, we have $d_t(u, l_i) \leq (1 + L)x_{l_i}$. Since $L = \frac{1}{A}$, we have

$$d_t(u, h_j) + A(y_{h_j} - y) \leq d_t(u, l_i) + d_t(l_i, l_j) + d_2(l_j, h_j) + A(y_{h_j} - y)$$
$$\leq (1 + \tfrac{1}{A})x_{l_i} + d_t(l_i, l_j) + d_2(l_j, h_j) + Ay_{h_j}.$$

Since $l_i, ..., l_{j-1} = l_j$ is a maximal low path, by Lemma 7 we know $d_t(l_i, l_j) \leq (x_{l_j} - x_{l_i}) + (y_{l_i} - y_{l_j})$. Because R_j is inductive, we know that edge (l_j, h_j) is gentle. Therefore, $d_2(l_j, h_j) \leq \sqrt{1 + \frac{1}{A^2}}(x_{h_j} - x_{l_j})$ and thus:

$$d_t(u, h_j) + A(y_{h_j} - y) \leq (1 + \tfrac{1}{A})x_{l_i} + (x_{l_j} - x_{l_i}) + (y_{l_i} - y_{l_j}) + d_2(l_j, h_j) + Ay_{h_j}$$
$$\leq (1 + \tfrac{1}{A})x_{l_i} + (x_{l_j} - x_{l_i}) + (y_{l_i} - y_{l_j}) + \sqrt{1 + \tfrac{1}{A^2}}(x_{h_j} - x_{l_j}) + Ay_{h_j}$$
$$\leq \tfrac{1}{A}x_{l_i} + (y_{l_i} - y_{l_j}) + \sqrt{1 + \tfrac{1}{A^2}}x_{h_j} + Ay_{h_j}.$$

Again because edge (l_j, h_j) is gentle, we have that $y_{h_j} - y_{l_j} \leq \frac{1}{A}(x_{h_j} - x_{l_j})$. Therefore $A(y_{h_j} - y_{l_j}) \leq (x_{h_j} - x_{l_j})$. We have $A \geq 1$ and therefore:

$$d_t(u, h_j) + A(y_{h_j} - y) \leq \frac{1}{A}x_{l_i} + A(y_{l_i} - y_{l_j}) + \sqrt{1 + \frac{1}{A^2}}x_{h_j} + Ay_{h_j}$$

$$\leq \frac{1}{A}x_{l_i} + Ay_{l_i} + (x_{h_j} - x_{l_j}) + \sqrt{1 + \frac{1}{A^2}}x_{h_j}.$$

Since $\frac{1}{A} \leq 1$ and $i \leq j$, we have $\frac{1}{A}x_{l_i} \leq x_{l_i} \leq x_{l_j}$. Therefore

$$d_t(u, h_j) + A(y_{h_j} - y) \leq Ay_{l_i} + x_{h_j} + \sqrt{1 + \frac{1}{A^2}}x_{h_j} \leq (1 + \sqrt{\frac{1}{A^2} + 1})x_{h_j}.$$

Property 2c: Let R_j be the first inductive rectangle in the sequence $R_1, ..., R_k$. Now, let $h_i, ..., h_{j-1} = h_j$ be the maximal high path ending at h_j, and recall that l_j is the inductive point of R_j. By Lemma 4, R_i has potential, and by Lemma 5, we have $d_t(u, h_i) \leq (1+L)x_{h_i}$. Since $L = A$,

$$d_t(u, l_j) - y_{l_j} \leq d_t(u, h_i) + d_t(h_i, h_j) + d_2(h_j, l_j) - y_{l_j}$$

$$\leq (1+A)x_{h_i} + d_t(h_i, h_j) + d_2(h_j, l_j) - y_{l_j}.$$

Since $h_i, ..., h_{j-1} = h_j$ is a maximal high path, by Lemma 7 we know $d_t(h_i, h_j) \leq (x_{h_j} - x_{h_i}) + (y_{h_j} - y_{h_i})$. It follows that:

$$d_t(u, l_j) - y_{l_j} \leq (1+A)x_{h_i} + (x_{h_j} - x_{h_i}) + (y_{h_j} - y_{h_i}) + d_2(h_j, l_j) - y_{l_j}$$

$$= Ax_{h_i} + x_{h_j} + (y_{h_j} - y_{h_i}) + d_2(h_j, l_j) - y_{l_j}.$$

Because R_j is inductive, we know that edge (l_j, h_j) is gentle. Therefore, $d_2(h_j, l_j) \leq \sqrt{1 + A^2}(x_{l_j} - x_{h_j})$ and thus:

$$d_t(u, l_j) - y_{l_j} \leq Ax_{h_i} + x_{h_j} + (y_{h_j} - y_{h_i}) + \sqrt{1 + A^2}(x_{l_j} - x_{h_j}) - y_{l_j}$$

$$\leq Ax_{h_i} + (y_{h_j} - y_{h_i}) + \sqrt{1 + A^2}x_{l_j} - y_{l_j}.$$

Furthermore, again because edge (l_j, h_j) is gentle, we have that $y_{h_j} - y_{l_j} \leq A(x_{l_j} - x_{h_j})$ and therefore:

$$d_t(u, l_j) - y_{l_j} \leq Ax_{h_i} - y_{h_i} + \sqrt{1 + A^2}x_{l_j} + A(x_{l_j} - x_{h_j})$$

$$\leq -y_{h_i} + \sqrt{1 + A^2}x_{l_j} + Ax_{l_j}.$$

Finally, since $R(u, v)$ is empty, h_i must lie above it and thus $y_{h_i} > 0$, which leads to $d_t(u, l_j) - y_{l_j} \leq (A + \sqrt{A^2 + 1})x_{l_j}$.

Property 2d: Let R_j be the first inductive rectangle in the sequence $R_1, ..., R_k$. Now, let $h_i, ..., h_{j-1} = h_j$ be the maximal high path ending at h_j, and recall that l_j is the inductive point of R_j. By Lemma 4, R_i has potential, and by Lemma 5, we have $d_t(u, h_i) \leq (1+L)x_{h_i}$. Since $L = \frac{1}{A}$,

$$d_t(u, l_j) - Ay_{l_j} \leq d_t(u, h_i) + d_t(h_i, h_j) + d_2(h_j, l_j) - Ay_{l_j}$$

$$\leq (1 + \frac{1}{A})x_{h_i} + d_t(h_i, h_j) + d_2(h_j, l_j) - Ay_{l_j}.$$

Since $h_i, ..., h_{j-1} = h_j$ is a maximal high path, by Lemma 7 we know $d_t(h_i, h_j) \leq (x_{h_j} - x_{h_i}) + (y_{h_j} - y_{h_i})$. Because edge (l_j, h_j) is gentle, we have that $d_2(h_j, l_j) \leq \sqrt{1 + \frac{1}{A^2}}(x_{l_j} - x_{h_j})$. It follows that:

$$d_t(u, l_j) - Ay_{l_j} \leq (1 + \tfrac{1}{A})x_{h_i} + (x_{h_j} - x_{h_i}) + (y_{h_j} - y_{h_i}) + d_2(h_j, l_j) - Ay_{l_j}$$

$$\leq (1 + \tfrac{1}{A})x_{h_i} + (x_{h_j} - x_{h_i}) + (y_{h_j} - y_{h_i}) + \sqrt{1 + \tfrac{1}{A^2}}(x_{l_j} - x_{h_j}) - Ay_{l_j}$$

$$\leq \tfrac{1}{A}x_{h_i} + (y_{h_j} - y_{h_i}) + \sqrt{1 + \tfrac{1}{A^2}}x_{l_j} - Ay_{l_j}.$$

Again because edge (l_j, h_j) is gentle, we have that $y_{h_j} - y_{l_j} \leq \tfrac{1}{A}(x_{l_j} - x_{h_j})$. Therefore $A(y_{h_j} - y_{l_j}) \leq (x_{l_j} - x_{h_j})$ and

$$d_t(u, l_j) - Ay_{l_j} \leq \tfrac{1}{A}x_{h_i} + A(y_{h_j} - y_{h_i}) + \sqrt{1 + \tfrac{1}{A^2}}x_{l_j} - Ay_{l_j}$$

$$\leq \tfrac{1}{A}x_{h_i} + (x_{l_j} - x_{h_j}) - Ay_{h_i} + \sqrt{1 + \tfrac{1}{A^2}}x_{l_j}.$$

Since $\tfrac{1}{A} \leq 1$, we have $\tfrac{1}{A}x_{h_i} \leq x_{h_i} \leq x_{h_j}$. Thus

$$d_t(u, l_j) - Ay_{l_j} \leq x_{l_j} - Ay_{h_i} + \sqrt{1 + \tfrac{1}{A^2}}x_{l_j}$$

$$\leq (1 + \sqrt{\tfrac{1}{A^2} + 1})x_{l_j}.$$

as required, completing our proof of Property 1, 2a, 2b, 2c and 2d. ◄

Our final ingredient determines the types of edges we can encounter when the y-coordinate of a vertex differs significantly from that of v.

▶ **Lemma 9.** *Let $R(u, v)$ not contain any vertices of P and let the coordinates of the inductive point c of R_i be such that it satisfies $0 < L(x - x_c) < |y - y_c|$.*
- *If $c = h_i$ and thus $0 < L(x - x_c) < y_c - y$, then let j be the smallest index larger than i such that $L(x - x_{h_j}) \geq y_c - y \geq 0$. All edges on the path $h_i, ..., h_j$ are NE edges.*
- *If $c = l_i$ and thus $0 < L(x - x_c) < y - y_c$, then let j be the smallest index larger than i such that $L(x - x_{l_j}) \geq y - y_c \geq 0$. All edges on the path $l_i, ..., l_j$ are SE edges.*

We now have all the ingredients needed to prove our main result. Recall that, up to Lemma 9, the (x, y)-coordinate system is fixed so that $Ld_x(u, v) > d_y(u, v)$, i.e. $Lx \geq y$. However, for ease of exposition, in Theorem 10 we instead fix the (x, y)-coordinate system so that all the homothet rectangles have their vertical sides being the long sides.

Note that in Lemma 8, we obtain different upper bounds depending on whether $L = A$ or $L = 1/A$. These two cases must be treated differently for the inductive proof of Theorem 10 to hold. In particular, in Theorem 10 the bound for $Ad_x(u, v) \geq d_y(u, v)$ does not coincide with the rotated version of the bound for $Ad_x(u, v) < d_y(u, v)$.

▶ **Theorem 10.** *Let u, v be any two vertices in the rectangle Delaunay triangulation. If $Ad_x(u, v) \geq d_y(u, v)$, then*

$$d_t(u, v) \leq (A + \sqrt{A^2 + 1})x + y.$$

Otherwise,

$$d_t(u, v) \leq Ax + \left(1 + \sqrt{\tfrac{1}{A^2} + 1}\right)y.$$

Proof. We consider all pairs of vertices (u, v) and order them by the size of the smallest scaled translate of R that has both u and v on its boundary. We perform induction based on the rank in this ordering.

The first pair (u, v) in this ordering has the smallest overall scaled translate of R and can thus contain no vertices of P, as any such vertex would imply the existence of a smaller rectangle with two vertices on its boundary, contradicting that we are considering the smallest one. Hence, by construction there exists an edge between u and v and thus $d_t(u, v) = d_2(u, v) \leq x + y$, satisfying the induction hypothesis, regardless of whether or not $Ad_x(u, v) \geq d_y(u, v)$.

Next, consider an arbitrary pair (u, v) and assume the theorem holds for all pairs (u, v) defining a smaller rectangle. We consider two cases: $R(u, v)$ does not contain any vertex of P, and $R(u, v)$ contains some vertices of P.

Case 1: There are no vertices inside $R(u, v)$. We distinguish two subcases, either $Ax \geq y$ or $Ax < y$.

Subcase $Ax \geq y$: Note that since the vertical side of the homothets is the longer side, for the (x, y)-coordinate system we have $L = A$, and $Lx \geq y$.

If (u, v) is an edge in the rectangle Delaunay triangulation, then $d_t(u, v) \leq x + y \leq (A + \sqrt{A^2 + 1})x + y$. Otherwise, if no rectangle in $R_1, ..., R_k$ is inductive then by Property 1 of Lemma 8 we know $d_t(u, v) \leq (A + \sqrt{A^2 + 1})x + y$.

Hence, we focus on the case where there is an inductive rectangle. Let R_i be the first inductive rectangle in the sequence $R_1, ..., R_k$. We distinguish the case where the inductive point is h_i and where it is l_i. If h_i is the inductive point of R_i then by Property 2a of Lemma 8 we know $d_t(u, h_i) + (y_{h_i} - y) \leq (A + \sqrt{A^2 + 1})x_{h_i}$ and thus $d_t(u, h_i) \leq (A + \sqrt{A^2 + 1})x_{h_i} - (y_{h_i} - y)$.

If $A(x - x_{h_i}) \geq y_{h_i} - y \geq 0$, we let $h_j = h_i$ in the remainder. Otherwise, we let j be the smallest index larger than i such that $A(x - x_{h_j}) \geq y_{h_j} - y \geq 0$. By Lemma 9, h_j exists and all edges on the path $h_i, ..., h_j$ are NE edges. By triangle inequality, $d_t(h_m, h_{m+1}) \leq (x_{h_{m+1}} - x_{h_m}) + (y_{h_m} - y_{h_{m+1}})$ for any h_m and h_{m+1} on this path. This implies that $d_t(h_i, h_j) \leq (x_{h_j} - x_{h_i}) + (y_{h_i} - y_{h_j})$. Since $A(x - x_{h_j}) \geq y_{h_j} - y \geq 0$ and the smallest scaled translate of R with h_j and v on its boundary is smaller than that of u and v, we can use induction to get $d_t(h_j, v) \leq (A + \sqrt{A^2 + 1})d_x(h_j, v) + d_y(h_j, v)$. Putting everything together, we obtain that

$$
\begin{aligned}
d_t(u, v) &\leq d_t(u, h_i) + d_t(h_i, h_j) + d_t(h_j, v) \\
&\leq (A + \sqrt{A^2 + 1})x_{h_i} - (y_{h_i} - y) + (x_{h_j} - x_{h_i}) + (y_{h_i} - y_{h_j}) \\
&\quad + (A + \sqrt{A^2 + 1})d_x(h_j, v) + d_y(h_j, v) \\
&= (A + \sqrt{A^2 + 1})x_{h_i} + (x_{h_j} - x_{h_i}) + (y - y_{h_j}) \\
&\quad + (A + \sqrt{A^2 + 1})d_x(h_j, v) + d_y(h_j, v) \\
&\leq (A + \sqrt{A^2 + 1})d_x(u, h_j) - d_y(h_j, v) + (A + \sqrt{A^2 + 1})d_x(h_j, v) + d_y(h_j, v) \\
&= (A + \sqrt{A^2 + 1})x.
\end{aligned}
$$

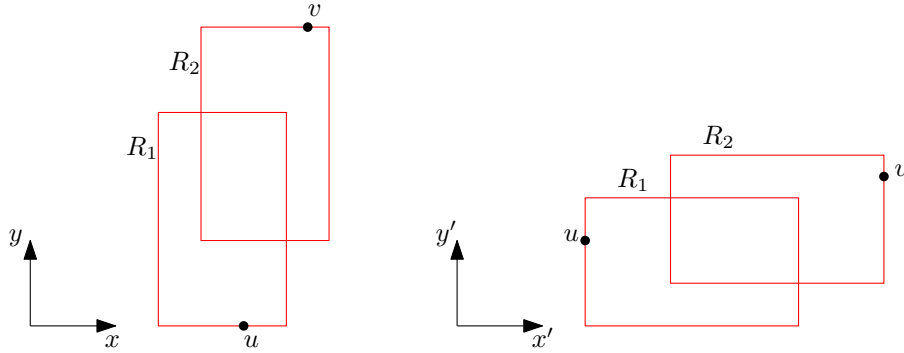proving the theorem when h_i is the inductive point of R_i.

If l_i is the inductive point of R_i then by Property 2c of Lemma 8 we know $d_t(u, l_i) - y_{l_i} \leq (A + \sqrt{A^2 + 1})x_{l_i}$ and thus $d_t(u, l_i) \leq (A + \sqrt{A^2 + 1})x_{l_i} + y_{l_i}$.

If $A(x - x_{l_i}) \geq y - y_{l_i}$, we let $l_j = l_i$ in the remainder. Otherwise, we let j be the smallest index larger than i such that $A(x - x_{l_j}) \geq y - y_c \geq 0$. By Lemma 9, l_j exists and all edges on the path $l_i, ..., l_j$ are SE edges. By triangle inequality, $d_t(l_m, l_{m+1}) \leq (x_{l_{m+1}} - x_{l_m}) + (y_{l_{m+1}} - y_{h_m})$ for any l_m and l_{m+1} on this path. This implies that $d_t(l_i, l_j) \leq (x_{l_j} - x_{l_i}) + (y_{l_j} - y_{l_i})$. Since $A(x - x_{l_j}) \geq y - y_{l_j} \geq 0$ and the smallest scaled translate of R with l_j and v on its boundary is smaller than that of u and v, we can use induction to get $d_t(l_j, v) \leq (A + \sqrt{A^2 + 1})d_x(l_j, v) + d_y(l_j, v)$. Putting everything together, this implies that

$$d_t(u, v) \leq d_t(u, l_i) + d_t(l_i, l_j) + d_t(l_j, v)$$
$$\leq (A + \sqrt{A^2 + 1})x_{l_i} + y_{l_i} + (x_{l_j} - x_{l_i}) + (y_{l_j} - y_{l_i})$$
$$+ (A + \sqrt{A^2 + 1})d_x(l_j, v) + d_y(l_j, v)$$
$$\leq (A + \sqrt{A^2 + 1})x_{l_i} + (x_{l_j} - x_{l_i}) + y_{l_j} + (A + \sqrt{A^2 + 1})d_x(l_j, v) + d_y(l_j, v)$$
$$\leq (A + \sqrt{A^2 + 1})x + y.$$

completing the proof of Case 1 when $Ax \geq y$.

Subcase $Ax < y$: Consider the (x', y')-coordinate system where the x'-axis equals the y-axis and the y'-axis equals the x-axis. See Figure 4. When we look at the homothet rectangles $R_1, \ldots, R_k$ intersecting the segment uv in the (x', y')-coordinate system, the horizontal side of the homothets is the longer side and we have $L = 1/A$. Therefore, $Ax < y$ implies $Lx' > y'$.

Figure 4 Homothet rectangles $R_1, \ldots, R_k$ in the (x', y')-coordinate system, for $k = 2$.

If (u, v) is an edge in the rectangle Delaunay triangulation, then $d_t(u, v) \leq x' + y' \leq (1 + \sqrt{\frac{1}{A^2} + 1})y + Ax$. If no rectangle in $R_1, \ldots, R_k$ is inductive then by Property 1 of Lemma 8 we know $d_t(u, v) \leq (\frac{1}{A} + \sqrt{\frac{1}{A^2} + 1})x' + y' \leq (1 + \sqrt{\frac{1}{A^2} + 1})y + Ax$.

When there is an inductive rectangle, define R_i, h_i and l_i as above. If h_i is the inductive point of R_i then by Property 2b of Lemma 8 we know $d_t(u, h_i) + A(y'_{h_i} - y') \leq (1 + \sqrt{\frac{1}{A^2} + 1})x'_{h_i}$.

If $\frac{1}{A}(x' - x'_{h_i}) \geq y'_{h_i} - y' \geq 0$, we let $h_j = h_i$ in the remainder. Otherwise, we let j be the smallest index larger than i such that $\frac{1}{A}(x' - x'_{h_j}) \geq y'_{h_j} - y' \geq 0$. By Lemma 9, h_j exists and all edges on the path $h_i, \ldots, h_j$ are NE edges. By triangle inequality, $d_t(h_i, h_j) \leq (x'_{h_j} - x'_{h_i}) + (y'_{h_i} - y'_{h_j})$. Since $\frac{1}{A}(x' - x'_{h_j}) \geq y'_{h_j} - y' \geq 0$ and the smallest scaled translate of R with h_j and v on its boundary is smaller than that of u and v, we can use induction to get $d_t(h_j, v) \leq (1 + \sqrt{\frac{1}{A^2} + 1})d_y(h_j, v) + Ad_x(h_j, v)$. Putting everything together, we obtain

$$d_t(u, v) \leq d_t(u, h_i) + d_t(h_i, h_j) + d_t(h_j, v)$$

$$\leq (1 + \sqrt{\tfrac{1}{A^2} + 1})x'_{h_i} - A(y'_{h_i} - y') + (x'_{h_j} - x'_{h_i}) + (y'_{h_i} - y'_{h_j})$$

$$+ (1 + \sqrt{\tfrac{1}{A^2} + 1})d_y(h_j, v) + Ad_x(h_j, v)$$

$$\leq (1 + \sqrt{\tfrac{1}{A^2} + 1})x'_{h_i} - A(y'_{h_i} - y') + (x'_{h_j} - x'_{h_i}) + A(y'_{h_i} - y'_{h_j})$$

$$+ (1 + \sqrt{\tfrac{1}{A^2} + 1})d_y(h_j, v) + Ad_x(h_j, v)$$

$$\leq (1 + \sqrt{\tfrac{1}{A^2} + 1})x'_{h_i} + (x'_{h_j} - x'_{h_i}) - Ad_{y'}(h_j, v)$$

$$+ (1 + \sqrt{\tfrac{1}{A^2} + 1})d_y(h_j, v) + Ad_x(h_j, v).$$

Recall that the y'-axis in the (x', y')-coordinate system equals the x-axis in the (x, y)-coordinate system, so $Ad_{y'}(h_j, v) = Ad_x(h_j, v)$. Thus

$$d_t(u, v) \leq (1 + \sqrt{\tfrac{1}{A^2} + 1})x'_{h_j} + (1 + \sqrt{\tfrac{1}{A^2} + 1})d_y(h_j, v)$$

$$= (1 + \sqrt{\tfrac{1}{A^2} + 1})y.$$

If l_i is the inductive point of R_i then by Property $2d$ of Lemma 8 we know $d_t(u, l_i) - Ay'_{l_i} \leq (1 + \sqrt{\tfrac{1}{A^2} + 1})x'_{l_i}$. Thus $d_t(u, l_i) \leq (1 + \sqrt{\tfrac{1}{A^2} + 1})x'_{l_i} + Ay'_{l_i}$.

If $\tfrac{1}{A}(x' - x'_{l_i}) \geq y' - y'_{l_i} \geq 0$, we let $l_j = l_i$ in the remainder. Otherwise, we let j be the smallest index larger than i such that $\tfrac{1}{A}(x' - x'_{l_j}) \geq y' - y'_{l_j} \geq 0$. By Lemma 9, l_j exists and all edges on the path $l_i, ..., l_j$ are SE edges. By triangle inequality, $d_t(l_i, l_j) \leq (x'_{l_j} - x'_{l_i}) + (y'_{l_j} - y'_{l_i})$. Since $\tfrac{1}{A}(x' - x'_{l_j}) \geq y' - y'_{l_j} \geq 0$ and the smallest scaled translate of R with l_j and v on its boundary is smaller than that of u and v, we can use induction to get $d_t(l_j, v) \leq (1 + \sqrt{\tfrac{1}{A^2} + 1})d_y(l_j, v) + Ad_x(l_j, v)$. Thus we obtain that

$$d_t(u, v) \leq d_t(u, l_i) + d_t(l_i, l_j) + d_t(l_j, v)$$

$$\leq (1 + \sqrt{\tfrac{1}{A^2} + 1})x'_{l_i} + Ay'_{l_i} + (x'_{l_j} - x'_{l_i}) + (y'_{l_j} - y'_{l_i})$$

$$+ (1 + \sqrt{\tfrac{1}{A^2} + 1})d_y(l_j, v) + Ad_x(l_j, v)$$

$$\leq (1 + \sqrt{\tfrac{1}{A^2} + 1})x'_{l_i} + Ay'_{l_i} + (x'_{l_j} - x'_{l_i}) + A(y'_{l_j} - y'_{l_i})$$

$$+ (1 + \sqrt{\tfrac{1}{A^2} + 1})d_y(l_j, v) + Ad_x(l_j, v)$$

$$= (1 + \sqrt{\tfrac{1}{A^2} + 1})x'_{l_i} + (x'_{l_j} - x'_{l_i}) + Ay'_{l_j}$$

$$+ (1 + \sqrt{\tfrac{1}{A^2} + 1})d_y(l_j, v) + Ad_x(l_j, v).$$

Using that $Ay'_{l_j} = Ax_{l_j}$, we obtain

$$d_t(u, v) \leq (1 + \sqrt{\tfrac{1}{A^2} + 1})y_{l_j} + (1 + \sqrt{\tfrac{1}{A^2} + 1})d_y(l_j, v) + Ax$$

$$= (1 + \sqrt{\tfrac{1}{A^2} + 1})y + Ax.$$

completing the proof of Case 1.

Case 2: There are vertices of P inside $R(u,v)$. We distinguish two subcases, either $Ax \geq y$ or $Ax < y$.

Subcase $Ax \geq y$: We split $R(u,v)$ into three regions formally defined as follows: $A = \{p \mid p \text{ is inside } R(u,v) \text{ such that } Ad_x(u,p) < d_y(u,p)\}$, $B = \{p \mid p \text{ is inside } R(u,v) \text{ such that } Ad_x(u,p) \geq d_y(u,p) \text{ and } Ad_x(p,v) \geq d_y(p,v)\}$, $C = \{p \mid p \text{ is inside } R(u,v) \text{ such that } Ad_x(p,v) < d_y(p,v)\}$. Informally, these three regions can be constructed by considering the line through u and the line through v parallel to the line through the diagonal of R and labelling the resulting regions A, B, and C from left to right (see Figure 5(a)).
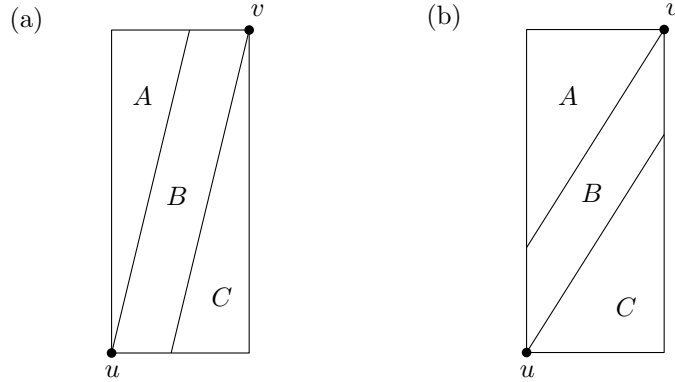
Figure 5 (a) The three regions in $R(u,v)$ when $Ax \geq y$. (b) The three regions in $R(u,v)$ when $Ax < y$.

If there exists a vertex p inside region B, then we can apply induction on the pairs (u,p), which satisfies $Ad_x(u,p) \geq d_y(u,p)$, and (p,v), which satisfies $Ad_x(p,v) \geq d_y(p,v)$:

$$d_t(u,v) \leq d_t(u,p) + d_t(p,v)$$
$$\leq (A + \sqrt{A^2 + 1})d_x(u,p) + d_y(u,p) + (A + \sqrt{A^2 + 1})d_x(p,v) + d_y(p,v)$$
$$= (A + \sqrt{A^2 + 1})x + y.$$

If there is no vertex inside region B, we define R_u to be the smallest scaled translate of R that has u on its lower left corner and some vertex $p \in A$ in $R(u,v)$ on its boundary. Similarly, we define R_v to be the smallest scaled translate of R that has v on its upper right corner and some vertex $q \in C$ in $R(u,v)$ on its boundary. Since $R(u,v)$ is not empty, at least one of p and q must exist. Assume without loss of generality that p exists. In this case we have that $Ad_x(p,v) > d_y(p,v)$ and the smallest homothet with p and v on its boundary is smaller than that of u and v. If (u,p) is an edge in the rectangle Delaunay triangulation, then we obtain that:

$$d_t(u,v) \leq d_t(u,p) + d_t(p,v)$$
$$= d_2(u,p) + d_t(p,v)$$
$$\leq d_x(u,p) + d_y(u,p) + (A + \sqrt{A^2 + 1})d_x(p,v) + d_y(p,v)$$
$$\leq (A + \sqrt{A^2 + 1})x + y.$$

An analogous argument can be used if q exists and (v,q) is an edge in the rectangle Delaunay triangulation.

Hence, it remains to consider the case where (u, p) is not an edge, in which case R_u is not empty. This implies that there exists a $p' \in C$ such that (u, p') is an edge. We have that $Ad_x(p', v) < d_y(p', v)$ and the smallest scaled translate of R with p' and v on its boundary is smaller than that of u and v. By the induction hypothesis, we have:

$$
\begin{aligned}
d_t(u, v) &\le d_t(u, p') + d_t(p', v) \\
&= d_2(u, p') + d_t(p', v) \\
&\le d_x(u, p') + d_y(u, p') + Ad_x(p', v) + \left(1 + \sqrt{\tfrac{1}{A^2} + 1}\right) d_y(p', v) \\
&\le Ax + \left(1 + \sqrt{\tfrac{1}{A^2} + 1}\right) y.
\end{aligned}
$$

Since $Ax \ge y$ and $\left(1 + \sqrt{\tfrac{1}{A^2} + 1}\right) > 1$, we have

$$
\begin{aligned}
d_t(u, v) &\le 1Ax + \left(1 + \sqrt{\tfrac{1}{A^2} + 1}\right) y \\
&\le \left(1 + \sqrt{\tfrac{1}{A^2} + 1}\right) Ax + 1y \\
&= \left(A + \sqrt{A^2 + 1}\right) x + y.
\end{aligned}
$$

Subcase $Ax < y$: We split $R(u, v)$ into three regions formally defined as follows: $A = \{p \mid p \text{ is inside } R(u, v) \text{ such that } Ad_x(v, p) \ge d_y(v, p)\}$, $B = \{p \mid p \text{ is inside } R(u, v) \text{ such that } Ad_x(v, p) < d_y(v, p) \text{ and } Ad_x(u, p) < d_y(u, p)\}$, $C = \{p \mid p \text{ is inside } R(u, v) \text{ such that } Ad_x(u, p) \ge d_y(u, p)\}$. See Figure 5(b).

If there exists a vertex p inside region B, then we can apply induction on the pairs (u, p), which satisfies $Ad_x(u, p) < d_y(u, p)$, and (p, v), which satisfies $Ad_x(v, p) < d_y(v, p)$:

$$
\begin{aligned}
d_t(u, v) &\le d_t(u, p) + d_t(p, v) \\
&\le Ad_x(u, p) + (1 + \sqrt{\tfrac{1}{A^2} + 1})d_y(u, p) + Ad_x(p, v) + (1 + \sqrt{\tfrac{1}{A^2} + 1})d_y(p, v) \\
&= Ax + (1 + \sqrt{\tfrac{1}{A^2} + 1})y.
\end{aligned}
$$

If there is no vertex inside region B, we define R_u to be the smallest scaled translate of R that has u on its lower left corner and some vertex $p \in A$ in $R(u, v)$ on its boundary. Similarly, we define R_v to be the smallest scaled translate of R that has v on its upper right corner and some vertex $q \in C$ in $R(u, v)$ on its boundary. Since $R(u, v)$ is not empty, at least one of p and q must exist. Assume without loss of generality that p exists. In this case we have that $Ad_x(p, v) > d_y(p, v)$ and the smallest rectangle with p and v on its boundary is smaller than that of u and v. If (u, p) is an edge in the rectangle Delaunay triangulation, then we obtain that:

$$
\begin{aligned}
d_t(u, v) &\le d_t(u, p) + d_t(p, v) \\
&= d_2(u, p) + d_t(p, v) \\
&\le d_x(u, p) + d_y(u, p) + (A + \sqrt{A^2 + 1})d_x(p, v) + d_y(p, v) \\
&\le (A + \sqrt{A^2 + 1})x + y.
\end{aligned}
$$

Since $Ax < y$, we have

$$d_t(u, v) \leq (A + \sqrt{A^2 + 1})x + y$$
$$= (1 + \sqrt{1 + \tfrac{1}{A^2}})Ax + 1y$$
$$\leq Ax + (1 + \sqrt{\tfrac{1}{A^2} + 1})y.$$

An analogous argument can be used if q exists and (v, q) is an edge in the rectangle Delaunay triangulation.

Hence, it remains to consider the case where (u, p) is not an edge, in which case R_u is not empty. This implies that there exists a $p' \in C$ such that (u, p') is an edge. We have that $Ad_x(p', v) < d_y(p', v)$ and the smallest scaled translate of R with p' and v on its boundary is smaller than that of u and v. By the induction hypothesis, we have:

$$d_t(u, v) \leq d_t(u, p') + d_t(p', v)$$
$$= d_2(u, p') + d_t(p', v)$$
$$\leq d_x(u, p') + d_y(u, p') + Ad_x(p', v) + \left(1 + \sqrt{\tfrac{1}{A^2} + 1}\right) d_y(p', v)$$
$$\leq Ax + \left(1 + \sqrt{\tfrac{1}{A^2} + 1}\right) y.$$

This completes the proof of Case 2 and the theorem.   ◄

We can now use Theorem 10 to show an upper bound of the spanning ratio of the rectangle Delaunay triangulation. For any pair of vertices u, v in the graph, if $Ad_x(u, v) \geq d_y(u, v)$ we have

$$\frac{d_t(u, v)}{d_2(u, v)} < \frac{(A + \sqrt{A^2 + 1})x + y}{\sqrt{x^2 + y^2}}.$$

This function is maximized when $y/x = 1/(A + \sqrt{A^2 + 1})$, where the function is equal to

$$\sqrt{2}\sqrt{1 + A^2 + A\sqrt{1 + A^2}}.$$

On the other hand, when $Ad_x(u, v) < d_y(u, v)$, we can get

$$\frac{d_t(u, v)}{d_2(u, v)} < \frac{Ax + (1 + \sqrt{\tfrac{1}{A^2} + 1})y}{\sqrt{x^2 + y^2}}.$$

This function is maximized when $y/x = (1 + \sqrt{\tfrac{1}{A^2} + 1})/A$, where the function value equals

$$\sqrt{A^2 + 2 + 2\sqrt{1 + \tfrac{1}{A^2}} + \tfrac{1}{A^2}},$$

which is at most $\sqrt{2}\sqrt{1 + A^2 + A\sqrt{1 + A^2}}$. This implies the main result of the paper.

▶ **Theorem 11.** *The spanning ratio of the rectangle Delaunay triangulation is at most $\sqrt{2}\sqrt{1 + A^2 + A\sqrt{1 + A^2}}$, where A is the aspect ratio of the rectangle used in its construction.*

Since it was already known that $\sqrt{2}\sqrt{1 + A^2 + A\sqrt{1 + A^2}}$ is a lower bound on the spanning ratio [3], we obtain that the bound of $\sqrt{2}\sqrt{1 + A^2 + A\sqrt{1 + A^2}}$ is tight.

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