Using Light Spanning Graphs for Passenger Assignment in Public Transport

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Abstract

In a public transport network a passenger's preferred route from a point x to another point y is usually the shortest path from x to y. However, it is simply impossible to provide all the shortest paths of a network via public transport. Hence, it is a natural question how a lighter sub-network should be designed in order to satisfy both the operator as well as the passengers.

We provide a detailed analysis of the interplay of the following three quality measures of lighter public transport networks:

- *building cost*: the sum of the costs of all edges remaining in the lighter network,
- *routing costs*: the sum of all shortest paths costs weighted by the demands,
- *fairness*: compared to the original network, for each two points the shortest path in the new network should cost at most a given multiple of the shortest path in the original network.

We study the problem by generalizing the concepts of optimum communication spanning trees (Hu, 1974) and optimum requirement graphs (Wu, Chao, and Tang, 2002) to *generalized optimum requirement graphs* (GORGs), which are graphs achieving the social optimum amongst all subgraphs satisfying a given upper bound on the building cost. We prove that the corresponding decision problem is NP-complete, even on *orb-webs*, a variant of grids which serves as an important model of cities with a center. For the case that the given network is a parametric city (cf. Fielbaum et. al., 2017) with a heavy vertex we provide a polynomial-time algorithm solving the GORG-problem. Concerning the fairness-aspect, we prove that *light spanners* are a strong concept for public transport optimization.

We underpin our theoretical considerations with integer programming-based experiments that allow us to compare the fairness-approach with the routing cost-approach as well as passenger assignment approaches from the literature.

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1 Introduction

In the light of climate change and the resulting aim to reduce greenhouse gas emissions, mobility has to be considered from a sustainability standpoint. A public transport system that is both cost-efficient and attractive to passengers can contribute to reducing the environmental impact of mobility by bundling demand efficiently. To achieve such a system, both objectives have to be considered throughout the planning process. From a passengers' perspective, individual door-to-door service for each passenger represents the best possible solution. However, such solutions are undesirable as they would result in very high operational costs and provide little benefit in comparison with individual transport. Thus, passenger routes have to be bundled to achieve the desired effect of reduced environmental impact.

In this paper, we consider the passenger assignment problem to bundle demand. This problem is one of the first stages in a traditional sequential planning process [6]. Passenger assignment can be part of usually heuristic approaches for transit route network design problems [14] but it is also considered on its own. In [10] the authors compare heuristic approaches from [19] and [10] to an integrated approach and analyzes the impact on the line planning costs and average travel time. Note that assigning routes to passengers can lead to large detours for some passengers and unrealistic assumptions on passenger behavior. We put special emphasis on the case that the considered networks are orb-webs or parametric-city networks (see Figure 1 for examples).

Our contribution. We provide a detailed analysis of two approaches for designing subnetworks of bounded *building cost* (the total cost of all edges remaining in the subnetwork). Our focus is on the following two measures of passenger-satisfaction

(A) **Fairness:** for each two network points the shortest path costs in the lighter network should be at most a given fixed multiple of the shortest path costs in the original network.

(B) **Total routing cost:** the sum of all shortest paths weighted by the demands should be minimal among all networks of a given maximal network size.

We analyze the optimization problems resulting from combining an upper-bounded building cost with (A) or (B), respectively.

For upper-bounded building costs in combination with (A) we show that the concept of *light spanners* from structural graph theory exactly mirrors our fairness-measure. As an interesting observation we obtain that fair sub-networks in orb-webs with a heavy center contain a star whose center vertex matches with the web-center. This nicely confirms common practice in public transport planning, where oftentimes cities with a center offer a star-shaped public transport network.

Concerning the combination of bounded building costs with (B) we prove that this problem is NP-hard, even if it is restricted to orb-webs. Moreover, we develop new exact algorithms for the problem on parametric city instances (see also Figure 1).

We complete our analysis with practical experiments. To this end, we provide IP formulations for both problems. Based on the integer programs, we analyze the influence of optimal solutions to the two problems on the quality of line plans and compare them to passenger assignment methods from the literature. Further, we run practical experiments on orb-web instances with a relaxed heavy center vertex in order to confirm also from the practical perspective that star-shaped solutions are often optimal or contained in an optimal public transport network for cities with one heavily demanded center.

(a) A parametric city network with eleven spokes. **(b)** An orb-web on five spokes and four rings.

Figure 1 Examples for public transport networks considered in this paper.

Further related work. An *optimum communication spanning tree*, as introduced in [18], is minimizing the total routing cost amongst all spanning trees of a given network. Based on [18] various results on optimum communication spanning trees were developed, including integer programming techniques, approximation algorithms, and exact algorithms for certain subclasses, cf. [8, 26, 23, 28]. The problem of finding a forest of minimum building cost which connects given vertex subsets is known as the *Steiner forest problem*, cf. [12, 15, 3].

2 Preliminaries

In this section, we introduce relevant notation and discuss quality measures for light spanning graphs. Additionally, we connect the problem of finding light spanning graphs to passenger assignment and line planning in public transport.

Sets. For an natural number k, we set $[k] = \{1, 2, \dots, k\}$. For a set A and a natural number k, we denote the set of all k-element subsets of A by $\binom{A}{k}$.

Graphs. A *weighted graph* is a tuple (G, c) where G is a graph and $c\colon E(G) \to \mathbb{R}_{\geq 0}$ is a function mapping each edge $e \in E(G)$ to its *weight* or *cost* $c_e \coloneqq c(e)$. For $u, v \in V(G)$, we denote the set of simple u-v paths by $\mathcal{P}_{u,v}$. A subgraph H of G is *spanning* if $V(H) = V(G)$. We denote the length of a shortest u-v-path in G with respect to c by $d_G(u, v)$ and the length of a shortest u-v-path in H with respect to $c_{|E(H)}$ is denoted by $d_H(u, v)$.

Orb-webs and Euclidean costs. An orb-web (cf. [7]) is a graph obtained from a cylindric grid by contracting the vertices of one of the border cycles to one vertex. More precisely, let r and s be positive integers. The $(r \times s)$ *orb-web* $\mathrm{W}_{r,s}$ is a graph on the vertex set $\{z\} \cup \{v_{i,j} : i \in [r], j \in [s]\}$ which decomposes into the cycles $R_i \coloneqq v_{i,1}v_{i,2} \dots v_{i,s}v_{i,1}$, one for each $i \in [r]$, and the paths $S_j \coloneqq zv_{1,j}v_{2,j} \dots v_{r,j}$, one for each $j \in [s]$. We call R_i a *ring* of $\mathrm{W}_{r,s}$ and S_j is a *spoke* of $\mathrm{W}_{r,s}$ for $i \in [r]$ and $j \in [s]$, respectively. The vertex z is the *center* of $\mathrm{W}_{r,s}$. An edge of $\mathrm{W}_{r,s}$ either belongs to a ring or to a spoke. We then call it a *ring-edge* or a *spoke-edge*, respectively.

We say that $\mathrm{W}_{r,s}$ is equipped wit a *Euclidean cost function* c whenever c can be obtained as follows: Embedd $\mathrm{W}_{r,s}$ into the plane such that for every $i \in [r]$ the vertices of R_i are of Euclidean distance i to the center vertex z and the Euclidean distance of two adjacent vertices on R_i is uniform on R_i. For every edge e of $\mathrm{W}_{r,s}$ we set $c(e)$ to be the Euclidean distance of the two endvertices of e.

Our public transport model. We assume that we are given

- a weighted graph (G, c) called *public transport network* (PTN), where the vertices of G represent traffic junctions (e.g. bus or tram stops) and the edges represent connections joining the junctions (e.g. streets or potential tracks) and c_{uv} represents the costs of traveling from u to v,
- *demand data* $a_{\{u,v\}} \in \mathbb{R}_{\geq 0}$ for $\{u, v\} \in \binom{V(G)}{2}$, which represents the number of passengers who want to travel between u and v per time unit. We often abbreviate $a_{\{u,v\}}$ to $a_{u,v}$. In particular, $a_{u,v} = a_{v,u}$.

Quality measures for light spanning graphs. While any connected spanning subgraph H of G can be used for designing a transportation supply, the choice of H has a large influence on the quality of the system, both from the operator's and the passengers' side. Ideally, the operator is able to keep the costs low by selecting a light spanning subgraph, while the travel times of the passengers do not grow too much compared to using the full graph G. We consider three measures for the quality of H:

- the *building cost* $c(H) \coloneqq \sum_{e \in E(H)} c(e)$ of H, which represents the operator's point of view,
- the *routing cost* $r(H) \coloneqq \sum_{\{u,v\} \in \binom{V(G)}{2}} a_{u,v} d_H(u, v)$ of H, which represent the social optimum from the passengers' point of view. Finally,
- the *maximum detour factor* $d(H) \coloneqq \max_{\{u,v\} \in \binom{V(G)}{2}} \frac{d_H(u,v)}{d_G(u,v)}$ of H, which represents the fairness aspect from the passengers' point of view.

From light spanning graphs to passenger assignments. For a public transport network (G, c) and demands $a_{u,v}$, $\{u, v\} \in \binom{V(G)}{2}$, a *passenger assignment* distributes the demand to feasible paths. Thus, it assigns for each two distinct vertices u, v a weight $w_P \geq 0$ to each u-v path $P \in \mathcal{P}_{u,v}$ such that $\sum_{P \in \mathcal{P}_{u,v}} w_P = a_{u,v}$. We evaluate the quality of a passenger assignment by considering the *average detour factor*

$$\frac{\sum_{\{u,v\} \in \binom{V(G)}{2}} \sum_{P \in \mathcal{P}_{u,v}} w_P c(P)}{\sum_{\{u,v\} \in \binom{V(G)}{2}} d_G(u, v)}.$$

In this paper we consider four variants of passenger assignments:

Shortest paths in spanning graphs (SPS): Given a spanning graph H of G, set $w_P = a_{u,v}$ for a shortest u-v path P in H.

Shortest paths (SP): Set $w_P = a_{u,v}$ for a shortest u-v path P in G.

REWARD: Iterative procedure from [19]: In each iteration k, passengers are routed on shortest paths in G according to weights c^k. Weights c^{k+1} are adapted to be lower on edges that are used by more passengers. After a fixed number of N iterations, edges that do not appear in a shortest path according to c^N are deleted to result in a spanning graph H'. Set $w_P = a_{u,v}$ for a shortest u-v path in H' for original weights c.

REDUCTION: Iterative procedure from [10] where w is assigned according to shortest paths in G and weights c^k. c^k is adapted in each iteration such that edges with spare capacity are rewarded, i.e., c^k is reduced on these edges.

Note that for SPS, the average detour factor is $\frac{r(H)}{r(G)}$, i.e., a normalization of the routing cost, and for SP, it is $\frac{r(G)}{r(G)} = 1$.

Evaluating passenger assignments by line planning. Line planning is a crucial step in public transport planning, where operating frequencies of lines are determined [25]. A line is a simple path in a public transport network that is operated by a vehicle end-to-end.

For a given vehicle capacity $\mathcal{K}$, we can easily compute *lower frequency constraints* $f^{\min}\colon E(G) \to \mathbb{N}_{>0}$ used in many line planning approaches [25, 4] as

$$f^{\min}(e) = \left\lceil \frac{\sum_{\{u,v\}\in\binom{V(G)}{2}} \sum_{P\in\mathcal{P}_{u,v}\colon e\in E(P)} w_P}{\mathcal{K}} \right\rceil .$$

Given a line pool $\mathcal{L}$, i.e., a set of lines, the cost model of line planning [25] is

$$\left\{ \min \sum_{\ell\in\mathcal{L}} \text{cost}_\ell f_\ell \colon \sum_{\ell\in\mathcal{L}\colon e\in E(\ell)} f_\ell \geq f^{\min}(e), e \in E(G);\ f_\ell \in \mathbb{N}_{\geq 0}, \ell \in \mathcal{L} \right\}$$

where cost_ℓ represents the cost of operating line ℓ once per planning period. In our experiments, we set $\text{cost}_\ell = c_{\text{fix}} + \alpha|E(\ell)| + \beta c(\ell)$, with $c_{\text{fix}} \in \mathbb{R}_{\geq 0}$ representing the fixed cost for operating a line and $\alpha, \beta \in \mathbb{R}_{\geq 0}$, see [13]. As the lower frequency constraints correspond to a passenger assignment, they guarantee that routing passengers with the average detour factor is possible. We evaluate a line plan by its cost, i.e., $\sum_{\ell\in\mathcal{L}} \text{cost}_\ell f_\ell$.

Outline. To construct light graphs with regard to these objectives, we consider two concepts from the literature. In Section 3, we consider *light* $(1+\varepsilon)$ *spanners*. Here, we are looking for a subgraph H of minimal building costs such that maximum detour factor is not exceeding $(1+\varepsilon)$. In Section 4, we consider a different perspective by computing *generalized optimum requirement graphs* for which we introduce two IP formulations in Section 5. Thus, we minimize the routing cost imposing an upper bound on the building cost. We evaluate both concepts experimentally in Section 6. The paper is concluded in Section 7.

3 Spanners

We first give the basic terminology for spanners and translate it to our public transport setting. Let (G, c) be a weighted graph and let H be a spanning subgraph of G. If $d_H(u, v) \leq (1+\varepsilon)d_G(u, v)$ for all $u, v \in V(G)$, then H is a $(1+\varepsilon)$-*spanner* of G. In this case, we say that H has *stretch* at most $(1+\varepsilon)$ and *lightness* at most $\frac{c(H)}{c_{\mathbf{MST}}}$, where $c_{\mathbf{MST}}$ is the weight of a minimum spanning tree of (G, c). Observe that the stretch directly corresponds to the maximum detour factor $d(H)$ and the lightness to the building costs $c(H)$.

Note that it is already NP-complete to decide whether a given graph has a 2-spanner [5, 22]. This directly yields the following statement:

▶ **Theorem 1.** *Given a public transport network, a bound K on the building cost, and a bound B' on the maximum detour factor it is NP-complete to decide whether there exists a sub-network of building cost at most K and maximum detour factor at most B'.*

At first glance, it seems to be a direct consequence of Theorem 1 that spanners are simply useless in any practical context. However, the following concept of greedy spanners (cf. [1]) gives some cause for hope. The *greedy* $(1+\varepsilon)$-*spanner* of a weighted graph (G, c) is defined to be the output of Algorithm 1.

Observe that the algorithm GreedySpanner is a straight-forward generalization of Kruskal's algorithm for finding minimum weight spanning forests. As Kruskal's algorithm also GreedySpanner runs in time $\mathcal{O}(m \log n)$ where m and n denote the size and the order

Algorithm 1 GREEDYSPANNER$((G, c), \varepsilon)$.

let $e_1, \ldots, e_m$ be an ordering of $E(G)$ such that $c(e_1) \leq c(e_2) \leq \cdots \leq c(e_m)$
let H be the edgeless graph on $V(G)$
for $i = 1, \ldots, m$ **do**
  if $d_H(u_i, v_i) > (1+\varepsilon)w(e_i)$, *where* $e_i = u_i v_i$, **then**
    add e_i to the edges of H
return H

of the input graph, respectively. Moreover, the greedy $(1+\varepsilon)$-spanner of (G, c) is indeed a $(1+\varepsilon)$-spanner, see also [2, 1]. The following statement makes the consideration of spanners as a concept for public transport networks of low building cost even more attractive since planarity (or, even more general, a low genus) is a realistic assumption in the public transport context.

▶ **Theorem 2** (Baligács et. al. [2])**.** *For every graph G of genus g and $\varepsilon > 0$, the greedy $(1+\varepsilon)$-spanner of G has lightness at most $(1 + \frac{2}{\varepsilon})\left(1 + \frac{2g}{1+\varepsilon}\right)$.*

Note that orb-webs are planar, that is, of genus 0. We close this section with an observation on star-shaped subgraphs of greedy spanners in orb-webs.

▶ **Corollary 3.** *For every $\varepsilon > 0$ and every two positive integers r and s there exists a greedy $(1+\varepsilon)$spanner H of $(W_{r,s}, \mathbb{1}_E)$ which contains all spoke-edges of $W_{r,s}$ and H has lightness at most $1 + \frac{2}{\varepsilon}$.*

Proof. Since all edges are of the same weight, we can choose an ordering of the edges such that the spoke-edges are of lesser order than the ring-edges of the orb-web. It follows immediately that the GREEDYSPANNER algorithm on the orb-web with this chosen ordering returns a subgraph containing all the spoke-edges. Since orb-webs are planar graphs we obtain the lightness as an immediate consequence of Theorem 2. ◀

▶ **Corollary 4.** *Let $\varepsilon > 0$ and $r, s \in \mathbb{N}_{\geq 1}$ with $s \leq 6$. If $W_{r,s}$ is equipped with an Euclidean costs c, then there exists a greedy $(1+\varepsilon)$-spanner H of $(W_{r,s}, c)$ which contains all spoke-edges of $W_{r,s}$ and H has lightness at most $1 + \frac{2}{\varepsilon}$.*

Proof. Since $s \leq 6$ and by the definition of the Euclidean costs we obtain $c(e_s) = 1 \leq c(e_r)$ for every spoke-edge e_s and every ring-edge e_r. In particular, we can proceed exactly as in the proof of Corollary 3. ◀

4 Generalized optimum requirement graphs

Given a weighted graph (G, c), a non-negative set of demands $\left\{a_{u,v} \colon \{u,v\} \in \binom{V(G)}{2}\right\}$, and a bound $K \in \mathbb{R}_{\geq 0}$ the *generalized optimum requirement graph problem* (GORG) is to find a spanning subgraph of G which minimizes the routing cost amongst all spanning subgraphs of G with building costs at most K.

In the literature, often either the demand is assumed to be uniform (*optimum distance graph problem*) or the edge-costs are assumed to be uniform (*optimum requirement graph problem*), cf. [18, 27]. Here, we consider the general problem (GORG) where both the demand and the cost can take on arbitrary non-negative values. This problem is shown to be NP-hard for general graphs in [20]. In the following, we refine this result by showing that the problem is even NP-hard when it is restricted orb-webs.

▶ **Theorem 5.** *The problem* (GORG) *is* NP-*hard, even for orb-webs.*

Proof. We show that the decision version of (GORG) is NP-complete by reducing the NP-complete decision version of the Knapsack problem (cf. [21, 11]) to the decision version of (GORG). Note that for a given subgraph H, varifying whether the routing and building costs are below given thresholds can be done in polynimal time by computing shortest paths. Thus, the decision version of (GORG) is in NP.

Consider an instance of the Knapsack problem, i.e., n items with weight $w_i \in \mathbb{N}$ and value $v_i \in \mathbb{N}$, $i \in [n]$, maximal weight W' and minimal value V'.

We construct an instance of (GORG) as follows: Let $\mathrm{W}_{1,2n}$ be an orb-web with one ring and $2n$ spokes and $K = 2\sum_{i\in[n]} w_i + W'$. For $i \in [n]$, we set $c(e) = w_i$ for $e \in \{zv_{1,2i-1},\, zv_{1,2i},\, v_{1,2i-1}v_{1,2i}\}$. The costs for the remaining ring-edges are set to $K+1$.

We define the demand a as follows: For $k \in [2n]$, $a_{z,v_{1,i}} = M$ with $M = 3\sum_{i\in[n]} v_i$, for $i \in [n]$, $a_{v_{1,2i-1},v_{1,2i}} = \frac{v_i}{w_i}$ and $a_{u,v} = 0$ otherwise. For $B = 4M\sum_{i\in[n]} w_i + 4\sum_{i\in[n]} v_i - 2V$, we show that there is a feasible solution of (GORG) with routing cost at most B if and only if there is a feasible solution of the Knapsack problem, i.e., a subset $S \subset [n]$ with $\sum_{i\in S} w_i \geq W$ and $\sum_{i\in S} v_i \leq V$.

First, consider a feasible solution S of the Knapsack problem. Construct a spanner H of $\mathrm{W}_{1,2n}$ by adding all spoke-edges as well as ring-edges $v_{1,2i-1}v_{1,2i}$ for $i \in S$. It is easy to see that the building costs of H satisfy

$$c(H) = 2\sum_{i\in[n]} w_i + \sum_{i\in S} w_i \leq 2\sum_{i\in[n]} w_i + W = K.$$

Additionally, the routing costs can be computed as

$$r(H) = 4M\sum_{i\in I} w_i + 2\sum_{i\in S} \frac{v_i}{w_i} w_i + 2\sum_{i\notin S} \frac{v_i}{w_i} 2w_i = 4M\sum_{i\in I} w_i + 4\sum_{i\in[n]} v_i - 2\sum_{i\in S} v_i \leq B.$$

Thus, H is a feasible solution of the decision version of (GORG). Second, consider a feasible solution H of (GORG). Note that due to the cost definition, only spoke-edges and ring-edges in $\{v_{1,2i-1}v_{1,2i}\}$ can be in $E(H)$. Additionally, all spoke-edges are in $E(H)$ as otherwise the routing costs would exceed

$$4M\sum_{i\in[n]} w_i + 2M \cdot \min_{i\in[n]} w_i \geq 4M\sum_{i\in[n]} w_i + 6\sum_{i\in[n]} v_i > B.$$

Let $S \subset [n]$ be the set of indices such that $v_{1,2i-1}v_{1,2i} \in E(H)$. Then the building cost satisfy

$$c(H) = 2\sum_{i\in[n]} w_i + \sum_{i\in S} w_i \leq K = 2\sum_{i\in[n]} w_i + W$$

such that $\sum_{i\in S} w_i \leq W$. The shortest u-w-path is uv for $u = z$, $v \neq z$ and $u = v_{1,2i-1}, v = v_{1,2i}$ with $i \in S$. However, for $u = v_{1,2i-1}, v = v_{1,2i}$ with $i \notin S$, the shortest route is uzv. Thus, the routing costs are

$$\begin{aligned} r(H) &= 4M\sum_{i\in I} w_i + 2\sum_{i\in S} \frac{v_i}{w_i} w_i + 2\sum_{i\notin S} \frac{v_i}{w_i} 2w_i = 4M\sum_{i\in I} w_i + 4\sum_{i\in[n]} v_i - 2\sum_{i\in S} v_i \\ &\leq B = 4M\sum_{i\in[n]} w_i + 4\sum_{i\in[n]} v_i - 2V \end{aligned}$$

such that $\sum_{i\in S} v_i \geq V$ and S is feasible for the Knapsack problem. ◀

▶ **Observation 6.** *Let (G, c) be weighted graph and $A \coloneqq \left\{a_{uv} : \{u, v\} \in \binom{V}{2}\right\}$ be a set of demands on G. If there exists a vertex $v \in V(G)$ with $a_{uv} = 0$ for all $u \in V(G) \setminus \{v\}$, then every optimal solution of* (GORG) *on $(G - v, c_{|V(G)\setminus\{v\}})$ with demands $\left\{a_{u,v} : \{u, v\} \in \binom{V(G)\setminus\{v\}}{2}\right\}$ is also optimal for* (GORG) *on the original instance.*

Hence, we assume from now on that for every vertex u of a considered (GORG) *instance there exists at least one other vertex v with strictly positive demand between u and v.*

4.1 Parametric cities

In this subsection, we consider parametric city networks which are introduced in [9] as an abstract representation of real city networks. A graph is a *parametric city of order s*, denoted by PC_s if it can be obtained from an orb-web $\mathrm{W}_{1,s}$ with just one ring by adding s new vertices and joining each of the new vertices to exactly one of the ring-vertices of $\mathrm{W}_{1,s}$, see Figure 1 for an example.

It is natural to assume that the demand towards the center vertex of a parametric city is high. In this context, we generalize the heavy-vertex condition introduced in [27] and, we prove that (GORG) can be solved in polynomial time on a parametric city with a heavy vertex.

The following lemma enables us to reduce (GORG) on parametric cities to (GORG) on orb-webs with precisely one ring.

▶ **Lemma 7.** *Let (G, c) be weighted graph and $A \coloneqq \{a_{uv} : \{u, v\} \in \binom{V}{2}\}$ be a set of demands on G. If w is a degree-1 vertex in G and w' denotes the neighbor of w, then an optimal solution for* (GORG) *on (G, c) with demands A can be obtained from an optimal solution for* (GORG) *on $G - w$ with demands*

$$a'_{u,v} = \begin{cases} a_{u,v} & \text{if } w' \notin \{u, v\}, \\ sa_{u,w'} + a_{u,w} & \text{otherwise.} \end{cases}$$

Adding the edge $w'w$ to an optimal solution of the smaller instance yields an optimal solution for the original instance.

Proof. Since w is a degree-1 vertex we have that for every $u \in V(G) \setminus \{w\}$ every shortest u-w-path in an optimal solution for (GORG) on G is of the form $v_1 v_2 \ldots v_{k-2} w' w$. In particular, it can be obtained simply by extending a $v_1 \ldots w'$-path in $G - w$ by w. In particular, we obtain that an optimal solution in G can be projected to a feasible solution of $G - w$ with the adapted demands. If there was a solution of $G - w$ with the adapted demands with a strictly better objective value than the projected solution, then this would yield to a better solution for G, a contradiction. We obtain a 1:1-correspondence of the optimal solutions for (G, c) with demands A and $(G - w, c_{|V\setminus\{w\}})$ with the adapted demands. This settles the claim. ◀

Let G be a graph and let $A \coloneqq \{a_{uv} : \{u, v\} \in \binom{V}{2}\}$ be a set of demands on G. A vertex h is *heavy* in G if $a_{u,h} \geq a_{u,v}$ for each two distinct vertices u and v of G. Along the same lines as the heavy-vertex proof on complete graphs in [27] we obtain the following statement.

▶ **Theorem 8.** *Let $s \in \mathbb{N}_{\geq 1}$. If the center of $W_{1,s}$ is heavy, then* (GORG) *can be solved in time $\mathcal{O}(n^2)$ on this instance.*

4.2 Symmetric generalized optimum requirement graphs

Let us consider an interesting special case of (GORG) on orb-webs where the solution H has to be rotationally symmetric. In this case, all connected solutions have a special structure. To ensure connectivity, all spoke-edge are in $E(H)$. Additionally, for each cycle R_i either all edges are in $E(H)$ or none of them are. Thus, the problem reduces to choosing the best subset of rings within the given budget. For unit weights, we thus have to choose p rings such that the routing costs are minimized. Note that for all demand where origin and destination are on the same spoke the shortest path in H is the same as the shortest path in G. Thus, we only have to consider demand where origin and destination are on separate spokes.

Given a demand structure with positive demand only between neighbors on the same ring, the problem is equivalent to a p-median problem on a line and can be solved in $\mathcal{O}(pr + rs)$. Note that $p \leq r$, i.e., the runtime is polynomial in $\mathcal{O}(r^2 + rs)$.

▶ **Theorem 9.** *Consider an orb-web $W_{r,s}$ with $c \equiv 1$ and $a_{u,v} = 0$ if u and v are not neighbors on the same ring. Then, a symmetric solution to* (GORG) *with at most p rings can be found in $\mathcal{O}(pr + rs)$.*

Proof. Consider a solution H of (GORG) where rings R_i, $i \in S \subset [r]$, are in $E(H)$ with $|S| < p$. For $a_{v_{k,l},v_{k,l'}}$ with $l' = l + 1$ or $l' = s$, $l = 1$, we can compute the routing costs as

$$\begin{aligned}
d_H(v_{k,l}, v_{k,l'}) &= \min\left\{ \underbrace{2k}_{c(v_{k,l} z v_{k,l'})}, \min\left\{ \underbrace{2|k-i|+1}_{c(v_{k,l} \dots v_{i,l} v_{i,l'} \dots v_{k,l'})} : i \in S \right\}\right\} \\
&= \min\left\{2|k-0.5|+1, \min\left\{2|k-i|+1 \colon i \in S\right\}\right\} \\
&= \min\left\{2|k-i| \colon i \in S \cup \{0.5\}\right\} + 1.
\end{aligned}$$

For ease of notation, we identify $a_{v_{k,1},v_{k,s}}$ with $a_{v_{k,s},v_{k,s+1}}$. The routing costs of H are

$$\begin{aligned}
r(H) &= \sum_{k=1}^{r} \sum_{l=1}^{s} a_{v_{k,l},v_{k,l+1}} d_H(v_{k,l}, v_{k,l+1}) \\
&= \sum_{k=1}^{r} \sum_{l=1}^{s} a_{v_{k,l},v_{k,l+1}} + 2 \sum_{k=1}^{r} \min\left\{|k-i| \colon i \in S \cup \{0.5\}\right\} \cdot \sum_{l=1}^{s} a_{v_{k,l},v_{k,l+1}}.
\end{aligned}$$

Thus, finding S with minimal routing costs is equivalent to solving a p-median problem on the line where 0.5 is fixed as a facility. Following the proof of Lemma 1 and Section 2 in [17], this problem can be solved in $\mathcal{O}(pr)$. The weights can be computed in $\mathcal{O}(rs)$. ◀

▶ Remark. The solution remains optimal if there is additional positive demand $a_{v_{k,l},v_{k',l'}}$ between arbitrary nodes, and there is at least one ring R_m, $k \leq m \leq k'$ in $E(H)$. In this case, $d_H(v_{k,l}, v_{k',l+1}) = d_G(v_{k,l}, v_{k',l+1})$ as a shortest path in G either contains only spoke-edges or spoke-edges and ring-edges for a ring R_m with $k \leq m \leq k'$.

5 IP formulation for (GORG)

In this section we present two integer programming formulations for (GORG). In both formulations we use binary variables x_e, $e \in E(G)$, to indication whether edge e is in $E(H)$. For convenience, we abbreviate $V := V(G)$ and $E := V(G)$ in the IP formulations. Additionally, we introduce an ordering $<$ on the finite set $V(G)$ to avoid computing both the shortest path from u to v and from v to u.

One-to-one IP formulation. For Model (1), we model the shortest paths for each pair of nodes $s < t \in V(G)$ separately as an s-t flow using binary variables y^{st}_{uv}, y^{st}_{vu}, $uv \in E(G)$. Note that we implicitly transform G to a directed graph to model the flow. A capacity constraint bounds the building costs and coupling constraints between x- and y-variables to ensure that only edges from H can be used.

$$
\begin{aligned}
\min \quad & \sum_{s<t\in V} \sum_{uv\in E} a_{u,v} c(uv)(y^{st}_{uv} + y^{st}_{vu}) \\
\text{s.t.} \quad & \sum_{e\in E} c(e)x_e \le K \\
& \sum_{w\in V:\, wu\in E} (y^{st}_{wu} + y^{st}_{uw}) \\
& - \sum_{w\in V:\, uw\in E} (y^{st}_{uw} + y^{st}_{wu}) = \begin{cases} -1, & \text{if } u = s \\ 1, & \text{if } u = t \\ 0, & \text{otherwise} \end{cases} && s < t \in V, u \in V \\
& y^{st}_{uv} \le x_{uv} && s < t \in V, uv \in E \\
& y^{st}_{vu} \le x_{uv} && s < t \in V, uv \in E \\
& x_e \in \{0,1\} && e \in E \\
& y^{st}_{uv}, y^{st}_{vu} \in \{0,1\} && s < t \in V, uv \in E
\end{aligned}
\tag{1}
$$

Note that the binary flow variables y^{st}_{uv}, y^{st}_{vu}, $uv \in E(G)$, can be relaxed to continuous variables.

One-to-many IP formulation. For Model (2), we replace the one-to-one flow formulation with a single-source-multiple-target flow formulation using variables y^{s}_{uv}, y^{s}_{vu}, $s \in V(G)$, $uv \in E(G)$. This reduces both the number of variables and the number of constraints significantly.

$$
\begin{aligned}
\min \quad & \sum_{s\in V} \sum_{uv\in E} c(uv)(y^{s}_{uv} + y^{s}_{vu}) \\
\text{s.t.} \quad & \sum_{e\in E} c(e)x_e \le K \\
& \sum_{w\in V:\, wu\in E} (y^{s}_{wu} + y^{s}_{uw}) \\
& - \sum_{w\in V:\, uw\in E} (y^{s}_{uw} + y^{s}_{wu}) = \begin{cases} -\sum_{t>s} a_{s,t}, & \text{if } u = s \\ a_{s,u}, & \text{if } u > s \\ 0, & \text{otherwise} \end{cases} && s \in V, u \in V \\
& y^{s}_{uv} \le x_{uv} && s < t \in V, uv \in E \\
& y^{s}_{vu} \le x_{uv} && s < t \in V, uv \in E \\
& x_e \in \{0,1\} && e \in E \\
& y^{s}_{uv}, y^{s}_{vu} \in \{0,1\} && s \in V, uv \in E
\end{aligned}
\tag{2}
$$

Adding valid inequalities. To improve the linear programming relaxation of an integer program, it is possible to add *valid inequalities* or *cuts* to the IP formulation. These cut off some of the LP feasible region without eliminating any of the integer feasible solutions, thus making the LP relaxation closer to the convex hull of the IP feasible set.

We present two sets of valid inequalities for the IP formulations (1) and (2) for the case that the demand requires a connected graph. The first set consists of general inequalities for graphs, and the second set exploits the properties of orb-webs. The general inequalities are

$$\sum_{uv\in E} x_{uv} \geq |V| - 1 \tag{3}$$

$$\sum_{s\in V:\ us\in E} x_{us} + \sum_{s\in V:\ sv\in E} x_{sv} \geq 1 \qquad s \in V. \tag{4}$$

Inequality (3) rules out solutions where the number of edges in $E(H)$ is not enough for connectivity while inequality (4) ensures local connectivity. The orb-web-specific inequalities are

$$\sum_{j\in[s]:\ zv_{1,j}\in E} x_{zv_{1,j}} \geq 1 \tag{5}$$

$$\sum_{j\in[s]:\ v_{i,j}v_{i+1,j}\in E} x_{v_{i,j}v_{i+1,j}} \geq 1 \qquad i \in [r-1] \tag{6}$$

$$x_{zv_{1,1}} = 1. \tag{7}$$

Inequalities (6) and (5) ensure connectivity between adjacent rings or the center z and the first ring, respectively. For rotationally symmetric orb-webs and the demand as described in Section 4.2, (7) fixes the spoke edge which connects the center to the first ring.

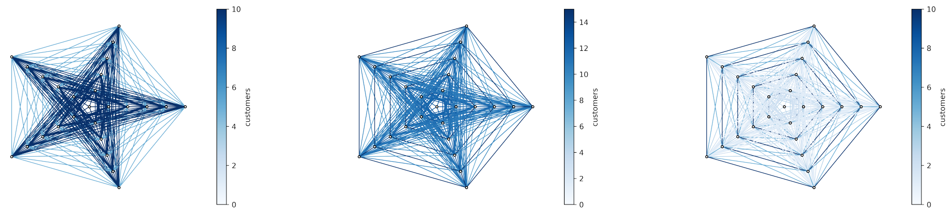
6 Experimental evaluation

We experimentally evaluate the performance of light spanning graphs for passenger assignment by comparing light spanners, (GORG) and passenger assignment methods from the literature. The implementations are done on a Intel(R) Core(TM) i5-1145G7 @ 2.60GHz machine with 32 GB RAM using Gurobi 10.01 [16] within the LinTim software framework [24].

Data. For the evaluation, we generate $(r \times s)$ orb-webs $\mathrm{W}_{r,s}$ for varying values of (r, s). We assume the costs to be either unit costs $c \equiv 1$ or to represent the Euclidean cost function defined in Section 2. We generate the demand $a_{u,v}$, $u, v \in V(G)$ as

$$a_{u,v} = \left\lceil M(d_r \frac{1}{r_{u,v}+1} + d_s \frac{1}{s_{u,v}+1}) \right\rceil$$

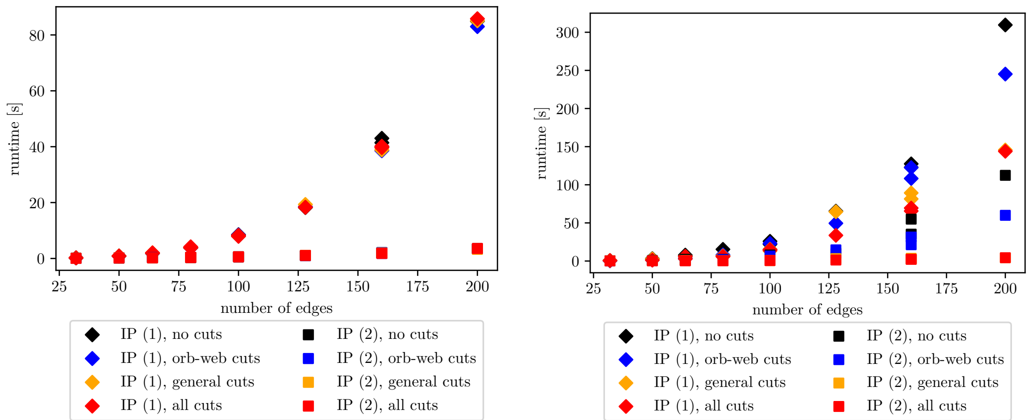
where $s_{u,v}$ is the number of spoke-edges on the shortest path from u to v according to Euclidean weights and $r_{u,v}$ is the number of ring-edges on this path. We set $M = 10$ here and vary (d_r, d_s) in $\{(1,0),(1,1),(0,1)\}$. For the (5×5) orb-web, the demand is represented in Figure 2. Note that in case $(d_r = 1, d_s = 0)$, the demand is highest between nodes which are on the same spoke and in case $(d_r = 0, d_s = 1)$ the demand is highest on nodes which are on the same ring. For $(d_r = 1, d_s = 1)$, we get a more balanced distribution of the demand.



(a) $(d_r = 1, d_s = 0)$. (b) $(d_r = 1, d_s = 1)$. (c) $(d_r = 0, d_s = 1)$.

Figure 2 Demand for a (5×5) orb-web. Edges uv represent demand from u to v where the shading corresponds to the amount of demand $a_{u,v}$. The darker the shading is, the higher is the demand.

Evaluation of formulation and cuts. We first analyze the runtime of the IP formulations (1) and (2) for (GORG) and the influence of the valid inequalities introduced in Section 5, see Figure 3. The demand is computed using $(d_r = 1, d_s = 1)$ and the size of the graphs is varied in {(4,4), (4,8), (8, 4), (5,5), (5,8), (8,5), (5,10), (10,5), (8,8), (8, 10), (10, 8), (10,10)}. The building cost bound K is derived from the building cost of lightest 1.5-spanner.



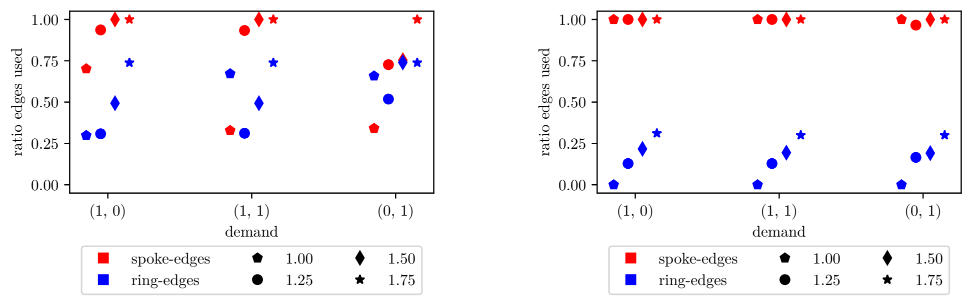
(a) Unit costs. (b) Euclidean costs.

Figure 3 Evaluating the runtime of both formulations (1) and (2) for (GORG) with and without the two sets of valid inequalities. The demand is computed according to $(d_r = 1, d_s = 1)$. The different graph sizes are aggregated by the number of edges $|E(G)|$.

Figure 3 shows that IP formulation (2) significantly outperforms IP formulation (1). Additionally, adding valid inequalities as described in Section 5 reduces the runtime. Here, the influence of the general cuts is higher than the influence of the orb-web specific cuts and the combination of both cuts yields even lower runtimes. For Euclidean costs, the improvement by using cuts is higher than for unit costs. Note that for demand $(d_r = 1, d_s = 1)$, instances with unit weights are considerably faster to solve than for Euclidean weights. However, the runtime for unit weights is highly dependent on the demand and the bound K.

Analyzing the structure of (GORG) solutions. Next, we analyze the structure of the solutions for (GORG) by considering the ratio of spoke and ring-edges in the optimal solution H compared to the original orb-web G. Figure 4 shows this for the demand settings $(d_r, d_s) \in \{(1,0),(1,1),(0,1)\}$ aggregated for orb-webs with varying size. Additionally, we investigate how the solution changes for increasing building cost bound K. Figure 4b shows that for the case of Euclidean costs, almost always all spoke-edges are in the optimal solution. Thus, increasing the building cost bound K leads to adding more ring-edges. Only for demand $(d_r = 0, d_s = 1)$, i.e., where most demand is on the same ring, there are solution where not all spokes edges are in $E(H)$. Note that also for greedy $(1+\epsilon)$-spanners, all spoke edges are in $E(H)$, see Corollary 4.

For unit costs, we get a different pattern. Here, the solution structure depends more on the demand. For $(d_r = 1, d_s = 0)$, i.e., when most demand is directed towards the center, the ratio of spoke-edges in the optimal solutions is highest. On the contrary, it is lowest for $(d_r = 0, d_s = 1)$, where instead there are more ring-edges in $E(H)$.



(a) Unit costs. **(b)** Euclidean costs.

Figure 4 Ratio of spoke and ring-edges in an optimal solution H of (GORG) compared to original orb-web G. The results are aggregated over orb-webs of varying sizes but split up according to the demand settings (d_r, d_s). For each demand scenario, four different bounds are used, i.e., $K = \alpha c_{\text{star}}$ where $\alpha \in \{1, 1.25, 1.5, 1.75\}$ and c_{star} is the weight of all spoke-edges.

The trade-off between routing costs, detour factor and building costs. In Figure 5, we consider the trade-off between the routing costs and the maximum detour factor for spanners, (GORG) and the passenger assignment model introduced in Section 2 for Euclidean costs. Note that the routing costs are normalized by the routing costs in the original graph G as we are considering orb-webs of different sizes. For each solution, the color represents the building costs of the solution, normalized by the building costs of a minimum spanning tree. As expected, allowing for higher building costs results in solutions dominating ones with lower building costs. The solutions computed by REWARD and REDUCTION have very low building costs but a high maximum detour factor and often also a high average detour factor, i.e., high routing costs. Routing passengers on shortest paths in G, i.e., using SP leads to a maximal and average detour factor of 1, but the building costs are high due to using all edges in the graph. Both spanners and (GORG) result in solutions which represent a reasonable trade-off between the solutions found by SP and REWARD and REDUCTION. Note that the points for spanners and (GORG) coincide, i.e., for Euclidean costs, spanners are a good approximation for (GORG). This fits to the results of Corollary 4 and the observations on Figure 4 as for both spanners and (GORG), all or almost all spoke edges are in an optimal solution.

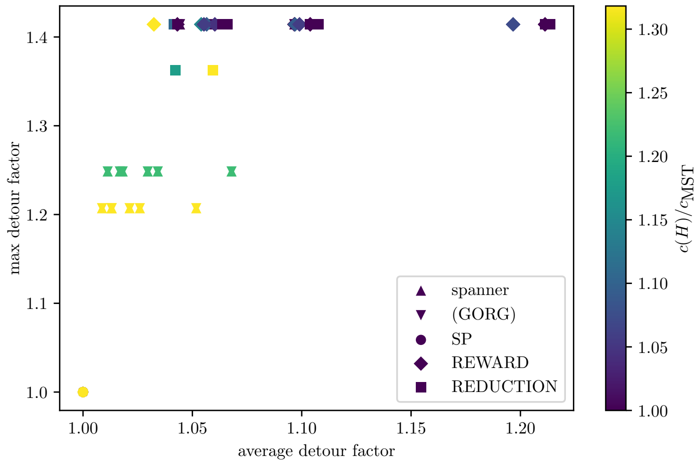
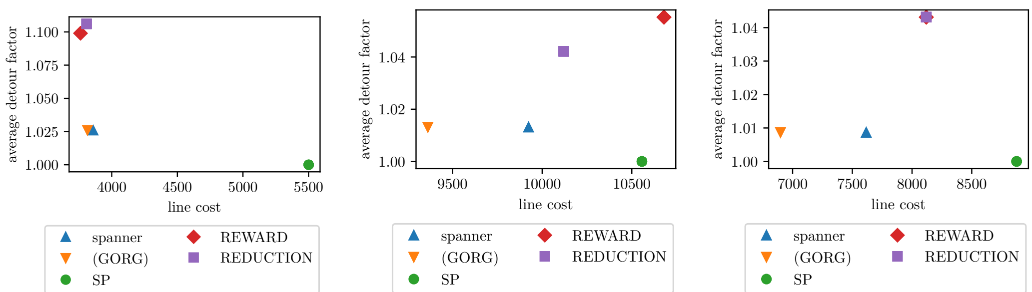


Figure 5 Trade-off between average detour factor, i.e., a normalization of the routing costs, and the maximum detour factor. For each solution, the color represents the building costs of the solution, normalized by the building costs of a minimum spanning tree. We are using orb-webs $W_{r,s}$ with Euclidean costs and $r, s \in \{5, 8\}$ and demand computed by $(d_r, d_s) \in \{(1,0), (1,1), (0,1)\}$.

Evaluation with line planning. Lastly, we evaluate the performance of light spanners according to the line planning objectives, average detour factor and line cost. Figure 6 shows this evaluation for a (8×8) orb-web with euclidean weights. We compute a 1.25-spanner, a solution for (GORG) for a building cost bound derived from the building cost of the spanner as well as passenger assignments using SP, REWARD and REDUCTION, see Section 2. For the resulting passenger assignment, we compute a line pool using the algorithm from [13] and a line plan according to the cost model [25]. While SP by definition always results in the lowest average detour factor and comparatively high line cost, the performance of the other approaches depends on the demand structure. Spanners and (GORG) always result in considerably lower average detour factor than REWARD and REDUCTION and for $(d_r, d_s) \in \{(1,1), (0,1)\}$ they even dominate those solution, i.e., they also result in lower line cost. For demand $(d_r, d_s) = (1, 0)$, REWARD results in slightly lower line cost. We conclude that using light spanning graphs for passenger assignment is a promising approach to find line plans that are satisfactory both from an operator's and a passengers' point of view.



(a) Demand $(d_r = 1, d_s = 0)$. **(b)** Demand $(d_r = 1, d_s = 1)$. **(c)** Demand $(d_r = 0, d_s = 1)$.

Figure 6 Evaluating the line cost and the average detour factor for solutions with $\epsilon = 0.25$ and resulting building cost bound. We use (8×8) orb-webs with Euclidean costs.

7 Conclusion and further research

In this paper, we apply the concept of light $(1+\epsilon)$-spanners and a generalization of optimum requirement graphs to passenger assignment in public transport planning. Therefore, we especially consider orb-webs and parametric city instances which represent a large class of real city networks with a high-demand center. Note that the concept of light $(1+\epsilon)$-spanners exactly mirrors the fairness measure in routing, which guarantees that the maximal detour factor over all passengers is bounded. Generalized optimum requirement graphs on the other hand represent a social optimum, where the total routing costs are minimized. Our experiments show that using light spanning graphs for passenger assignment can be beneficial for finding line plans that are attractive both from an operator's and a passengers' perspective.

While both considered problems are NP-hard in general, we identify polynomially solvable cases for greedy spanners and symmetric optimum requirement graphs on orb-webs. In future work, we aim to analyze the *price of symmetry*, i.e., how much optimal non-symmetric solutions differ from symmetric ones. Due to the reduced solution space, we expect that finding symmetric solutions is considerably easier in practice. Another interesting aspect is to improve the solution approaches, especially the IP-based approaches for generalized optimum requirement graphs. Here, it might be beneficial to consider Benders' decomposition approaches as well as a path-based reformulation which can be solved by column generation.

While the concept of light spanners is very well researched, there is little literature on generalized optimum requirement graphs. Only the case of finding trees with minimal routing costs is well understood. Thus, it is a natural extension to consider the theoretical properties of generalized optimum requirement graphs in future work. Especially in the context of public transport planning, moving from trees to general light spanning graphs is an important step towards applicability.

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