A Tour on Ecumenical Systems

Elaine Pimentel
Department of Computer Science, University College London, UK

Luiz Carlos Pereira
Department of Philosophy, UERJ, Brazil

Abstract

Ecumenism can be understood as a pursuit of unity, where diverse thoughts, ideas, or points of view coexist harmoniously. In logic, ecumenical systems refer, in a broad sense, to proof systems for combining logics. One captivating area of research over the past few decades has been the exploration of seamlessly merging classical and intuitionistic connectives, allowing them to coexist peacefully. In this paper, we will embark on a journey through ecumenical systems, drawing inspiration from Prawitz' seminal work [35]. We will begin by elucidating Prawitz' concept of "ecumenism" and present a pure sequent calculus version of his system. Building upon this foundation, we will expand our discussion to incorporate alethic modalities, leveraging Simpson's meta-logical characterization. This will enable us to propose several proof systems for ecumenical modal logics. We will conclude our tour with some discussion towards a term calculus proposal for the implicational propositional fragment of the ecumenical logic, the quest of automation using a framework based in rewriting logic, and an ecumenical view of proof-theoretic semantics.

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1 Introduction

What is a proof? In the context of logic and mathematics, a proof is a logical argument that establishes the correctness of a claim based on a set of assumed axioms and definitions, together with previously proven statements. Nevertheless, since the construction methods of these arguments may vary, a proof that appears satisfactory to a classical logician may not necessarily meet the criteria for an intuitionistic logician. For instance, constructive logicians do not accept mathematical proofs that explicitly employ the principle of excluded middle. But does this discrepancy solely pertain to proof methods? What is the real nature of this disagreement?

According to Prawitz [35] the accuracy of an inference relies on the assigned meaning of the *logical constants*, and classical and intuitionistic logicians differ in their interpretations to some of them. The case of disjunction is central in this discussion, since asserting that

$A \vee B$ is valid only if it is possible to give a proof of either A or B

often claimed to be enough for determining meaning of disjunction in intuitionistic logic, clearly does not correctly determine the meaning of the classical disjunction. In Prawitz' view, classical and intuitionistic logicians would also not agree on the meanings for the implication and existential quantifier, while they would share the same view regarding conjunction, negation, the constant for the absurd and the universal quantifier.

To explore the meanings of all these connectives collectively, Prawitz proposed an all-encompassing language known as *ecumenical logic*, which codifies both classical and intuitionistic reasoning based on a uniform pattern of meaning explanations. In the ecumenical language, the classical and intuitionistic constants coexist harmoniously: the subscript c is added when denoting the classical meaning, while the subscript i represents the intuitionistic meaning. This provides a neutral ground for the contestants, as described by Prawitz

> "The classical logician is not asserting what the intuitionistic logician denies. For instance, the classical logician asserts $A \vee_c \neg A$ to which the intuitionist does not object; he objects to the universal validity of $A \vee_i \neg A$, which is not asserted by the classical logician."

We embraced Prawitz's agenda in a series of works, delving into various aspects of ecumenism. In [32], we presented LE, a single-conclusion sequent calculus for Prawitz' original natural deduction ecumenical system. Using proof-theoretic methods, we showed that the ecumenical entailment is intrinsically intuitionistic, but it turns classical in the presence of classical succedents. We then produced a nested sequent version of the original sequent system and showed all of them sound and complete with respect to (first-order extension of) the ecumenical Kripke semantics [31]. Finally, we analysed fragments of the systems presented, coming to well known intuitionistic calculi and a sequent system for classical logic amenable to a treatment by goal directed proof search.

In [22], we lifted this discussion to modal logics, presenting an extension of LE with the alethic modalities of *necessity* and *possibility.* Our proposal for ecumenical modal logics comes in the light of Simpson's meta-logical interpretation of modalities [40] by embedding the expected semantical behavior of the modal operator into ecumenical first-order logic. This resulted in a labelled ecumenical modal system, amenable for modal extensions.

It turns out that the inference rules in the systems presented in [32, 22] are not *pure* [11] or *separable* [25], in the sense that the introduction rules for some connectives strongly depend on the presence of negation. In [23] we presented a *pure label free* calculus for ecumenical modalities, where every basic object of the calculus can be read as a formula in the language of the logic. For that, we used *nested systems* [7, 16, 6, 33] with a stoup [14], together with a new notion of *polarities* for ecumenical formulae.

Recently [24] all these aforementioned studies were revisited, and we started from a pure ecumenical first-order system and naturally expanded it to the modal case. Such pure systems allowed for a clearer notion of the *meaning* for connectives (including modalities), faithfully matching Prawitz' original intention, and the tradition of the *proof-theoretic semantics' school* [38, 39].

Proof-theoretic semantics aims not only to elucidate the meaning of a logical proof, but also to provide means for its use as a basic concept of semantic analysis. Hence while logical ecumenism provides a medium in which meaningful interactions may occur between classical

Intuitionistic and neutral Rules

$$\frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\ \wedge L \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ \wedge R \qquad \frac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{A \vee_i B, \Gamma \Rightarrow C}\ \vee_i L$$

$$\frac{\Gamma \Rightarrow A_j}{\Gamma \Rightarrow A_1 \vee_i A_2}\ \vee_i R_j \qquad \frac{A \rightarrow_i B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow C}{\Gamma, A \rightarrow_i B \Rightarrow C}\ \rightarrow_i L \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow_i B}\ \rightarrow_i R$$

$$\frac{\neg A, \Gamma \Rightarrow A}{\neg A, \Gamma \Rightarrow \bot}\ \neg L \qquad \frac{\Gamma, A \Rightarrow \bot}{\Gamma \Rightarrow \neg A}\ \neg R \qquad \frac{}{\bot, \Gamma \Rightarrow A}\ \bot L$$

$$\frac{A[y/x], \forall x.A, \Gamma \Rightarrow C}{\forall x.A, \Gamma \Rightarrow C}\ \forall L \qquad \frac{\Gamma \Rightarrow A[y/x]}{\Gamma \Rightarrow \forall x.A}\ \forall R \qquad \frac{A[y/x], \Gamma \Rightarrow C}{\exists_i x.A, \Gamma \Rightarrow C}\ \exists_i L \qquad \frac{\Gamma \Rightarrow A[y/x]}{\Gamma \Rightarrow \exists_i x.A}\ \exists_i R$$

Classical rules

$$\frac{A, \Gamma \Rightarrow \bot \quad B, \Gamma \Rightarrow \bot}{A \vee_c B, \Gamma \Rightarrow \bot}\ \vee_c L \qquad \frac{\Gamma, \neg A, \neg B \Rightarrow \bot}{\Gamma \Rightarrow A \vee_c B}\ \vee_c R \qquad \frac{A \rightarrow_c B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow \bot}{A \rightarrow_c B, \Gamma \Rightarrow \bot}\ \rightarrow_c L$$

$$\frac{\Gamma, A, \neg B \Rightarrow \bot}{\Gamma \Rightarrow A \rightarrow_c B}\ \rightarrow_c R \qquad \frac{p_i, \Gamma \Rightarrow \bot}{p_c, \Gamma \Rightarrow \bot}\ L_c \qquad \frac{\Gamma, \neg p_i \Rightarrow \bot}{\Gamma \Rightarrow p_c}\ R_c$$

$$\frac{A[y/x], \Gamma \Rightarrow \bot}{\exists_c x.A, \Gamma \Rightarrow \bot}\ \exists_c L \qquad \frac{\Gamma, \forall x.\neg A \Rightarrow \bot}{\Gamma \Rightarrow \exists_c x.A}\ \exists_c R$$

Initial, cut and Structural Rules

$$\frac{}{p_i, \Gamma \Rightarrow p_i}\ \mathsf{init} \qquad \frac{\Gamma \Rightarrow A \quad A, \Gamma \Rightarrow C}{\Gamma \Rightarrow C}\ \mathsf{cut} \qquad \frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow A}\ \mathsf{W}$$

Figure 1 Ecumenical sequent system LE. In rules $\forall R, \exists_i L, \exists_c L$, the eigenvariable y is fresh; p is atomic.

and intuitionistic logic, proof-theoretic semantics provides a way of clarifying what is at stake when one accepts or denies *reductio ad absurdum* as a meaningful proof method. In [26] we closed this circle, by showing how to coherently combine both approaches by providing not only a medium in which classical and intuitionistic *logics* may coexist, but also one in which classical and intuitionistic *notions of proof* may coexist.

Finally, building on Girard's original idea of stoup, we presented in [30] an ecumenical pure natural deduction system (NE_p) for the propositional fragment, which seems to be a promising step towards the proposal of a ecumenical term calculus.

In this text, we will synthesise the main aspects of the *op. cit.*, thus providing a tour on ecumenical systems inspired by Prawitz seminal work [35].

2 Ecumenical systems

In [35] Dag Prawitz proposed a natural deduction system where classical and intuitionistic logics could coexist in peace. The language $\mathcal{L}$ used for ecumenical systems is described as follows. We will use a subscript c for the classical meaning and i for the intuitionistic one, dropping such subscripts when formulae/connectives can have either meaning.

Classical and intuitionistic n-ary predicate symbols $(p_c, p_i, \ldots)$ co-exist in $\mathcal{L}$ but have different meanings. The neutral logical connectives $\{\bot, \neg, \wedge, \forall\}$ are common for classical and intuitionistic fragments, while $\{\rightarrow_i, \vee_i, \exists_i\}$ and $\{\rightarrow_c, \vee_c, \exists_c\}$ are restricted to intuitionistic and classical interpretations, respectively.

In [32] we presented the system LE (Figure 1), the sequent counterpart of Prawitz' natural deduction system. Sequents are build over $\mathcal{L}$-formulae, and have the form $\Gamma \Rightarrow A$, where Γ is a multiset. Moving from natural deduction to sequent systems allowed us to carefully analyse the ecumenical notion of *entailment*.

Denoting by $\vdash_{\mathsf{S}} A$ the fact that the formula A is a theorem in the proof system S, we showed that the ecumenical entailment $\Gamma \Rightarrow A$ is intrinsically intuitionistic, in the following sense.

▶ **Theorem 1.** *Let Γ, A be a multiset of ecumenical formulae. Then $\Gamma \Rightarrow A$ is provable in the system* LE *iff* $\vdash_{\mathsf{LE}} \bigwedge \Gamma \rightarrow_i A$

But when A is classical, that is, built from classical atomic predicates using only the connectives: $\rightarrow_c, \vee_c, \exists_c, \neg, \wedge, \forall$ and the unit $\bot$, then entailments can be read classically.

▶ **Theorem 2.** *Let A_c be a classical formula and Γ be a multiset of ecumenical formulae. Then*

$$\vdash_{\mathsf{LE}} \bigwedge \Gamma \rightarrow_c A_c \text{ iff } \vdash_{\mathsf{LE}} \bigwedge \Gamma \rightarrow_i A_c.$$

This justifies the ecumenical view of entailments in Prawitz's original proposal.

In [32] the system LE was presented also in a nested sequent version, and all the systems were shown sound and complete w.r.t. (the first-order extension of) the ecumenical Kripke semantics in [31]. Finally, in that work we analysed several fragments of the systems presented.

3 The quest for purity

Although being a powerful tool for describing proof-theoretical properties of Prawitz' ecumenical logic, LE is not satisfactory as a logical system since it is not *pure* [11]: the definition of classical connectives depend on other connectives. For example, introducing $\exists_c$ on the right depends on the presence of negation and the universal quantifier.

One way of *purifying* systems is by introducing the notion of *polarities*. As in linear logic [13], it is possible to polarise formulae [1] into *positive* and *negative* in both classical [14, 18] and intuitionistic [19] logics, where the application of rules is determined by the polarity of the active formula.

The choice of polarization of formulae may vary from system to system, though, as it depends on their intended behaviour. The following rules for the conjunction of positive/negative formulae, represented by P, Q and N, M respectively, are characteristic examples of the use of polarities in sequent systems

$$\frac{\Gamma_1 \Rightarrow \Delta_1, P \quad \Gamma_2 \Rightarrow \Delta_2, Q}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, P \wedge Q} \wedge_P \qquad \frac{\Gamma \Rightarrow \Delta, N \quad \Gamma \Rightarrow \Delta, M}{\Gamma \Rightarrow \Delta, M \wedge N} \wedge_N$$

In this case, polarities determine the multiplicative/additive behaviour of the rules for conjunction.

Another way of controlling rule applications is by separating the contexts into *bins*. For example, sequents may be restricted for having the form $\Gamma \Rightarrow \Delta; \Sigma$, where Γ, Δ, Σ represent sets or multisets of formulae, and the *stoup* Σ is limited to containing *at most* one formula. In such systems, it is common that the active formula in the conclusion of a rule is placed in the stoup.

Usually, in sequent systems polarities and stoup come together. Structural rules then control the movement of formulae in derivations, as in the following *decision* and *store* rules

$$\frac{\Gamma \Rightarrow \Delta; P}{\Gamma \Rightarrow \Delta, P; \cdot} \mathsf{D} \qquad \frac{\Gamma \Rightarrow \Delta, N; \cdot}{\Gamma \Rightarrow \Delta; N} \mathsf{store}$$

On a bottom-up reading of these rules, while in D positive formulae can be chosen to be "focused on", in store negative formulae are stored in the classical context. This often enables for a *two-phase* proof construction, where the focused formula P is systematically decomposed until reaching a leaf or a negative sub-formula N. In this last case, focusing is lost and N is stored, allowing for the beginning of a new focused phase.

Finally, in sequent systems combining polarities and stoup the cut rule can assume different forms, depending on the polarity or the placement of the cut-formula (or both). The following are typical examples of positive and negative cut rules.

$$
\dfrac{\Gamma_1 \Rightarrow \Delta_1; P \quad P, \Gamma_2 \Rightarrow \Delta_2; \Sigma}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2; \Sigma}\ \mathsf{cut}_P \qquad\qquad \dfrac{\Gamma_1 \Rightarrow \Delta_1, N; \cdot \quad N, \Gamma_2 \Rightarrow \Delta_2; \Sigma}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2; \Sigma}\ \mathsf{cut}_N
$$

In [24] we made use of polarities and stoups for proposing the pure ecumenical first-order sequent system LCE. Sequents with a stoup in LCE are built over $\mathcal{L}$-formulae and have the form $\Gamma \Rightarrow \Delta; \Sigma$. Intuitionistic formulae are *positive* and dealt in the stoup, while classical formulae are *negative* and their rules are handled by the classical context Δ.

The following states that LCE is correct and complete w.r.t. LE.

▶ **Theorem 3.** *The sequent $\Gamma \Rightarrow \Delta; \Sigma$ is provable in LCE iff $\Gamma, \neg\Delta \Rightarrow \Sigma$ is provable in LE.*

Moreover, it shows that a formula in the classical context actually corresponds to its negated version in the left context. This is justified by the fact that if A_c is classical, then $\vdash_{\mathsf{LE}} \Gamma, \neg A_c \Rightarrow \bot$ iff $\vdash_{\mathsf{LE}} \Gamma \Rightarrow A_c$[1].

Moving now to the natural deduction setting, in [30] we gave an ecumenical view to Parigot's natural deduction stoup mechanism [29]. This allowed the definition of the *pure harmonic* natural deduction system NE_p (depicted in Figure 2) for the propositional fragment of Prawitz' ecumenical logic.

While polarities are not considered in NE_p, the stoup controls the shape of derivations. The inference rules manipulate *stoups with a context*, which are expressions of the form $\Delta; \Sigma$, extensions of natural deduction formulae where Σ is the stoup and Δ is its accompanying context (similar to *alternatives* in [36]).

As a derivation example, the following version of Peirce's Law is provable in NE_p.

$$
3\ \dfrac{\dfrac{2\ \dfrac{[\cdot; ((A \to_c B) \to_c A)]^3 \qquad 1\ \dfrac{\dfrac{\dfrac{[\cdot; A]^1}{A; \cdot}\ \mathsf{D}}{A, B; \cdot}\ \mathsf{W}_c}{A; (A \to_c B)}\ {\to_c}\text{-int} \qquad \dfrac{[\cdot; A]^2}{A; \cdot}\ \mathsf{D}}{A, A; \cdot}\ {\to_c}\text{-elim}}{A; \cdot}\ \mathsf{C}}{\cdot; (((A \to_c B) \to_c A) \to_c A)}\ {\to_c}\text{-int}
$$

More interestingly, any sequent of the form $(((A \to_j B) \to_k A) \to_c A)$ with $j, k \in \{i, c\}$ is provable in NE_p. That is, provability is maintained if the outermost implication is classical.

NE_p's normalisation procedure is really interesting, since the presence of stoups enables two kinds of compositions on derivations: in the stoup or in the classical context (see [30] for the details). This reflects, in the natural deduction setting, the two forms of cut for sequent systems with stoup shown above.

[1] Actually the "if" part is valid for any ecumenical formula.

Intuitionistic and neutral Rules

$$\dfrac{\begin{array}{c}\Gamma \qquad [\cdot;A]\\ \Pi\\ \Delta;B\end{array}}{\Delta;A\to B}\ \to_i\text{-int} \qquad \dfrac{\begin{array}{c}\Gamma_1\\ \Pi_1\\ \Delta_1;A\to_i B\end{array} \qquad \begin{array}{c}\Gamma_2\\ \Pi_2\\ \Delta_2;A\end{array}}{\Delta_1,\Delta_2;B}\ \to_i\text{-elim}$$

$$\dfrac{\begin{array}{c}\Gamma\\ \Pi\\ \Delta;A_i\end{array}}{\Delta;A_1\vee_i A_2}\ \vee_i\text{-int} \qquad \dfrac{\begin{array}{c}\Gamma_1\\ \Pi_1\\ \Delta_1;A\vee_i B\end{array} \qquad \begin{array}{c}[\cdot;A] \qquad \Gamma_2\\ \Pi_2\\ \Delta_2;\Sigma\end{array} \qquad \begin{array}{c}[\cdot;B] \qquad \Gamma_3\\ \Pi_3\\ \Delta_3;\Sigma\end{array}}{\Delta_1,\Delta_2,\Delta_3;\Sigma}\ \vee_i\text{-elim}$$

Classical rules

$$\dfrac{\begin{array}{c}[\cdot;A] \qquad \Gamma\\ \Pi\\ \Delta,B;\cdot\end{array}}{\Delta;A\to_c B}\ \to_c\text{-int} \qquad \dfrac{\begin{array}{c}\Gamma_1\\ \Pi_1\\ \Delta_1;A\to_c B\end{array} \qquad \begin{array}{c}\Gamma_2\\ \Pi_2\\ \Delta_2;A\end{array} \qquad \begin{array}{c}[\cdot;B] \qquad \Gamma_3\\ \Pi_3\\ \Delta_3;\cdot\end{array}}{\Delta_1,\Delta_2,\Delta_3;\cdot}\ \to_c\text{-elim}$$

$$\dfrac{\begin{array}{c}\Gamma\\ \Pi\\ \Delta,A,B;\cdot\end{array}}{\Delta;A\vee_c B}\ \vee_c\text{-int} \qquad \dfrac{\begin{array}{c}\Gamma_1\\ \Pi_1\\ \Delta_1;A\vee_c B\end{array} \qquad \begin{array}{c}[\cdot;A] \qquad \Gamma_2\\ \Pi_2\\ \Delta_2;\cdot\end{array} \qquad \begin{array}{c}[\cdot;B] \qquad \Gamma_3\\ \Pi_3\\ \Delta_3;\cdot\end{array}}{\Delta_1,\Delta_2,\Delta_3;\cdot}\ \vee_c\text{-elim}$$

Hypothesis formation and structural rules

$$\cdot;A \qquad \dfrac{\begin{array}{c}\Gamma\\ \Pi\\ \Delta;A\end{array}}{\Delta,A;\cdot}\ \mathsf{D} \qquad \dfrac{\begin{array}{c}\Gamma\\ \Pi\\ \Delta;\cdot\end{array}}{\Delta;A}\ \mathsf{W}_i \qquad \dfrac{\begin{array}{c}\Gamma\\ \Pi\\ \Delta;\Sigma\end{array}}{\Delta,A;\Sigma}\ \mathsf{W}_c \qquad \dfrac{\begin{array}{c}\Gamma\\ \Pi\\ \Delta,A,A;\Sigma\end{array}}{\Delta,A;\Sigma}\ \mathsf{C}$$

Figure 2 Ecumenical natural deduction system with stoup NE_p.

The design of the proof system is not only a matter of taste: it also allows for adequate proposals for extensions and/or applications. As an example, in [29] Parigot shows that, when trying to establish a link between control operators and classical constructs, a satisfactory notion of reduction for usual natural deduction (with the classical absurdity rule [34]) is hard to achieve. According to him

> "The difficulties met in trying to use $\neg\neg A\to A$ (or the classical absurdity rule) as a type for control operators is not really due to classical logic, but much more to the deduction system in which it is expressed. It is not easy to find a satisfactory notion of reduction in usual natural deduction because of the restriction to one conclusion which forbids the most natural transformations of proofs (they often generate proofs with more than one conclusion). Of course, as a by-product of our work, we can get possible adequate reductions for usual natural deduction, but none of them can be called "the" canonical one."

Parigot's solution for tackling the subject reduction problem was exactly to adopt a system with stoup, where the double negated formulae are stored in the classical context. This served as inspiration to the ongoing work on an ecumenical term calculus, where the $\lambda\mu$ internalization of *stoups* and the continuation-passing aspect of general rules [37] are naturally mixed together.

On the other hand, the use of polarities and stoup in the sequent setting not only allows for a better proof theoretic view of Prawitz' original proposal, but it also serves as a solid ground for smoothly accommodating modalities [24].

4 Ecumenical modalities

In [22] we lifted the discussion about ecumenism to modal logics, by presenting an extension of EL with the alethic modalities of *necessity* and *possibility*. On doing so, there were many choices to be made and many relevant questions to be asked, *e.g.*: what is the ecumenical interpretation of ecumenical modalities? Should we add classical, intuitionistic, or neutral versions for modal connectives? We proposed an answer for these questions in the light of Simpson's meta-logical interpretation of modalities [40] by embedding the expected semantical behavior of the modal operator into the ecumenical first order logic.

Formally, the language of *(propositional, normal) modal formulae* consists of the propositional fragment of the classical language enhanced with the unary *modal operators* $\Box$ and $\Diamond$ concerning necessity and possibility, respectively [2]. Given a variable x, we recall the standard translation $[\cdot]_x$ from modal formulae into first-order formulae with at most one free variable, x, as follows: if p is atomic, then $[p]_x = p(x)$; $[\bot]_x = \bot$; for any binary connective $\star$, $[A \star B]_x = [A]_x \star [B]_x$; for the modal connectives

$$[\Box A]_x = \forall y(R(x, y) \rightarrow [A]_y) \qquad [\Diamond A]_x = \exists y(R(x, y) \wedge [A]_y)$$

where $R(x, y)$ is a binary predicate. $R(x, y)$ then represents the *accessibility relation* R in a Kripke frame.

A (object-)modal logic OL is then characterized by the respective interpretation of the modal model in the meta-theory ML (called *meta-logical characterization* [40]) as follows

$$\vdash_{OL} A \quad \text{iff} \quad \vdash_{ML} \forall x.[A]_x$$

Hence, if ML is classical logic (CL), the former definition characterizes the classical modal logic K [2], while if it is intuitionistic logic (IL), then it characterizes the intuitionistic modal logic IK [40]. In [22], we adopted EL as the meta-theory, hence characterizing the ecumenical modal logic EK.

The ecumenical translation $[\cdot]^e_x$ from propositional ecumenical formulae into LE is defined in the same way as the modal translation $[\cdot]_x$. For the case of modal connectives, observe that, due to Theorem 1, the interpretation of ecumenical consequence should be essentially *intuitionistic*. This implies that the box modality is a *neutral connective*. The diamond, on the other hand, has two possible interpretations: classical and intuitionistic, since its leading connective is an existential quantifier. Hence we should have the ecumenical modalities: $\Box, \Diamond_i, \Diamond_c$, determined by the translations

$$[\Box A]^e_x = \forall y(R(x, y) \rightarrow_i [A]^e_y)$$
$$[\Diamond_i A]^e_x = \exists_i y(R(x, y) \wedge [A]^e_y) \qquad [\Diamond_c A]^e_x = \exists_c y(R(x, y) \wedge [A]^e_y)$$

Setting $\mathcal{L}_{\mathcal{M}}$ as the ecumenical modal language (that is, built from $\mathcal{L}$ with ecumenical modalities), the translation above naturally induces the labelled language $\mathcal{L}_{\mathcal{L}}$ of *labelled modal formulae*, determined by *labelled formulae* of the form $x : A$ with $A \in \mathcal{L}_{\mathcal{M}}$ and *relational atoms* of the form xRy, where x, y range over a set of variables.

In [22] we proposed a non-pure labelled calculus for ecumenical modal logic. In [24] we achieved purity, as expected, by using polarities and sequents with stoup. *Labelled sequents with stoup* have the form $\Gamma \Rightarrow \Delta; x : A$, where Γ is a multiset containing labelled modal formulae and relational atoms, and Δ is a multiset containing labelled modal formulae. The notion of polarities can be lifted from LCE to modalities smoothly, both for labelled and non-labelled calculi. In the former, relational atoms are not polarizable.

In Figure 3 we present the pure, labelled ecumenical modal system labEK [24]. Observe that

$$\vdash_{\mathsf{labEK}} x : \Diamond_c A \leftrightarrow_i x : \neg\Box\neg A$$

On the other hand, $\Box$ and $\Diamond_i$ are not inter-definable. However, if A_c is classical, then

$$\vdash_{\mathsf{labEK}} x : \Box A_c \leftrightarrow_i x : \neg\Diamond_c\neg A_c$$

This means that, when restricted to the classical fragment, $\Box$ and $\Diamond_c$ are duals. This reflects well the ecumenical nature of the defined modalities.

We conclude this section by showing the delicate line separating ecumenical and classical systems. We show how even slight alterations within ecumenical systems can lead to their eventual breakdown and a collapse into the classical framework.

The first example is valid for first-order and modal cases (see [24]).

▶ **Example 4.** If the cut rule

$$\frac{\Gamma \Rightarrow \Delta, x : A; \Pi^* \quad x : A, \Gamma \Rightarrow \Delta; \Pi}{\Gamma \Rightarrow \Delta; \Pi}\ \mathsf{cut}$$

was admissible in labEK for an arbitrary formula A, then $\cdot \Rightarrow \cdot; x : A \vee_i \neg A$ would have the proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x : A \Rightarrow x : A \vee_i \neg A; x : A}\ \mathsf{init_i}}{x : A \Rightarrow x : A \vee_i \neg A; x : A \vee_i \neg A}\ \vee_i}{x : A \Rightarrow x : A \vee_i \neg A; \cdot}\ \mathsf{D}}{\cdot \Rightarrow x : A \vee_i \neg A; \neg A}\ \neg R}{\cdot \Rightarrow x : A \vee_i \neg A; x : A \vee_i \neg A}\ \vee_i}{\cdot \Rightarrow x : A \vee_i \neg A; \cdot}\ \mathsf{D} \quad \dfrac{}{x : A \vee_i \neg A, \Gamma \Rightarrow \cdot; x : A \vee_i \neg A}\ \mathsf{init_i}}{\cdot \Rightarrow \cdot; x : A \vee_i \neg A}\ \mathsf{cut}$$

Remember that $x : A \vee_i \neg A$ is positive, hence it can not be the cut-formula in cut_N.

The second interesting example regards extensions of the modal logic EK, which can be defined by adding extra modal axioms. Many of such axioms can be specified as formulas in first-order logic. For example, in the ecumenical setting, the axiom $\mathsf{T} : \Box A \rightarrow_i A \wedge A \rightarrow_i \Diamond_i A$ is specified by the first-order formula $\forall x.R(x, x)$, which corresponds to the rule[2]

$$\frac{xRx, \Gamma \Rightarrow \Delta; \Pi}{\Gamma \Rightarrow \Delta; \Pi}\ \mathsf{T}$$

The addition of T to EK yields the system EKT [22]. The next example shows that adding the axiom $\neg\Diamond_i\neg A \rightarrow_i \Box A$ to EKT has a disastrous propositional consequence.

[2] See [21] for a general framework using polarities and focusing for transforming axioms into rules in the first-order setting and [40, 43, 27] for other seminal works on the subject.

▶ **Example 5.** The following is a derivation of $x : A \vee_i \neg A$ in EKT, assuming $\neg\Diamond_i\neg A \to_i \Box A$ as an axiom[3]

$$
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\cfrac{}{xRy, y : A, y : \neg(A \vee_i \neg A) \Rightarrow \cdot; y : A}\ \mathsf{init}
}{xRy, y : A, y : \neg(A \vee_i \neg A) \Rightarrow \cdot; y : \bot}\ \neg L, \vee_i R_1
}{xRy, y : \neg(A \vee_i \neg A) \Rightarrow \cdot; x : \bot}\ \neg L, \vee_i R_2, \neg R
}{x : \Diamond_i\neg(A \vee_i \neg A) \Rightarrow \cdot; x : \bot}\ \Diamond_i L
}{\Rightarrow \cdot; x : \neg\Diamond_i\neg(A \vee_i \neg A)}\ \neg R
}{\Rightarrow \cdot; x : \Box(A \vee_i \neg A)}\ eq
\qquad
\cfrac{
\cfrac{
\cfrac{}{xRx, x : (A \vee_i \neg A) \Rightarrow \cdot; x : A \vee_i \neg A}\ \mathsf{init}
}{xRx, x : \Box(A \vee_i \neg A) \Rightarrow \cdot; x : A \vee_i \neg A}\ \Box L
}{x : \Box(A \vee_i \neg A) \Rightarrow \cdot; x : A \vee_i \neg A}\ \mathsf{T}
}{\Rightarrow \cdot; x : A \vee_i \neg A}\ \mathsf{cut}_P
$$

where eq represents the proof steps of the substitution of a boxed formula for its diamond version.[4] That is, if $\Box$ and $\Diamond_i$ are inter-definable, then $A \vee_i \neg A$ is a theorem and EKT collapses to classical KT.

4.1 A nested system for ecumenical modal logic

In [23] we went one step ahead and proposed a *pure label free* calculus for ecumenical modalities, where every basic object of the calculus can be read as a formula in the language of the logic. The price to pay for getting rid of labels was having to extend sequent systems with *nestings* [7, 16, 6, 33].

This not only allowed for establishing the meaning of modalities via the rules that determine their correct use (*logical inferentialism* [5]), but it also places ecumenical systems as a unifying framework for modalities of which well known modal systems are fragments.

We shall briefly describe the general idea behind a pure label free calculus for ecumenical modalities. First of all, inspired by [41], we adopt the following notation for (one-sided) sequents with stoup:

- formulae in the left context Γ (*left inputs*) will be marked with a full circle $^\bullet$;
- formulae in the classical right context Δ (*right inputs*) will be marked with a triangle $^\triangledown$;
- the formula in the stoup Σ (*right output*) will be marked with a white circle $^\circ$.

Hence, for example, the sequent with stoup $C \wedge D \Rightarrow \Diamond_c A; \neg B$ will be written as $C \wedge D^\bullet, \Diamond_c A^\triangledown, \neg B^\circ$.

Second, we substitute labels for nestings, where a single sequent is replaced with a tree of sequents, whose nodes are multisets of formulae, with the relationship between parent and child in the tree represented by bracketing $[\cdot]$.

For example, the labelled sequent with stoup $xRy, xRz, z : C \wedge D \Rightarrow x : \Diamond_c A; y : \neg B$ corresponds to the nested sequent $\Diamond_c A^\triangledown, [\neg B^\circ], [C \wedge D^\bullet]$, which in turn represents the following tree of sequents with stoup

$$
\begin{tikzcd}
\Rightarrow \cdot; \neg B \arrow[dr, dash] & & C \wedge D \Rightarrow \cdot; \cdot \arrow[dl, dash] \\
& \Rightarrow \Diamond_c A; \cdot &
\end{tikzcd}
$$

[3] Observe that $\vdash_{\mathsf{labEK}} x : \Box A \to_i \neg\Diamond_i\neg A$.

[4] We have presented a proof with cut for clarity, but remember that labEK has the cut-elimination property [24].

INTUITIONISTIC AND NEUTRAL RULES

$$\frac{\Gamma, x : A, x : B \Rightarrow \Delta; \Pi}{\Gamma, x : A \wedge B \Rightarrow \Delta; \Pi}\ \wedge L \qquad \frac{\Gamma \Rightarrow \Delta; x : A \quad \Gamma \Rightarrow \Delta; x : B}{\Gamma \Rightarrow \Delta; x : A \wedge B}\ \wedge R$$

$$\frac{\Gamma, x : A \Rightarrow \Delta; \Pi \quad \Gamma, x : B \Rightarrow \Delta; \Pi}{\Gamma, x : A \vee_i B \Rightarrow \Delta; \Pi}\ \vee_i L \qquad \frac{\Gamma \Rightarrow \Delta; x : A_j}{\Gamma \Rightarrow \Delta; x : A_1 \vee_i A_2}\ \vee_i R_j$$

$$\frac{\Gamma, x : A \rightarrow_i B \Rightarrow \Delta; x : A \quad \Gamma, x : B \Rightarrow \Delta; \Pi}{\Gamma, x : A \rightarrow_i B \Rightarrow \Delta; \Pi}\ \rightarrow_i L \qquad \frac{\Gamma, x : A \Rightarrow \Delta; x : B}{\Gamma \Rightarrow \Delta; x : A \rightarrow_i B}\ \rightarrow_i R$$

$$\frac{\Gamma, x : \neg A \Rightarrow \Delta; x : A}{\Gamma, x : \neg A \Rightarrow \Delta; \cdot}\ \neg L \qquad \frac{\Gamma, x : A \Rightarrow \Delta; \cdot}{\Gamma \Rightarrow \Delta; x : \neg A}\ \neg R$$

CLASSICAL RULES

$$\frac{\Gamma, x : A \rightarrow_c B \Rightarrow \Delta; x : A \quad \Gamma, x : B \Rightarrow \Delta; \cdot}{\Gamma, x : A \rightarrow_c B \Rightarrow \Delta; \cdot}\ \rightarrow_c L \qquad \frac{\Gamma, x : A \Rightarrow x : B, \Delta; \cdot}{\Gamma \Rightarrow x : A \rightarrow_c B, \Delta; \cdot}\ \rightarrow_c R$$

$$\frac{\Gamma, x : A \Rightarrow \Delta; \cdot \quad \Gamma, x : B \Rightarrow \Delta; \cdot}{\Gamma, x : A \vee_c B \Rightarrow \Delta; \cdot}\ \vee_c L \qquad \frac{\Gamma \Rightarrow x : A, x : B, \Delta; \cdot}{\Gamma \Rightarrow x : A \vee_c B, \Delta; \cdot}\ \vee_c R$$

$$\frac{\Gamma, x : p_i \Rightarrow \Delta; \cdot}{\Gamma, x : p_c \Rightarrow \Delta; \cdot}\ L_c \qquad \frac{\Gamma \Rightarrow x : p_i, \Delta; \cdot}{\Gamma \Rightarrow x : p_c, \Delta; \cdot}\ R_c$$

MODAL RULES

$$\frac{xRy, y : A, x : \Box A, \Gamma \Rightarrow \Delta; \Pi}{xRy, x : \Box A, \Gamma \Rightarrow \Delta; \Pi}\ \Box L \qquad \frac{xRy, \Gamma \Rightarrow \Delta; y : A}{\Gamma \Rightarrow \Delta; x : \Box A}\ \Box R \qquad \frac{xRy, y : A, \Gamma \Rightarrow \Delta; \Pi}{x : \Diamond_i A, \Gamma \Rightarrow \Delta; \Pi}\ \Diamond_i L$$

$$\frac{xRy, \Gamma \Rightarrow \Delta; y : A}{xRy, \Gamma \Rightarrow \Delta; x : \Diamond_i A}\ \Diamond_i R \qquad \frac{xRy, y : A, \Gamma \Rightarrow \Delta; \cdot}{x : \Diamond_c A, \Gamma \Rightarrow \Delta; \cdot}\ \Diamond_c L \qquad \frac{xRy, \Gamma \Rightarrow y : A, x : \Diamond_c A, \Delta; \cdot}{xRy, \Gamma \Rightarrow x : \Diamond_c A, \Delta; \cdot}\ \Diamond_c R$$

INITIAL, DECISION AND STRUCTURAL RULES

$$\frac{}{\Gamma, x : A \Rightarrow \Delta; x : A}\ \mathsf{init}_i \qquad \frac{}{\Gamma, x : A \Rightarrow x : A, \Delta; \Pi}\ \mathsf{init}_c$$

$$\frac{\Gamma \Rightarrow x : P, \Delta; x : P}{\Gamma \Rightarrow x : P, \Delta; \cdot}\ \mathsf{D} \qquad \frac{\Gamma \Rightarrow x : N, \Delta; \cdot}{\Gamma \Rightarrow \Delta; x : N}\ \mathsf{store} \qquad \frac{\Gamma \Rightarrow \Delta; \cdot}{\Gamma \Rightarrow \Delta; x : A}\ \mathsf{W}$$

CUT RULES

$$\frac{\Gamma \Rightarrow \Delta; x : P \quad x : P, \Gamma \Rightarrow \Delta; \Pi}{\Gamma \Rightarrow \Delta; \Pi}\ \mathsf{cut}_P \qquad \frac{\Gamma \Rightarrow \Delta, x : N; \Pi^* \quad x : N, \Gamma \Rightarrow \Delta; \Pi}{\Gamma \Rightarrow \Delta; \Pi}\ \mathsf{cut}_N$$

Figure 3 Ecumenical pure modal labelled system labEK. In rules $\Box R, \Diamond_i L, \Diamond_c L$, the eigenvariable y does not occur free in any formula of the conclusion; N is negative and P is positive; p is atomic; Π^* is either empty or some $z : P \in \Delta$.

The modal rules in nested systems then govern the transfer of (modal) formulae between the different sequents, and they are *local*, in the sense that it is sufficient to transfer only one formula at a time.

In [24] we presented the nested ecumenical modal system nEK. We will highlight next some of its rules. Starting with modalities, the nested rules for the intuitionistic diamond are

$$\frac{\Gamma\{[A^\bullet]\}}{\Gamma\{\Diamond_i A^\bullet\}}\ \Diamond_i^\bullet \qquad \frac{\Lambda_1\{[A^\circ,\Lambda_2]\}}{\Lambda_1\{\Diamond_i A^\circ,[\Lambda_2]\}}\ \Diamond_i^\circ$$

where Λ represents a nested context containing only input formulae[5]. In the worlds-as-nestings interpretation [12], doing proof search in a system containing these rules actually corresponds to moving bottom-up on a Kripke structure: in rule $\Diamond_i^\bullet$, assuming $\Diamond_i A$ in a certain nesting (corresponding to a world, say, x) is equivalent to creating a new nesting (corresponding to a fresh world, say, y related to x) and assuming A there (compare with rule $\Diamond_i L$ in Figure 3).

Polarities determine the mobility of formulae between contexts, via the decision and store rules.

$$\frac{\Gamma^*\{P^\triangledown,P^\circ\}}{\Gamma^{\perp^\circ}\{P^\triangledown\}}\ \mathsf{D} \qquad \frac{\Lambda\{N^\triangledown,\perp^\circ\}}{\Lambda\{N^\circ\}}\ \mathsf{store}$$

In a bottom-up reading, a positive formula is chosen to be "focused on" in the decision rule D, while a negative formula in the stoup can be stored in the classical context by using the rule store, just as described in Section 3.

Finally, the positive and negative nested versions of the cut rule are given by

$$\frac{\Gamma^*\{P^\circ\} \quad \Gamma\{P^\bullet\}}{\Gamma\{\varnothing\}}\ \mathsf{cut}^\circ \qquad \frac{\Gamma^P\{N^\triangledown\} \quad \Gamma\{N^\bullet\}}{\Gamma\{\varnothing\}}\ \mathsf{cut}^\triangledown$$

where Γ^P denotes that the context contains either $\perp^\circ$ or P° for some $P^\triangledown \in \Gamma$. In [24] showed that both cut rules are admissible in nEK. Moreover, nEK was shown to be sound and complete w.r.t. an ecumenical birelational model. Since the same result holds for the labelled system labEK, the two systems are equivalent. Finally, the *op.cit.* also brings a discussion about fragments, axioms and extensions of ecumenical modal logics.

5 What is next?

There are still many paths to be traversed on this journey. We finish this text by discussing some future ideas and presenting related work.

Computational interpretation. As mentioned at the end of Section 3, we have been exploring the computational counterpart of the implicational fragment of the ecumenical logic, extending the paradigm "proofs-as-programs" to ecumenical proofs. There are two main challenges on this enterprise: (i) finding an adequate deduction system in which the classical and intuitionistic logical behaviours can be faithfully captured in a term calculus; (ii) dealing with general ecumenical natural deduction rules. In [30] we tackled part (i) by proposing the ecumenical pure natural deduction system NE_p, where the $\lambda\mu$ internalization of *stoups* can

[5] Observe that rules are applied anywhere in the nesting structure, which is represented by *contexts with a hole* of the form $\Gamma\{\,\}$.

be easily adapted to the ecumenical case. Regarding, (ii), we are currently investigating the possibility of formulating an ecumenical version of the call-by-name lambda-calculus with generalized applications presented in [37] which integrates a notion of distant reduction that allows to unblock β-redexes without resorting to the permutative conversions of generalized applications.

Automated theorem proving. In [28] we developed an algorithmic-based approach for proving inductive properties of propositional sequent systems such as admissibility, invertibility, cut-elimination, and identity expansion. The proposed algorithms are based on rewrite and narrowing techniques. They have been fully mechanized in the `L-Framework`, thus offering both proof-search algorithms and semi-decision procedures for proving theorems and meta-theorems of several logical systems. We have started implementing the sequent-based systems mentioned in this text in the `L-Framework`, proving proof-theoretic properties to some of them. The next step is to specify nested sequent systems, which turns out to be a real challenge.

Proof-theoretic semantics. Together with logical ecumenism, *proof-theoretic semantics* [38, 39] is another approach to logic currently providing interesting contributions to the debate concerning philosophical grounds for the validity of classical and intuitionistic logics. While logical ecumenism proposes an unified framework in which two "rival" logics may peacefully coexist, proof-theoretic semantics aims not only to elucidate the meaning of a logical proof, but also to provide means for its use as a basic concept of semantic analysis. In [26] we showed how to coherently combine both approaches by providing not only a medium in which classical and intuitionistic *logics* may coexist, but also one in which classical and intuitionistic *notions of proof* may coexist. We did not, however, provided a proof-theoretic semantics for Prawitz' original system, or any of the systems presented here – this is future work.

Related work

Given that ecumenical systems refer, in a broad sense, to proof systems for combining logics, the related work on this subject is extensive and encompasses numerous other works. We will mention few which serve as reference to the present work.

Peter Krauss [17] and Gilles Dowek [10] explored the same ecumenical ideas as the ones shown in this text. Their main motivation was mathematical: to explore the possibility of hybrid readings of axioms and proofs in mathematical theories, *i.e.*, the occurrences of classical and intutionistic operators in mathematical axioms and proofs, in order to propose a new and original method of constructivisation of classical mathematics. Krauss applied these ideas in basic algebraic number theory and Dowek considered the example of an ecumenical proof of a simple theorem in basic set theory.

Dowek's original work has been further explored in [3] and [4]. In that works, a (type) theory in $\lambda\Pi$-calculus modulo theory is investigated, where proofs of several logical systems can be expressed.

Regarding proof systems, there is the seminal work of Girard in [15] and the more recent work of Liang and Miller [18]. Their work is based on *polarities* and *focusing*, using translations into linear logic.

A complete different approach comes from the school of combining logics [9, 20, 8], where Hilbert like systems are built from a combination of axiomatic systems.

Finally, we would like to cite Tesi and Negri's work on an ecumenical approach to infinitary logic [42], where a labelled sequent calculus combining classical and intuitionistic connectives is proposed.

References

1 Jean-Marc Andreoli. Logic programming with focusing proofs in linear logic. *J. Log. Comput.*, 2(3):297–347, 1992. `doi:10.1093/logcom/2.3.297`.

2 Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic.* Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2001. `doi:10.1017/CBO9781107050884`.

3 Frédéric Blanqui, Gilles Dowek, Émilie Grienenberger, Gabriel Hondet, and François Thiré. Some axioms for mathematics. In Naoki Kobayashi, editor, *6th International Conference on Formal Structures for Computation and Deduction, FSCD 2021, July 17-24, 2021, Buenos Aires, Argentina (Virtual Conference)*, volume 195 of *LIPIcs*, pages 20:1–20:19. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021. `doi:10.4230/LIPIcs.FSCD.2021.20`.

4 Frédéric Blanqui, Gilles Dowek, Émilie Grienenberger, Gabriel Hondet, and François Thiré. A modular construction of type theories. *Log. Methods Comput. Sci.*, 19(1), 2023. `doi:10.46298/lmcs-19(1:12)2023`.

5 Robert Brandom. *Articulating Reasons: An Introduction to Inferentialism.* Harvard University Press, 2000.

6 Kai Brünnler. Deep sequent systems for modal logic. *Arch. Math. Log.*, 48:551–577, 2009.

7 Robert A. Bull. Cut elimination for propositional dynamic logic wihout *. *Zeitschr. f. math. Logik und Grundlagen d. Math.*, 38:85–100, 1992.

8 Carlos Caleiro and Jaime Ramos. Combining classical and intuitionistic implications. In Boris Konev and Frank Wolter, editors, *Frontiers of Combining Systems, 6th International Symposium, FroCoS 2007, Liverpool, UK, September 10-12, 2007, Proceedings*, volume 4720 of *Lecture Notes in Computer Science*, pages 118–132. Springer, 2007. `doi:10.1007/978-3-540-74621-8_8`.

9 Luis Fariñas del Cerro and Andreas Herzig. Combinig classical and intuitionistic logic, or: Intuitionistic implication as a conditional. In Franz Baader and Klaus U. Schulz, editors, *Frontiers of Combining Systems, First International Workshop FroCoS 1996, Munich, Germany, March 26-29, 1996, Proceedings*, volume 3 of *Applied Logic Series*, pages 93–102. Kluwer Academic Publishers, 1996.

10 Gilles Dowek. On the definition of the classical connectives and quantifiers. *Why is this a Proof?, Festschrift for Luiz Carlos Pereira*, 27:228–238, 2016.

11 Michael Dummett. *The Logical Basis of Metaphysics.* Cambridge, Mass.: Harvard University Press, 1991.

12 Melvin Fitting. Nested sequents for intuitionistic logics. *Notre Dame Journal of Formal Logic*, 55(1):41–61, 2014. `doi:10.1215/00294527-2377869`.

13 Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

14 Jean-Yves Girard. A new constructive logic: Classical logic. *Math. Struct. Comput. Sci.*, 1(3):255–296, 1991. `doi:10.1017/S0960129500001328`.

15 Jean-Yves Girard. On the unity of logic. *Ann. Pure Appl. Logic*, 59(3):201–217, 1993. `doi:10.1016/0168-0072(93)90093-S`.

16 Ryo Kashima. Cut-free sequent calculi for some tense logics. *Studia Logica*, 53(1):119–136, 1994. `doi:10.1007/BF01053026`.

17 Peter Krauss. A constructive refinement of classical logic. Draft, 1992.

18 Chuck Liang and Dale Miller. A focused approach to combining logics. *Ann. Pure Appl. Logic*, 162(9):679–697, 2011. `doi:10.1016/j.apal.2011.01.012`.

19 Chuck C. Liang and Dale Miller. Focusing and polarization in intuitionistic logic. In Jacques Duparc and Thomas A. Henzinger, editors, *Computer Science Logic, 21st International Workshop, CSL 2007, 16th Annual Conference of the EACSL, Lausanne, Switzerland, September 11-15, 2007, Proceedings*, volume 4646 of *Lecture Notes in Computer Science*, pages 451–465. Springer, 2007. doi:10.1007/978-3-540-74915-8_34.

20 Paqui Lucio. Structured sequent calculi for combining intuitionistic and classical first-order logic. In Hélène Kirchner and Christophe Ringeissen, editors, *Frontiers of Combining Systems, Third International Workshop, FroCoS 2000, Nancy, France, March 22-24, 2000, Proceedings*, volume 1794 of *Lecture Notes in Computer Science*, pages 88–104. Springer, 2000. doi:10.1007/10720084_7.

21 Sonia Marin, Dale Miller, Elaine Pimentel, and Marco Volpe. From axioms to synthetic inference rules via focusing. *Ann. Pure Appl. Log.*, 173(5):103091, 2022. doi:10.1016/j.apal.2022.103091.

22 Sonia Marin, Luiz Carlos Pereira, Elaine Pimentel, and Emerson Sales. Ecumenical modal logic. In *DaLí 2020*, volume 12569 of *LNCS*, pages 187–204. Springer, 2020. doi:10.1007/978-3-030-65840-3_12.

23 Sonia Marin, Luiz Carlos Pereira, Elaine Pimentel, and Emerson Sales. A pure view of ecumenical modalities. In Alexandra Silva, Renata Wassermann, and Ruy J. G. B. de Queiroz, editors, *Logic, Language, Information, and Computation - 27th International Workshop, WoLLIC 2021, Virtual Event, October 5-8, 2021, Proceedings*, volume 13038 of *Lecture Notes in Computer Science*, pages 388–407. Springer, 2021. doi:10.1007/978-3-030-88853-4_24.

24 Sonia Marin, Luiz Carlos Pereira, Elaine Pimentel, and Emerson Sales. Separability and harmony in ecumenical systems. *CoRR*, abs/2204.02076, 2022. doi:10.48550/arXiv.2204.02076.

25 Julien Murzi. Classical harmony and separability. *Erkenntnis*, 85:391–415, 2020.

26 Victor Nascimento, Luiz Carlos Pereira, and Elaine Pimentel. An ecumenical view of proof-theoretic semantics, 2023. arXiv:2306.03656.

27 Sara Negri. Proof analysis in modal logic. *J. Philosophical Logic*, 34(5-6):507–544, 2005. doi:10.1007/s10992-005-2267-3.

28 Carlos Olarte, Elaine Pimentel, and Camilo Rocha. A rewriting logic approach to specification, proof-search, and meta-proofs in sequent systems. *CoRR*, abs/2101.03113, 2021. arXiv:2101.03113.

29 Michel Parigot. $\lambda\mu$-calculus: An algorithmic interpretation of classical natural deduction. In *LPAR: Logic Programming and Automated Reasoning, International Conference*, volume 624 of *LNCS*, pages 190–201. Springer, 1992.

30 Luiz Carlos Pereira and Elaine Pimentel. On an ecumenical natural deduction with stoup - part I: the propositional case. *CoRR*, abs/2204.02199, 2022. doi:10.48550/arXiv.2204.02199.

31 Luiz Carlos Pereira and Ricardo Oscar Rodriguez. Normalization, soundness and completeness for the propositional fragment of Prawitz' ecumenical system. *Revista Portuguesa de Filosofia*, 73(3-3):1153–1168, 2017.

32 Elaine Pimentel, Luiz Carlos Pereira, and Valeria de Paiva. An ecumenical notion of entailment. *Synthese*, 198(22-S):5391–5413, 2021. doi:10.1007/s11229-019-02226-5.

33 Francesca Poggiolesi. The method of tree-hypersequents for modal propositional logic. In *Towards Mathematical Philosophy*, volume 28 of *Trends In Logic*, pages 31–51. Springer, 2009.

34 Dag Prawitz. *Natural Deduction, volume 3 of Stockholm Studies in Philosophy.* Almqvist and Wiksell, 1965.

35 Dag Prawitz. Classical versus intuitionistic logic. In Bruno Lopes Edward Hermann Haeusler, Wagner de Campos Sanz, editor, *Why is this a Proof?, Festschrift for Luiz Carlos Pereira*, volume 27, pages 15–32. College Publications, 2015.

36 Greg Restall. Speech acts & the quest for a natural account of classical proof. Available at https://consequently.org/writing/, 2021.

37 José Espírito Santo, Delia Kesner, and Loïc Peyrot. A faithful and quantitative notion of distant reduction for generalized applications. In Patricia Bouyer and Lutz Schröder, editors, *Foundations of Software Science and Computation Structures - 25th International Conference, FOSSACS 2022, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2022, Munich, Germany, April 2-7, 2022, Proceedings*, volume 13242 of *Lecture Notes in Computer Science*, pages 285–304. Springer, 2022. `doi:10.1007/978-3-030-99253-8_15`.

38 Peter Schroeder-Heister. Uniform proof-theoretic semantics for logical constants (abstract). *Journal of Symbolic Logic*, 56:1142, 1991.

39 Peter Schroeder-Heister. Validity concepts in proof-theoretic semantics. *Synthese*, 148(3):525–571, 2006.

40 Alex K. Simpson. *The Proof Theory and Semantics of Intuitionistic Modal Logic*. PhD thesis, College of Science and Engineering, School of Informatics, University of Edinburgh, 1994.

41 Lutz Straßburger. Cut elimination in nested sequents for intuitionistic modal logics. In *Proceedings of FOSSACS 2013*, pages 209–224, 2013. `doi:10.1007/978-3-642-37075-5_14`.

42 Matteo Tesi and Sara Negri. Neighbourhood semantics and labelled calculus for intuitionistic infinitary logic. *J. Log. Comput.*, 31(7):1608–1639, 2021. `doi:10.1093/logcom/exab040`.

43 Luca Viganò. *Labelled Non-Classical Logics*. Kluwer Academic Publishers, 2000.