More Than 0s and 1s: Metric Quantifiers and Counting over Timed Words

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Abstract

We study the expressiveness of the *pointwise* interpretations (i.e. over timed words) of some predicate and temporal logics with metric and counting features. We show that counting in the unit interval $(0, 1)$ is strictly weaker than counting in $(0, b)$ with arbitrary $b \geq 0$; moreover, allowing the latter indeed leads to expressive completeness for the metric predicate logic Q2MLO, recovering the corresponding result for the continuous interpretations (i.e. over signals). Exploiting this connection, we show that in contrast to the continuous case, adding "punctual" predicates into Q2MLO is still insufficient for the full expressive power of the Monadic First-Order Logic of Order and Metric (FO$[<, +1]$). Finally, we propose a generalisation of the recently proposed Pnueli automata modalities and show that the resulting metric temporal logic is expressively complete for FO$[<, +1]$.

2012 ACM Subject Classification Theory of computation → Logic and verification

Keywords and phrases Temporal Logic, Expressiveness, Automata

Digital Object Identifier 10.4230/LIPIcs.TIME.2023.7

1 Introduction

Timed logics. *Metric Temporal Logic* (MTL) [22] is a natural extension of *Linear Temporal Logic* (LTL) [31] with the capability of expressing real-time constraints by allowing intervals I to be specified with the "until" ($\mathbf{U}$) and "since" ($\mathbf{S}$) modalities of LTL. Intuitively, $p\,\mathbf{U}_I\,q$ holds at a position i if there is a position j in the future where q holds, the time difference between i and j is within I, and p holds at all the points between i and j. While MTL provides a convenient and intuitive syntax for timing constraints, the problem of whether a given MTL formula has a model (behaviour) that satisfies it is undecidable [3, 29] – this makes MTL infeasible as a specification formalism for practical verification tasks. To remedy this issue, Alur, Feder, and Henzinger proposed in a seminal work [1] a syntactic fragment of MTL called *Metric Interval Temporal Logic* (MITL) where intervals associated with modalities are "*non-punctual*", i.e. non-singular. They showed that the satisfiability and model-checking problems for MITL are decidable with ExpSpace-complete complexity. In other words, by sacrificing perfect timing precision, we obtain a fully decidable timed specification formalism capable of expressing many practical properties of interest (see, e.g., [35]).

Expressiveness. Pnueli conjectured in the early 1990s that the trivial property "p and then q will happen in the next time unit' is not expressible in timed temporal logics like MTL and MITL. The conjecture (in different forms) is proved in [5, 12, 13, 30] and has led to several decidable extensions of MITL; one of the most notable extensions amongst them is Hirshfeld and Rabinovich's Q2MLO [12]. It is straightforward to express the *counting modalities* and *Pnueli modalities* (a more general form of the aforementioned conjecture) in Q2MLO, and it admits a very simple and natural metric temporal logic characterisation: the extension of MITL with counting modalities is *expressively complete* for Q2MLO [16]. However, most of these results only hold for the *continuous* interpretations (i.e. over signals) of these logics

30th International Symposium on Temporal Representation and Reasoning (TIME 2023).
Editors: Alexander Artikis, Florian Bruse, and Luke Hunsberger; Article No. 7; pp. 7:1–7:15
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

$$\mathsf{C}_{(0,1)}\mathsf{MTL} \overset{①}{\equiv} \mathsf{C}_0\mathsf{MTL} \overset{②}{\equiv} \mathsf{CMTL} \overset{③}{\equiv} \mathsf{PQ2MLO} \overset{④}{\equiv} \mathsf{FO}[<,+1]\,.$$
$$\mathsf{C}_{(0,1)}\mathsf{MITL} \overset{⑤}{\equiv} \mathsf{C}_0\mathsf{MITL} \overset{⑥}{\equiv} \mathsf{CMITL} \overset{⑦}{\equiv} \mathsf{Q2MLO}\,.$$

Figure 1 The relevant expressiveness results in the continuous semantics. ② is trivial, e.g., $\mathbf{C}^2_{(1,2)}\, p \iff \mathbf{F}_{=1}\,\mathbf{C}^2_{(0,1)}\, p$. ① can similarly be seen to hold by an easy case analysis, e.g., $\mathbf{C}^2_{(0,2)}\, p \iff \mathbf{C}^2_{(0,1)}\, p \vee \mathbf{F}_{=1}(\mathbf{C}^2_{(0,1)}\, p) \vee \big(\mathbf{C}^1_{(0,1)}\, p \wedge \mathbf{F}_{=1}(p \vee \mathbf{C}^1_{(0,1)}\, p)\big)$. ③ and ④ are proved in [20]. ⑤, ⑥, and ⑦ follow from [14, 16] and [19].

$$\mathsf{C}_{(0,1)}\mathsf{MTL}_{\mathsf{fut}} \overset{①}{\subsetneq} \mathsf{C}_0\mathsf{MTL}_{\mathsf{fut}} \overset{②}{\subsetneq} \mathsf{CMTL}_{\mathsf{fut}} \overset{③}{\subseteq} \mathsf{PQ2MLO} \subseteq \mathsf{PGQMLO} \subseteq \mathsf{FO}[<,+1]\,.$$
$$\mathsf{C}_{(0,1)}\mathsf{MITL}_{\mathsf{fut}} \overset{④}{\subsetneq} \mathsf{C}_0\mathsf{MITL}_{\mathsf{fut}} \overset{⑤}{\subsetneq} \mathsf{CMITL}_{\mathsf{fut}} \overset{⑥}{\subseteq} \mathsf{Q2MLO}\,.$$

Figure 2 The relevant known expressiveness results in the pointwise semantics (where the subscript "fut" stands for the future-only fragments). ①, ②, ④, and ⑤ are proved in [24]. ③ and ⑥ follow from [19]. The rest are syntactic inclusions.

and do not hold for the *pointwise* interpretations (i.e. over timed words). This is unfortunate from a practical point of view, as the latter is usually more amenable to automata-based implementations (e.g., UPPAAL [27]).

Contributions. The present work focusses on the expressiveness of these logics. We show that, as opposed to the situation in the continuous semantics, counting in $(0, b\rangle$ is strictly more expressive than counting in $(0, 1)$, and by allowing this modest generalisation we can actually recover the expressive completeness result for Q2MLO; this is also in stark contrast with the future-only fragments of these logics in the pointwise semantics, where counting in $(0, b\rangle$ is still insufficient for the expressiveness of (future) Q2MLO [24]. Similarly, we show that Q2MSO (the second-order version of Q2MLO) is characterised by MITL with counting modalities and *untimed* automata modalities. Finally, we show that Q2MLO with punctual predicates is still strictly less expressive than $\mathsf{FO}[<,+1]$ (once again in stark contrast with the continuous case), and we propose an extension to achieve the full expressiveness of $\mathsf{FO}[<,+1]$.

Related work. Compared to the situation in the continuous semantics, there are very few expressive completeness results regarding timed temporal logics like MTL and MITL in the pointwise semantics in the literature. D'Souza and Tabareau [8] showed that "vanilla" MITL is expressively complete for a restricted fragment of the *Monadic First-Order Logic of Order and Metric* ($\mathsf{FO}[<,+1]$) in the pointwise semantics. It is shown in [17] that MTL with counting modalities is still strictly less expressive than $\mathsf{FO}[<,+1]$ in the pointwise semantics. On the practical side, counting modalties appear to be amenable to implementations, e.g., Bersani, Rossi, and San Pietro [4] proposed an SMT-based tool for deciding the satisfiability of MITL with counting modalities.

2 Preliminaries

We give a brief introduction to (linear-time) timed logics and some technical tools and notations used in the paper. For more detailed reviews and comparisons of relevant results, we refer the readers to [6, 15].

$$\mathsf{C}_{(0,1)}\mathsf{MTL} \overset{①}{\subsetneq} \mathsf{C}_0\mathsf{MTL} \equiv \mathsf{CMTL} \overset{②}{\equiv} \mathsf{PQ2MLO} \overset{③}{\subsetneq} \mathsf{PGQMLO} \subseteq \mathsf{PGQMLO}^{\mathsf{frac}} \overset{④}{\equiv} \mathsf{FO}[<,+1]\,.$$

$$\mathsf{C}_{(0,1)}\mathsf{MITL} \overset{⑤}{\subsetneq} \mathsf{C}_0\mathsf{MITL} \equiv \mathsf{CMITL} \overset{⑥}{\equiv} \mathsf{Q2MLO}\,.$$

Figure 3 The results of this paper (in the pointwise semantics). ① and ⑤ follow from Theorem 9. ② and ⑥ follow from Theorem 13. ③ is Corollary 16, and ④ is Theorem 17.

Timed languages. A *timed word* over a finite alphabet Σ is an ω-sequence of *events* $(\sigma_i,\tau_i)_{i\geq 1}$ over $\Sigma \times \mathbb{R}_{\geq 0}$ with $(\tau_i)_{i\geq 1}$ an increasing sequence of non-negative real numbers ("*timestamps*") such that for any $r \in \mathbb{R}_{\geq 0}$, there is some *position* $j \geq 1$ with $\tau_j \geq r$ (i.e. we consider *strictly monotonic* timed words and require them to be "*non-Zeno*").[1] We denote by $\rho[i,j]$ the *finite* timed word formed by the sequence of events $(\sigma_\ell,\tau_\ell)_{i\leq \ell \leq j}$. We denote by $T\Sigma^\omega$ the set of all timed words over Σ. A *timed language* is a subset of $T\Sigma^\omega$.

Metric predicate logics. We start by defining *Monadic Second-Order Logic of Order and Metric* (MSO$[<,+1]$), which encompasses all the timed logics discussed in this paper.

▶ **Definition 1** (MSO$[<,+1]$ [3, 33])**.** Monadic Second-Order Logic of Order and Metric (MSO$[<,+1]$) *formulae are generated by*

$$\vartheta ::= \top \mid X(x) \mid x < x' \mid d(x,x') \in I \mid \vartheta_1 \wedge \vartheta_2 \mid \neg\vartheta \mid \exists x\, \vartheta \mid \exists X\, \vartheta$$

where X is an atomic proposition, x, x' are first-order variables, d is the distance predicate, $I \subseteq \mathbb{R}_{\geq 0}$ is an interval with endpoints in $\mathbb{N}_{\geq 0} \cup \{\infty\}$, and $\exists x$, $\exists X$ are first- and second-order quantifiers, respectively.[2]

As a convention we write, e.g., $(0,b\rangle$, to refer to $(0,b)$ or $(0,b]$. The fragment of MSO$[<,+1]$ without second-order quantifiers is the *Monadic First-Order Logic of Order and Metric* (FO$[<,+1]$). The fragment of MSO$[<,+1]$ without the distance predicate is the *Monadic Second Logic of Order* (MSO$[<]$). The fragment of FO$[<,+1]$ without the distance predicate is the *Monadic First-Order Logic of Order* (FO$[<]$).

▶ **Definition 2** (Q2MLO [12])**.** Q2MLO *is the smallest fragment of* FO$[<,+1]$ *obtained from* FO$[<]$ *by the following rules:*

- *All* FO$[<]$ *formulae with a single free variable are* Q2MLO *formulae (note that they may use* Q2MLO *formulae as atomic propositions).*
- *If $\vartheta(x_0,x)$ is an* FO$[<]$ *formula where x_0 and x are the only free first-order variables, then $\exists x \big(x_0 < x \wedge d(x_0,x) \in I \wedge \vartheta(x_0,x)\big)$ and $\exists x \big(x < x_0 \wedge d(x_0,x) \in I \wedge \vartheta(x_0,x)\big)$, where I is non-singular, are also* Q2MLO *formulae (with free first-order variable x_0).*

We denote by Q2MLO$_{0,\infty}$ the fragment of Q2MLO with only intervals of the forms $(0,b\rangle$ or $\langle a,\infty)$, and Q2MLO$_0$ is the even more restricted fragment where only intervals of the form $(0,b\rangle$ are allowed.[3] We also define Q2MSO [26], the smallest fragment of MSO$[<,+1]$ obtained from MSO$[<]$ by the rules in the previous definition (replacing FO$[<]$ by MSO$[<]$).

[1] We restrict ourselves to strictly monotonic timed words to simplify the definitions of metric predicate logics; all the results carry over to the case of non-strictly monotonic timed words as well.
[2] Following [33], we use $d(x,x')$ in place of a "+1" function symbol.
[3] Note that non-metric FO$[<]$ formulae are still allowed in these fragments.

▶ **Definition 3** (PQ2MLO [20]). PQ2MLO *(where "P" stands for "punctual") is obtained from* Q2MLO *by adding the rule:*

- $\exists x \left(x_0 < x \land d(x_0, x) \in I \land \vartheta(x)\right)$

where I is a singular interval and $\vartheta(x)$ is a Q2MLO *formula with a single free variable x.*

Metric temporal logics. We start by defining *Extended Metric Temporal Logic* (EMTL) [33] where all operators are defined by *non-deterministic finite automata* (NFAs). An NFA over Σ is a tuple $\mathcal{A} = \langle \Sigma, S, s_0, \Delta, F \rangle$ where S is a finite set of locations, $s_0 \in S$ is the initial location, $\Delta \subseteq S \times \Sigma \times S$ is the transition relation, and F is the set of final locations. We say that $\mathcal{A}$ is *deterministic* (a DFA) iff for each $s \in S$ and $\sigma \in \Sigma$, $|\{(s, \sigma, s') \mid (s, \sigma, s') \in \Delta\}| \leq 1$. A *run* of $\mathcal{A}$ on $\sigma_1 \ldots \sigma_n \in \Sigma^+$ is a sequence of locations $s_0 s_1 \ldots s_n$ where there is a transition $(s_i, \sigma_{i+1}, s_{i+1}) \in \Delta$ for each i, $0 \leq i < n$. A run of $\mathcal{A}$ is *accepting* iff it ends in a final location. A finite word is *accepted* by $\mathcal{A}$ iff $\mathcal{A}$ has an accepting run on it.

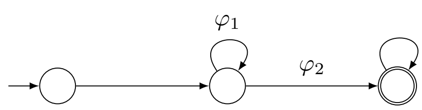
▶ **Definition 4** (EMTL [33]). Extended Metric Temporal Logic (EMTL) *formulae over a finite set of atomic propositions* AP *are generated by*

$$\varphi ::= \top \mid p \mid \varphi_1 \land \varphi_2 \mid \neg\varphi \mid \mathcal{A}_I(\varphi_1, \ldots, \varphi_n) \mid \overleftarrow{\mathcal{A}}_I\,(\varphi_1, \ldots, \varphi_n)$$

where $p \in$ AP, $\mathcal{A}$ is an NFA *over the n-ary alphabet $\{1, \ldots, n\}$*[4]*, and $I \subseteq \mathbb{R}_{\geq 0}$ is an interval with endpoints in $\mathbb{N}_{\geq 0} \cup \{\infty\}$*

As a convention, modalities with left arrows above them denote their "past" versions [2, 33]. We omit the subscript I when $I = (0, \infty)$ and write pseudo-arithmetic expressions for lower or upper bounds, e.g., "< 3" for $(0, 3)$. We also omit the arguments $\varphi_1, \ldots, \varphi_n$ and simply write $\mathcal{A}_I$ or $\overleftarrow{\mathcal{A}}_I$, if clear from the context. EMITL [33] is the fragment of EMTL with only non-singular intervals. $\mathsf{EMITL}_{0,\infty}$ is the fragment of EMITL with only intervals of the forms $(0, b\rangle$ or $\langle a, \infty)$.

▶ **Definition 5** (MTL [22]). Metric Temporal Logic (MTL) *is the fragment of* EMTL *with only the "until" and "since" modalities defined by the* NFA $\mathcal{A}^{\mathbf{U}}$ *below:*



MTL formulae are usually written in infix notation as $\varphi_1 \mathbf{U}_I \varphi_2$ and $\varphi_1 \mathbf{S}_I \varphi_2$. We also use the usual shortcuts like $\mathbf{F}_I \varphi \equiv \top \mathbf{U}_I \varphi$ and $\mathbf{G}_I \varphi \equiv \neg \mathbf{F}_I \neg\varphi$. *Metric Interval Temporal Logic* (MITL) [1] is the fragment of MTL with only non-singular intervals (or, equivalently, the fragment of EMITL with only the "until" and "since" modalities). $\mathsf{MITL}_{0,\infty}$ is the fragment of MITL with only intervals of the forms $(0, b\rangle$ or $\langle a, \infty)$ (or, equivalently, the fragment of $\mathsf{EMITL}_{0,\infty}$ with only the "until" and "since" modalities). *Linear Temporal Logic* (LTL) [31] is the fragment of $\mathsf{MITL}_{0,\infty}$ where all operators are labelled by $(0, \infty)$.[5]

▶ **Definition 6** (CMTL [14, 16]). CMTL *is obtained from* MTL *by adding the* counting modalities $\mathbf{C}_I^k$ *defined by the* $\mathsf{MSO}[<, +1]$ *formula*

$$\vartheta_I^{\mathbf{C},k}(x, X) = \exists x_1 \ldots \exists x_k \left(x < x_1 < \cdots < x_k \land d(x, x_1) \in I \land d(x, x_k) \in I \land \bigwedge_{1 \leq i \leq k} X(x_i)\right)$$

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[4] For clarity, we use $\varphi_1, \ldots, \varphi_n$ directly as transition labels (instead of $1, \ldots, n$) in the figures.

[5] We adopt the *strict* semantics for $\mathbf{U}$ and $\mathbf{S}$, which subsumes the usual "next" and "previous" operators.

as well as $\overleftarrow{\mathbf{C}}_I^k$ *defined by the past counterpart of* $\vartheta_I^{\mathbf{C},k}(x, X)$.[6] $\mathsf{C}_0\mathsf{MTL}$ *is the fragment of* CMTL *where the counting modalities use only intervals of the form* $(0, b\rangle$ *where* $b \in \mathbb{N}_{>0} \cup \{\infty\}$.

$\mathsf{C}_{(0,1)}\mathsf{MTL}$ is the fragment of $\mathsf{C}_0\mathsf{MTL}$ where the counting modalities use only $(0, 1)$. We will freely combine notations to refer to various fragments of metric temporal logics, e.g., $\mathsf{C}_{(0,1)}\mathsf{MITL}$ is obtained from MITL by adding $\mathbf{C}_I^k$ and $\overleftarrow{\mathbf{C}}_I^k$ with $I = (0, 1)$.

Semantics of $\mathsf{MSO}[<, +1]$. With each timed word $\rho = (\sigma_i, \tau_i)_{i\geq 1}$ over $\Sigma_{\mathsf{AP}} = 2^{\mathsf{AP}}$ we associate a structure M_ρ whose universe U_ρ is $\{i \mid i \geq 1\}$. The order relation $<$ and atomic propositions in AP are interpreted in the expected way, e.g., $P(i)$ holds in M_ρ iff $P \in \sigma_i$. The distance predicate $d(x, x') \in I$ holds iff $|\tau_x - \tau_{x'}| \in I$. The satisfaction relation for $\mathsf{MSO}[<, +1]$ is defined inductively in the usual way. We write $\rho, j_1, \ldots, j_m, J_1, \ldots, J_n \models \vartheta(x_1, \ldots, x_m, X_1, \ldots, X_n)$ if $j_1, \ldots, j_m \in U_\rho$, $J_1, \ldots, J_n \subseteq U_\rho$, and $\vartheta(j_1, \ldots, j_m, J_1, \ldots, J_n)$ holds in M_ρ. We say that two $\mathsf{MSO}[<, +1]$ formulae $\vartheta_1(x)$ and $\vartheta_2(x)$ are *equivalent* if for all timed words $\rho = (\sigma_i, \tau_i)_{i\geq 1}$ and $j \in U_\rho$,

$$\rho, j \models \vartheta_1(x) \iff \rho, j \models \vartheta_2(x) \ .$$

Semantics of EMTL. EMTL can be embedded into $\mathsf{MSO}[<, +1]$ through Büchi-Elgot-Trakhtenbrot theorem [25], but we can also define the satisfaction relation directly. Given an EMTL formula φ over AP, a timed word $\rho = (\sigma_i, \tau_i)_{i\geq 1}$ over Σ_{AP} and $i \geq 1$, define $\rho, i \models \varphi$ as follows:

- $\rho, i \models \top$;
- $\rho, i \models p$ iff $p \in \sigma_i$;
- $\rho, i \models \varphi_1 \wedge \varphi_2$ iff $\rho, i \models \varphi_1$ and $\rho, i \models \varphi_2$;
- $\rho, i \models \neg\varphi$ iff $\rho, i \not\models \varphi$;
- $\rho, i \models \mathcal{A}_I(\varphi_1, \ldots, \varphi_n)$ iff there exists $j \geq i$ such that (i) $\tau_j - \tau_i \in I$ and (ii) there is an accepting run of $\mathcal{A}$ on $a_i \ldots a_j$ where $\rho, \ell \models \varphi_{a_\ell}$ ($a_\ell \in \{1, \ldots, n\}$) for each ℓ, $i \leq \ell \leq j$.
- $\rho, i \models \overleftarrow{\mathcal{A}}_I(\varphi_1, \ldots, \varphi_n)$ is defined symmetrically.

We say that ρ *satisfies* φ (written $\rho \models \varphi$) iff $\rho, 1 \models \varphi$.

Ehrenfeucht-Fraïssé games for CMTL. An m-round CMTL Ehrenfeucht-Fraïssé (EF) game starts with round 0 and ends with round m. The game is played by two players (*Spoiler* and *Duplicator*) on a pair of timed words $\rho = (\sigma_i, \tau_i)_{i\geq 1}$ and $\rho' = (\sigma'_i, \tau'_i)_{i\geq 1}$. A *configuration* is a pair of positions (i, j), respectively in ρ and ρ'. In each round r ($0 \leq r \leq m$), the game proceeds as follows. *Spoiler* first checks whether the two events that correspond to the current configuration (i_r, j_r) in ρ and ρ' satisfy the same atomic propositions. If this is not the case then *Spoiler* wins the game. Otherwise if $r < m$, *Spoiler* chooses $I \subseteq \mathbb{R}_{\geq 0}$ with endpoints in $\mathbb{N}_{\geq 0} \cup \{\infty\}$ and plays either of the following moves:

- $\mathbf{U}_I$-*move*: *Spoiler* chooses one of the two timed words (say ρ) and picks i'_r such that $i_r < i'_r$ and $\tau_{i'_r} - \tau_{i_r} \in I$ (if there is no such i'_r then *Duplicator* wins the game). *Duplicator* must choose j'_r such that $\tau'_{j'_r} - \tau'_{j_r} \in I$ – if this is not possible then *Spoiler* wins the game. Otherwise, *Spoiler* plays either of the following "parts":
  - $\mathbf{F}$-*part*: The game proceeds to the next round with $(i_{r+1}, j_{r+1}) = (i'_r, j'_r)$.

[6] Note that $\mathbf{C}_I^k$ and $\overleftarrow{\mathbf{C}}_I^k$ are subsumed by EMTL even when $\inf I \neq 0$ [19].

- **U**-*part*: If $j'_r = j_r + 1$ the game proceeds to the next round with $(i_{r+1}, j_{r+1}) = (i'_r, j'_r)$. If $i'_r = i_r + 1$ but $j'_r \neq j_r + 1$ then *Spoiler* wins the game. Otherwise, *Spoiler* picks j''_r such that $j_r < j''_r < j'_r$; *Duplicator* has to choose i''_r such that $i_r < i''_r < i'_r$ in response – if this is not possible then *Spoiler* wins the game. Otherwise, the game proceeds to the next round with $(i_{r+1}, j_{r+1}) = (i''_r, j''_r)$.
- $\mathbf{S}_I$-*move*: Defined symmetrically.
- $\mathbf{C}_I^k$-*move*: *Spoiler* chooses one of the two timed words (say ρ) and picks $i_r^1, \ldots, i_r^k$ such that $i_r < i_r^1 < \cdots < i_r^k$ and $\tau_{i_r^\ell} - \tau_{i_r} \in I$ for all ℓ, $1 \leq \ell \leq k$ (if there are no such $i_r^1, \ldots, i_r^k$ then *Duplicator* wins the game); *Duplicator* must choose $j_r^1, \ldots, j_r^k$ such that $\tau'_{j_r^\ell} - \tau'_{j_r} \in I$ for all ℓ, $1 \leq \ell \leq k$ – if this is not possible then *Spoiler* wins the game. *Spoiler* then picks $j''_r = j_r^\ell$ for some ℓ, $1 \leq \ell \leq k$, *Duplicator* chooses $i''_r = i_r^\ell$ for some ℓ, $1 \leq \ell \leq k$, and the game proceeds to the next round with $(i_{r+1}, j_{r+1}) = (i''_r, j''_r)$.
- $\overleftarrow{\mathbf{C}}_I$-*move*: Defined symmetrically.

We say that *Duplicator* has a *winning strategy* for the m-round CMTL EF game on ρ and ρ' that starts from configuration (i, j) if and only if, no matter how *Spoiler* plays, *Duplicator* can always win the m-round CMTL EF game on ρ and ρ' with $(i_0, i_0) = (i, j)$. If this is not the case then we say that *Spoiler* has a winning strategy. The following theorem relates the number of rounds of CMTL EF games to the modal depth (i.e., the maximal depth of nesting of modalities) of CMTL formulae.

▶ **Theorem 7** ([24, 30]). *For timed words ρ, ρ' and a* CMTL *formula φ of modal depth $\leq m$, if* Duplicator *has a winning strategy for the m-round* CMTL *EF game on ρ, ρ' with $(i_0, j_0) = (1, 1)$, then*

$$\rho \models \varphi \iff \rho' \models \varphi\,.$$

Note that the theorem above can also be specialised to sublogics of CMTL; for example, the corresponding theorem for $\mathsf{C}_{(0,1)}\mathsf{MITL}$ is obtained by forcing $I = (0, 1)$ in $\mathbf{C}_I^k$-moves.

Expressiveness. We say that a metric logic L' is *expressively complete* for a metric logic L iff for any formula $\vartheta(x) \in L$, there is an equivalent formula $\varphi(x) \in L'$.[7] We say that L' is *at least as expressive as* (or *more expressive than*) L (written $L \subseteq L'$) iff for any formula $\vartheta(x) \in L$, there is an *initially equivalent* formula $\varphi(x) \in L'$ (i.e., $\vartheta(1)$ and $\varphi(1)$ evaluate to the same truth value for any timed word). We say that L' and L are *equally expressive* (written $L' \equiv L$) iff $L \subseteq L'$ and $L' \subseteq L$. If $L \subseteq L'$ but $L' \nsubseteq L$ then we say that L' is *strictly more expressive than* L (or L is *strictly less expressive than* L').

3 Expressive completeness for Q2MLO

Counting in $(0, 1)$. We argue that counting in $(0, 1)$ is not sufficiently expressive in the pointwise semantics; in particular, counting in $(0, b\rangle$ cannot be expressed in MTL extended with $\mathbf{C}_{(0,1)}^k$ and $\overleftarrow{\mathbf{C}}_{(0,1)}^k$, and it turns out to be essential for achieving the full expressiveness of $\mathsf{Q2MLO}$. This is in stark contrast with the situation in the continuous semantics, where LTL extended with $\mathbf{C}_{(0,1)}^k$ and $\overleftarrow{\mathbf{C}}_{(0,1)}^k$ is expressively complete for $\mathsf{Q2MLO}$ [14, 16]. We show this by constructing two families of timed words $(M_{m,c})$ and $(N_{m,c})$ over $\Sigma_{\{p,q\}}$ (inspired by [30]) that can be told apart easily by a $\mathsf{C}_0\mathsf{MTL}$ formula using $\mathbf{C}_{(0,b\rangle}^k$, yet they are indistinguishable by all $\mathsf{C}_{(0,1)}\mathsf{MTL}$ formulae of modal depth $\leq m$, all constants $\leq c$, and where all occurrences of counting modalities $\mathbf{C}_I^{k'}$ and $\overleftarrow{\mathbf{C}}_I^{k'}$ have $k' \leq k$.

[7] Formulae of metric temporal logics are $\mathsf{MSO}[<, +1]$ formulae with a single free first-order variable.

We start by describing $N_{m,c}$ for some fixed $m, c \in \mathbb{N}_{\geq 0}$. Let c' be the least integer greater than $\frac{5}{4} \cdot (c+3)+1$ and $\epsilon = \frac{1}{6}$. We put an $\emptyset$-event at time 0, and then a number of overlapping *segments* start at time $(c+1)$ where each segment consists of a $\{p\}$-event and a $\{q\}$-event (note that each $\{p\}$-event or $\{q\}$-event uniquely identifies a segment). If the $\{p\}$-event in the i^{th} segment is at, say, t, then its $\{q\}$-event is at $t + 2 + \frac{i}{3 \cdot m \cdot c' + 3} \cdot \epsilon$ (see Figure 4). We put a total of $2 \cdot m \cdot c' + 1$ segments where $\{p\}$-events in neighbouring segments are separated by $\frac{4}{5}$. Finally, we put an infinite sequence of $\emptyset$-events, equally separated by $(c+1)$ and starting at $(c+1)$ after the $\{q\}$-event in the last segment. $M_{m,c}$ is almost identical to $N_{m,c}$, except for the middle (i.e., $(m \cdot c' + 1)^{th}$) segment – say this segment starts at t, then in $M_{m,c}$ we shift the corresponding $\{q\}$-event to $t + 2 - \frac{m \cdot c' + 1}{3 \cdot m \cdot c' + 3} \cdot \epsilon$ instead. For convenience, we write t_a for the timestamp of the $\{p\}$-event in the middle segment (i.e. $t_a = (c+1) + \frac{4}{5} \cdot m \cdot c'$), $t_b = t_a + 2$, and denote the corresponding $\{q\}$-events in $M_{m,c}$ and $N_{m,c}$ by x and y respectively with timestamps t_x and t_y (see Figure 5). It is easy to see that no $\{q\}$-event is at an integer distance to some other $\{p\}$-event or $\{q\}$-event. This completes the description of $M_{m,c}$ and $N_{m,c}$. We say a configuration (i, j) is *identical* if $i = j$. For a position $i \geq 1$ in $M_{m,c}$ or $N_{m,c}$, we write $\mathsf{seg}(i)$ for the segment to which the i^{th} event belongs. For convenience we define $\mathsf{seg}(i) = 0$ if the i^{th} event is an $\emptyset$-event.

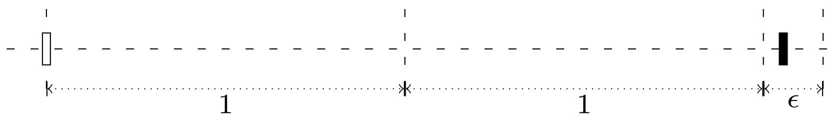

Figure 4 A segment in $N_{m,c}$. The white box is the $\{p\}$-event and the black box is the $\{q\}$-event.

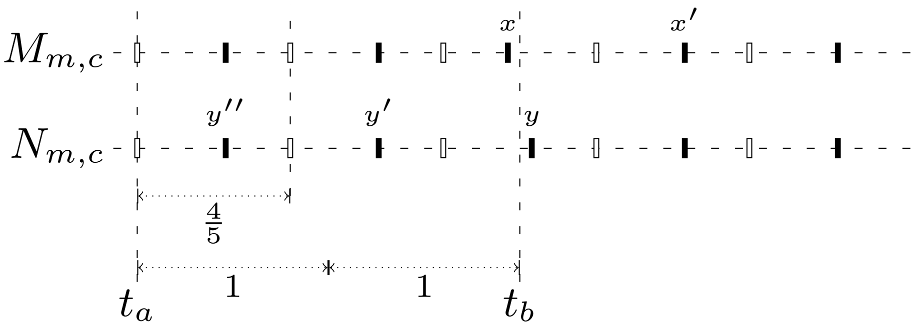

Figure 5 The events near the middle segments of $M_{m,c}$ and $N_{m,c}$. White boxes are $\{p\}$-events and black boxes are $\{q\}$-events.

We are now ready to state the main technical lemma, which intuitively says that *Duplicator* can either keep the configuration identical or far enough from the beginnings and the ends of both $M_{m,c}$ and $N_{m,c}$ (where *Spoiler* can easily win the EF game).

▶ **Lemma 8.** *In the m-round* $\mathsf{C}_{(0,1)}\mathsf{MTL}$ EF *game on $M_{m,c}$, $N_{m,c}$ starting from* $(1, 1)$, Duplicator *has a winning strategy such that for each round $0 \leq r \leq n$, the i_r^{th}-event in $M_{m,c}$ and the j_r^{th}-event in $N_{m,c}$ satisfy the same atomic propositions and*

- *if* $\mathsf{seg}(i_r) \neq \mathsf{seg}(j_r)$, *then $r \geq 1$ and*
  $\mathsf{seg}(i_r), \mathsf{seg}(j_r) \in \big[(m - r + 1) \cdot c' - 1, (m + r - 1) \cdot c' + 3\big]$.

Proof. We describe a winning strategy for *Duplicator* by induction on r. The basic idea is to make the resulting configuration identical whenever possible (and thus the induction hypothesis trivially holds); otherwise we use a copy-cat strategy (i.e. try to make $\mathsf{seg}(i_{r+1})$ –

$\mathsf{seg}(i_r) = \mathsf{seg}(j_{r+1}) - \mathsf{seg}(j_r)$). If that is also not possible, we must choose another event that satisfies the same atomic propositions. In the following, we refer to the timed word that *Spoiler* first chooses as $\rho^s = (\sigma_i^s, \tau_i^s)_{i\geq 0}$ ($\rho^d = (\sigma_i^d, \tau_i^d)_{i\geq 0}$ for that of *Duplicator*).

- *Base step.* The induction hypothesis holds trivially for $(i_0, j_0) = (1, 1)$.
- *Induction step.* Suppose the claim holds for $r < m$. We prove it also holds for $r + 1$.
  - $(i_r, j_r) = (1, 1)$:
    Since all segments happen at time $> c$, *Duplicator* can always make (i_{r+1}, j_{r+1}) an identical configuration, if necessary.
  - $(i_r, j_r) \neq (1, 1)$ is identical:
    We may assume $r > 0$. Observe from Figure 5 that any two $\{p\}$-events that are $5n$ segments away are separated by $4n$. More specifically, since $t_b - t_a = 2$, $\{p\}$-events whose distances to t_a are integers will also have integer distances to t_b. We consider the following cases:
    * (i_r, j_r) both correspond to $\emptyset$-events: since they are separated from any other events by $> c$, *Duplicator* can always make (i_{r+1}, j_{r+1}) identical if necessary.
    * (i_r, j_r) both correspond to $\{p\}$-events and *Spoiler* plays an $\mathbf{U}_I$-move or $\mathbf{S}_I$-move and picks (say) $i'_r = x$. *Duplicator* may either choose $j'_r = y$ (then *Duplicator* can surely make (i_{r+1}, j_{r+1}) identical later) or if that is not possible, choose event $j'_r = y'$. In the latter case, if *Spoiler* plays the **F**-part, it is obvious that the resulting configuration (i_{r+1}, j_{r+1}) would satisfy the claim. If *Spoiler* plays **U**-part, *Duplicator* may either make (i_{r+1}, j_{r+1}) identical or $\mathsf{seg}(j_{r+1}) - \mathsf{seg}(i_{r+1}) = -1$. In this latter case it is clear that the claim still holds ($\mathsf{seg}(i_{r+1}) = m \cdot c' + 2$ or $\mathsf{seg}(i_{r+1}) = m \cdot c' + 4$). If *Spoiler* plays a $\mathbf{C}_I^k$-move or $\overleftarrow{\mathbf{C}}_I^k$-move, as $I = (0, 1)$, *Duplicator* can always make (i_{r+1}, j_{r+1}) identical if necessary.
    * (i_r, j_r) corresponds to $\{q\}$-events except x and y, and *Spoiler* chooses, say, event $i'_r = x$. The reasoning is exactly similar to the case above.
    * (i_r, j_r) corresponds to events x and y. If *Spoiler* plays an $\mathbf{U}_I$-move or $\mathbf{S}_I$-move, chooses some event z, and forces *Duplicator* not to choose the corresponding event but another one in a neighbouring segment, then that event z must be less than $(c + 1)$ away from t_b. If it happens before t_b, then t_a would have distance $< (c - 1)$ to it. If it happens after t_b, then t_a would be $< (c + 3)$ away from it. Assume that z happens before t_b. If z is a $\{p\}$-event, we divide $(c - 1)$ by $\frac{4}{5}$ to obtain $\frac{5}{4} \cdot (c - 1) > |\mathsf{seg}(z) - \mathsf{seg}(i_r)|$ where $\mathsf{seg}(i_r) = m \cdot c' + 1$. Observe that the $\{p\}$-event z' that *Duplicator* chooses as the response will be at most one more segment away. Then the claim holds regardless of *Spoiler* plays **F**-part or **U**-part (may cause a drift of two more segments) later. If z is a $\{q\}$-event, observe that its corresponding $\{p\}$-event in the same segment must be less than $2 + \frac{1}{5} < 3 \cdot \frac{4}{5}$ away from z. Add this to $(c - 1)$ and divide the result by $\frac{4}{5}$ gives $\frac{5}{4} \cdot (c - 1) + 3 < \frac{5}{4} \cdot (c + 2)$. Again, the $\{q\}$-event z' that *Duplicator* chooses will be at most one more segment away. The case for z happens after t_b is similar. If *Spoiler* plays a $\mathbf{C}_I^k$-move or $\overleftarrow{\mathbf{C}}_I^k$-move, as $I = (0, 1)$, *Duplicator* can always make (i_{r+1}, j_{r+1}) identical if necessary.
  - (i_r, j_r) is not identical:
    We claim that no matter how *Spoiler* plays, *Duplicator* can always either make (i_{r+1}, j_{r+1}) identical or, ensure that (i_{r+1}, j_{r+1}) has not moved towards the nearest end by $\geq c'$ segments. In the latter case the claim holds by the induction hypothesis. If *Spoiler* plays an $\mathbf{C}_I^k$-move or $\overleftarrow{\mathbf{C}}_I^k$-move, it is once again clear that *Duplicator* can follow a copy-cat strategy if necessary, but this is not always the case for $\mathbf{U}_I$-moves and $\mathbf{S}_I$-moves. In the following, we focus on $\mathbf{U}_I$-moves and $\mathbf{S}_I$-moves and assume that

Spoiler always chooses some event that is more than two events away from the current event, e.g., $j'_r > j_r + 2$. If $j'_r \leq j_r + 2$, it is easy to see that *Duplicator* can simply choose $i'_r = i_r + (j'_r - j_r)$ (unless (i_r, j_r) are very close to one of the ends, which will not happen).

Assume that (i_r, j_r) corresponds to a pair of $\{p\}$-events and (without loss of generality) assume that *Spoiler* chooses a position j'_r such that $j'_r > j_r$. If *Duplicator* can choose i'_r such that $i'_r = j'_r$, *Duplicator* chooses $i'_r = j'_r$. Then, if *Spoiler* plays **F**-part, it is immediate that $i_{r+1} = j_{r+1}$. If *Spoiler* plays **U**-part, then *Duplicator* makes $i_{r+1} = j_{r+1}$ whenever possible. Otherwise, for example, if $i_r < j_r$ and *Spoiler* chooses some $\{p\}$-event in $(\tau^d_{i_r}, \tau^d_{j_r})$ as i_{r+1}, then *Duplicator* chooses $j_{r+1} = j_r + 2$. Observe that i_{r+1} has moved towards j_r (and away from the nearest end). The claim holds by the induction hypothesis. If *Duplicator* cannot choose i'_r such that $i'_r = j'_r$, consider the following cases:

- *Duplicator* can choose i'_r such that $i'_r = i_r + (j'_r - j_r)$: If *Duplicator* cannot choose $i'_r = j'_r$, then *Duplicator* chooses $i'_r = i_r + (j'_r - j_r)$. As before, we know that $\tau^s_{j'_r} < \tau^s_{j_r} + (c + 1)$. It is easy to see that $\mathsf{seg}(i_{r+1}) - \mathsf{seg}(i_r) < c'$ and $\mathsf{seg}(j_{r+1}) - \mathsf{seg}(j_r) < c'$, and hence the claim holds by the induction hypothesis.
- *Duplicator* cannot choose i'_r such that $i'_r = i_r + (j'_r - j_r)$: This can only happen when j'_r corresponds to a $\{q\}$-event. Observe that all $\{p\}$-events in neighbouring segments are separated by $\frac{4}{5}$. These imply that there exists t such that $t - \tau^s_{j_r} = n = n' \cdot \frac{1}{5}$ for some $n, n' \in \mathbb{N}_{>0}$, and there exists $|k_1|, |k_2| < 1$, $k_1, k_2 \neq 0$ such that $t - \tau^s_{j_r}$ lies between
  - $\tau^s_{j'_r} - \tau^s_{j_r} = n_1 \cdot \frac{1}{5} + k_1 \cdot \epsilon, n_1 \in \mathbb{N}_{>0}$ and
  - $\tau^d_{i_r + (j'_r - j_r)} - \tau^d_{i_r} = n_2 \cdot \frac{1}{5} + k_2 \cdot \epsilon, n_2 \in \mathbb{N}_{>0}$.

  It is obvious that $n_1 = n_2$. If $k_1 \cdot k_2 > 0$, since there is no integer multiple of $\frac{1}{5}$ that lies between, e.g., $n_1 \cdot \frac{1}{5}$ and $n_1 \cdot \frac{1}{5} + \epsilon$, this is a contradiction. If $k_1 \cdot k_2 < 0$, we must have $n' = n_1 = n_2$. This only happens when $i_r + (j'_r - j_r)$ in ρ^d corresponds to event x. In this case, *Duplicator* chooses the corresponding event in a neighbouring segment. For example, if (i_r, j_r) corresponds to a pair of $\{p\}$-events, $\mathsf{seg}(i_r) = m \cdot c' + 1, \mathsf{seg}(j_r) = m \cdot c'$, $I = (2, 3)$ and $j'_r = y'$, then *Duplicator* chooses $i'_r = x'$. Now if *Spoiler* plays **F**-part, since we know that $\tau^s_{j'_r} < \tau^s_{j_r} + (c + 1)$, the claim holds. If *Spoiler* plays **U**-part, e.g., in the aforementioned example, *Spoiler* chooses $i_{r+1} = x$, then *Duplicator* chooses $j_{r+1} = y''$ – the claim also holds.

Now assume that (i_r, j_r) corresponds to a pair of $\{q\}$-events and assume that the *Spoiler* chooses a position j'_r such that $j'_r < j_r$. Most cases can be argued in very similar ways. We consider the situation when *Duplicator* cannot choose i'_r such that $i'_r = i_r + (j'_r - j_r)$. If j'_r corresponds to a $\{p\}$-event then the argument is exactly similar to above. Otherwise if j'_r corresponds to a $\{q\}$-event, observe the fact that all $\{q\}$-events in neighbouring segments, except x, are separated by $\frac{4}{5} + \frac{1}{3 \cdot m \cdot c' + 3} \cdot \epsilon$. By a similar argument, if $k_1 \cdot k_2 < 0$, *Duplicator* chooses the corresponding event in a neighbouring segment. It can be argued in the same way that the claim holds regardless of *Spoiler* plays **F**-part or **U**-part later. ◀

Lemma 8 implies that any $\mathsf{C}_{(0,1)}\mathsf{MTL}$ formula of modal depth $\leq m$ and largest constant $\leq c$ cannot distinguish $M_{m,c}$ and $N_{m,c}$. However, from Figure 5 it is obvious that

$$M_{m,c} \models \mathbf{F}(p \wedge \mathbf{C}^3_{(0,2)}\, q) \wedge N_{m,c} \not\models \mathbf{F}(p \wedge \mathbf{C}^3_{(0,2)}\, q) \ ,$$

as each interval like (t_a, t_b) in $N_{m,c}$ contains at most two $\{q\}$-events. We thus have the theorem below, which can be seen as a strengthened version of a corresponding result in [24] (which holds for the future-only fragments).

▶ **Theorem 9.** $\mathsf{C}_0\mathsf{MTL} \subsetneq \mathsf{C}_{(0,1)}\mathsf{MTL}$.

Counting in $\langle 0, b\rangle$. We now show that once we bridge the expressiveness gap indicated by Theorem 9, we can derive a corresponding expressive completeness result for Q2MLO in the pointwise semantics. Before we give the main proof, let us state a crucial observation.

▶ **Theorem 10.** $\mathsf{Q2MLO}_0 \equiv \mathsf{Q2MLO}$.

Proof. We first note that $\mathsf{Q2MLO}_{0,\infty}$ is equally expressive as Q2MLO; this can be obtained as a simple corollary of the main result of [18] ($\mathsf{EMITL}_{0,\infty}$ is already as expressive as full EMITL), since all the automata modalities involved in the proof are counter free (aperiodic) and thus equivalent to FO[<] formulae of the form $\vartheta(x_0, x)$. To see that $\mathsf{Q2MLO}_0 \equiv \mathsf{Q2MLO}_{0,\infty}$, note that, e.g., the Q2MLO formula

$$\exists x \left(x_0 < x \wedge d(x_0, x) \in (a, \infty) \wedge \vartheta(x_0, x)\right)$$

is equivalent to an EMITL formula $\mathcal{A}_{(a,\infty)}$ where $\mathcal{A}$ is the automaton equivalent of $\vartheta(x_0, x)$; we assume (without loss of generality [34]) that $\mathcal{A} = \langle \Sigma, S, s_0, \Delta, F\rangle$ is a DFA and in particular, at most one of the arguments holds at any position. Let $\mathcal{B}^{s,\varphi}$ be the automaton obtained from $\mathcal{A}$ by adding a new location s_F, declaring it as the only final location, and adding new transitions $s' \xrightarrow{\varphi_a \wedge \varphi} s_F$ for every $s' \xrightarrow{\varphi_a} s$ in $\mathcal{A}$. Let $\mathcal{C}^s$ be the automaton obtained from $\mathcal{A}$ by adding new non-final locations s'_0 and s'_1, adding new transitions $s'_0 \to s'_1$ (i.e. labelled with $\top$) and $s'_1 \xrightarrow{\varphi_a} s''$ for every $s \xrightarrow{\varphi_a} s''$ in $\mathcal{A}$, and setting the initial location to s'_0. Intuitively, $\mathcal{B}^{s,\varphi}$ enforces φ at the point when s is reached in $\mathcal{A}$ and $\mathcal{C}^s$ "runs' $\mathcal{A}$ from s. We can argue that $\mathcal{A}_{(a,\infty)}$ is equivalent to

$$\mathcal{A}_{(0,\infty)} \wedge \neg \bigvee_{s \in S} \mathcal{B}^{s,\varphi}_{(0,a]}$$

where $\varphi = \neg \mathcal{C}^s$. This can be translated into a $\mathsf{Q2MLO}_0$ formula. ◀

We have thus reduced the problem to expressing $\mathsf{Q2MLO}_0$ formulae in $\mathsf{C}_0\mathsf{MITL}$. The proof below essentially follows [14, 16] with the exception that instead of the composition method [32] we use Myhill-Nerode congruence, which appears to be more natural in a pointwise setting. It suffices to show that we can use a $\mathsf{C}_0\mathsf{MITL}$ formula to express a $\mathsf{Q2MLO}_0$ formula of the form

$$\exists x \left(x_0 < x \wedge d(x_0, x) \in (0, b\rangle \wedge \vartheta(x_0, x)\right) \tag{1}$$

where $\vartheta(x_0, x)$ is an FO[<] formula, as we can repeatedly apply the equivalence on the minimal subformula until the whole formula is turned into a $\mathsf{C}_0\mathsf{MITL}$ formula.

We say an FO[<] formula $\vartheta(x_0, x)$ is *functional* if for any given timed word ρ and positions i_0, i, if we have $\rho, i_0, i \models \vartheta(x_0, x)$ then $i_0 < i$ and i is unique for i_0: if $\rho, i_0, i' \models \vartheta(x_0, x)$ then it must be the case that $i' = i$. It is not hard to see that (1) remains equivalent if we replace $\vartheta(x_0, x)$ by its "functional' counterpart

$$\vartheta'(x_0, x) = x_0 < x \wedge \vartheta(x_0, x) \wedge \forall x' \left(x_0 < x' < x \implies \neg\vartheta(x_0, x')\right) .$$

We recall some facts about functional formulae before stating the main theorem. Intuitively, once we restrict ourselves to the case of functional $\vartheta(x_0, x)$, then for any given position i_0, there can be only a bounded number of pairs of positions (i, j) such that $i < i_0 < j$ and $\rho, i, j \models \vartheta(x_0, x)$. In particular if $\rho, i_0, i \models \vartheta(x_0, x)$, we can make use of counting modalities to enforce that $\tau_i - \tau_{i_0} \in (0, b\rangle$.

▶ **Lemma 11.** *If $\vartheta(x_0, x)$ is functional and i_0 is a position in the timed word ρ, then $|\{j \mid \rho, i, j \models \vartheta(x_0, x)$ and $i < i_0 < j\}| \leq r$ where r is the number of locations of the minimal* DFA *equivalent to $\vartheta(x_0, x)$.*

Proof. Suppose to the contrary that there exists a set $\{(i_1, j_1), \dots, (i_{r+1}, j_{r+1})\}$ of $r + 1$ distinct pairs of positions (i, j) (where $j_1, \dots, j_{r+1}$ are all distinct) that satisfy the condition; $i_1, \dots, i_{r+1}$ must also be all distinct as $\vartheta(x_0, x)$ is functional. Let $\mathcal{D}$ be the minimal DFA equivalent to $\vartheta(x_0, x)$. As there are only r locations in $\mathcal{D}$, it must be the case that $\mathcal{D}$ reaches some specific location s after reading $\rho[i_u, i_0]$ and $\rho[i_v, i_0]$ for some $u \neq v$, and it follows that $\rho, i_u, j_u \models \vartheta(x_0, x)$ and $\rho, i_u, j_v \models \vartheta(x_0, x)$. This contradicts the fact that $\vartheta(x_0, x)$ is functional. ◀

If $\vartheta(x_0, x)$ is functional, we say that a pair of positions (i_1, j_1) such that $\rho, i_1, j_1 \models \vartheta(x_0, x)$ is of *ϑ-nesting depth at least m* in ρ if there exist positions $i_1 < \dots < i_m < j_m < \dots < j_1$ such that $\rho, i_\ell, j_\ell \models \vartheta(x_0, x)$ for all $\ell \in \{1, \dots, m\}$. We say (i_1, j_1) is of *ϑ-nesting depth m* in ρ if it is of ϑ-nesting depth at least m but not $m + 1$ in ρ. Let

$$R_\vartheta^{\geq m}(y_1) = \exists x_1, x_2, \dots, x_m, y_2, \dots, y_m \big(x_1 < x_2 < \dots < x_m < y_m < \dots < y_2 < y_1 \\ \wedge \vartheta(x_1, y_1) \wedge \vartheta(x_2, y_2) \wedge \dots \wedge \vartheta(x_m, y_m)\big)$$

and $R_\vartheta^m(y_1) = R_\vartheta^{\geq m}(y_1) \wedge \neg R_\vartheta^{\geq m+1}(y_1)$. Intuitively, $\rho, j_1 \models R_\vartheta^{\geq m}(y_1)$ iff there exists i_1 such that (i_1, j_1) is of ϑ-nesting depth at least m in ρ.

▶ **Lemma 12.** *If $\vartheta(x_0, x)$ is functional and (i, j) is of ϑ-nesting depth m in the timed word ρ, then if (i', j') where $j' < j$ is also of ϑ-nesting depth m in ρ (i.e. $\rho, j' \models R_\vartheta^m(y_1)$), we necessarily have $i' < i$.*

Proof. $i' > i$ contradicts the fact that (i, j) is of ϑ-nesting depth m in ρ, and $i' = i$ contradicts the fact that $\vartheta(x_0, x)$ is functional. ◀

▶ **Theorem 13.** $\mathsf{C_0MITL} \equiv \mathsf{Q2MLO}$.

Proof. Fix a functional formula $\vartheta(x_0, x)$ and a timed word ρ. Let $R_\vartheta^{m,\ell}(x_0)$ be the formula that says x_ℓ, the ℓ-th point $> x_0$ satisfying R_ϑ^m, also happens to satisfy $\vartheta(x_0, x_\ell)$, i.e.

$$R_\vartheta^{m,\ell}(x_0) = \exists x_1, \dots, x_\ell \Big(x_0 < x_1 < \dots < x_\ell \wedge \vartheta(x_0, x_\ell) \\ \wedge \forall x \big(x \in (x_0, x_\ell] \implies (R_\vartheta^m(x) \iff \bigvee_{i \in \{1,\dots,\ell\}} x = x_i)\big)\Big) .$$

By Lemma 11 and Lemma 12, we know that ℓ can at most be $r + 1$ (where r is the number of locations of the minimal DFA equivalent to $\vartheta(x_0, x)$). If (i_0, i) satisfies $\vartheta(x_0, x)$, then (i_0, i) must be of ϑ-nesting depth m in ρ for some $m \leq r$. To express

$$\exists x \big(x_0 < x \wedge d(x_0, x) \in (0, b\rangle \wedge \vartheta(x_0, x)\big) ,$$

we take the disjunction over all the possible choices of m's and ℓ's:

$$\bigvee_{m \in \{1,\dots,r\}} \Bigg(\bigvee_{\ell \in \{1,\dots,r+1\}} \Big(\exists x_1, \dots, x_\ell \big(x_0 < x_1 < \dots < x_\ell \wedge d(x_0, x_\ell) \in (0, b\rangle \\ \wedge \bigwedge_{i \in \{1,\dots,\ell\}} R_\vartheta^m(x_i) \wedge R_\vartheta^{m,\ell}(x_0)\big)\Big)\Bigg) .$$

The formula above is equivalent to

$$\bigvee_{m\in\{1,\ldots,r\}} \Big(\bigvee_{\ell\in\{1,\ldots,r+1\}} \big((\mathbf{C}^{\ell}_{\langle 0,b\rangle}\, R^m_\vartheta) \wedge R^{m,\ell}_\vartheta\big)\Big)$$

where R^m_ϑ, $R^{m,\ell}_\vartheta$ are the LTL equivalents of $R^m_\vartheta(y_1)$ and $R^{m,\ell}_\vartheta(x_0)$, respectively. ◀

▶ **Corollary 14.** C_0MITL *with untimed automata modalities is expressively complete for* Q2MSO.

4 Expressive completeness for FO[<, +1]

Generalising EMITL. We know that in the continuous semantics PQ2MLO [20] is expressively complete for FO[<, +1]; in other words, the only expressiveness gap between (decidable) Q2MLO and (undecidable) FO[<, +1] is the capability to express punctualities. Unfortunately, this pleasant result does not hold in the pointwise semantics.

▶ **Theorem 15.** PQ2MLO *is strictly less expressive than* FO[<, +1].

Proof. Thanks to Theorem 13, it suffices to show that C_0MTL is strictly less expressive than FO[<, +1]. In fact, we can prove the stronger result that MTL with arbitrary rational endpoints (which subsumes $\mathbf{C}^k_I$) is still insufficient for expressing the property below ("X holds at the first event in I from now'):

$$\mathbf{B}^{\rightarrow}_I(x, X) = \exists x' \big(x < x' \wedge d(x, x') \in I \wedge X(x') \wedge \neg\exists x'' \left(x < x'' < x' \wedge d(x, x'') \in I\right)\big) \quad (2)$$

The detailed proof can be found in the full version of this paper. ◀

The theorem above suggests that we need more involved extensions to make Q2MLO as expressive as FO[<, +1] in the pointwise semantics; at least we must be able to specify (2). PnEMTL [23] is a generalisation of EMTL where instead of just between the current point and a single witness point, one can use "Pnueli automata' modalities to specify behaviours between multiple witness points as well. More precisely, the semantics of Pnueli automata modalities are defined as follows:

- $\rho, i \models \mathcal{F}_{I_1,\ldots,I_k}(\mathcal{A}_1, \ldots, \mathcal{A}_k)$ iff there exists $j_1, \ldots, j_k$ such that
  1. $i < j_1 < \cdots < j_k$.
  2. For each $\ell \in \{1, \ldots, k\}$, $\tau_{j_\ell} - \tau_i \in I_\ell$.
  3. For each $\ell \in \{1, \ldots, k\}$, there is an accepting run of $\mathcal{A}_\ell$ on $a_{j_{\ell-1}} \ldots a_{j_\ell}$ ($\ell > 1$) or $a_i \ldots a_{j_\ell}$ ($\ell = 1$) such that for each m, $j_{\ell-1} \le m \le j_\ell$ (or $i \le m \le j_\ell$), $\rho, m \models \varphi_{a_m}$ ($a_m \in \{1, \ldots, n_\ell\}$ where n_ℓ is the arity of the alphabet of $\mathcal{A}_\ell$).
- $\rho, i \models \mathcal{P}_{I_1,\ldots,I_k}(\mathcal{A}_1, \ldots, \mathcal{A}_k)$ (the past counterpart) is defined symmetrically.

In [23], it is also shown that PnEMTL is expressively equivalent to PGQMSO, a generalisation of PQ2MSO with the following rule:

- if $\vartheta_1(x_0, x_1), \ldots, \vartheta_k(x_0, x_k)$ are MSO[<] formulae where for each $\vartheta_\ell(x_0, x_\ell)$ ($\ell \in \{1, \ldots, k\}$), x_0 and x_ℓ are the only free first-order variables, then $\exists x_1 \ldots \exists x_k \big(x_0 < x_1 < \cdots < x_k \wedge d(x_0, x_1) \in I_1 \wedge \cdots \wedge d(x_0, x_k) \in I_k \wedge \vartheta(x_0, x_1) \wedge \cdots \wedge \vartheta(x_0, x_k)\big)$ and the past counterpart, where $I_1, \ldots, I_k$ are (possibly singular) intervals with endpoints in $\mathbb{N}_{\ge 0} \cup \{\infty\}$, are also PGQMSO formulae (with free first-order variable x_0).

As we can easily express (2) in PGQMLO (the first-order fragment of PGQMSO) [23, Theorem 6.4], we have the following corollary.

▶ **Corollary 16.** PQ2MLO $\subsetneq$ PGQMLO.

Order of fractional parts. While we have not been able to prove or disprove whether PGQMLO $\equiv$ FO$[<,+1]$, we can show that a simple extension of PnEMTL, where one is allowed to specify orders of fractional parts of witnesses, can capture the full expressiveness of FO$[<,+1]$. Let $\mathcal{F}^{\mathit{frac},N}_{I_1,\ldots,I_k}(\mathcal{A}_1,\ldots,\mathcal{A}_k)$ be the new modalities where

- $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are all counter free (aperiodic).
- Each of $I_1, \ldots, I_k$ is a left-closed, right-open subinterval of $[-N, N)$ with integer endpoints and length 1 (e.g., $[3,4)$ or $[-7,-6)$).

The intended semantics when evaluated at position i_0 is as follows:

- There exists k "witness' points $i_1, \ldots, i_k$ such that $i_\ell \in I_\ell$ for all $\ell \in \{1,\ldots,k\}$.
- The fractional parts of the witnesses are in this order, i.e. $\mathit{frac}(\tau_{i_1}) < \cdots < \mathit{frac}(\tau_{i_k})$.
- For each $\ell \in \{1,\ldots,k\}$, $\mathcal{A}_\ell$ has an accepting run on the "stacked' word [17] formed by all events in $\tau_{i_0} + [-N,N)$ with the fractional parts in $[\tau_{i_{\ell-1}}, \tau_{i_\ell})$. More precisely, the transitions of $\mathcal{A}$ are partitioned into $2N$ sets, where each set is only enabled for events in the corresponding unit subinterval of $\tau_{i_0} + [-N,N)$.

In the same way we define the past counterpart $\mathcal{P}^{\mathit{frac}}$ and its semantics, and denote by PGQMLO$^{\mathsf{frac}}$ the extension of PGQMLO with these modalities.

▶ **Theorem 17.** PGQMLO$^{\mathit{frac}}$ $\equiv$ FO$[<,+1]$.

Proof (sketch). Following [21], the main challenge is to express formulae of the form

$$\begin{aligned}\exists z_0 \ldots \exists z_{n-1} \Big(x = z_0 < \cdots < z_{n-1} \wedge d(x, z_{n-1}) < 1 \\ \wedge \bigwedge \{\Phi_i(z_i) : 0 \le i < n\} \\ \wedge \bigwedge \{\forall u \left(z_i < u < z_{i+1} \implies \Psi_i(u)\right) : 0 \le i < n-1\} \\ \wedge \forall u \left(z_{n-1} < u \wedge d(x,u) < 1 \implies \Psi_{n-1}(u)\right)\Big)\end{aligned}$$

where Φ_i and Ψ_i are Boolean combinations of atomic formulae. This is readily possible with $\mathcal{F}^{\mathit{frac}}$ and subformulae of the forms $\mathbf{F}_{=1}\,p$ and $\overleftarrow{\mathbf{F}}_{=1}\,p$. ◀

5 Conclusion and future work

The general consensus in the real-time verification community is that the continuous interpretations of timed logics are more well behaved and admit more robust characterisations. The present paper showed that by allowing a mild generalisation of the counting modalities, we can recover the pleasant expressive completeness result for Q2MLO – one of the most expressive decidable fragments of FO$[<,+1]$ – in the pointwise semantics as well. On the other hand, we also showed that as opposed to the situation in the continuous semantics, the full expressiveness of FO$[<,+1]$ cannot be achieved by simply adding punctual predicates – we remedy this by proposing a more involved variant of PnEMTL, which we showed to be expressively complete for FO$[<,+1]$. We list some possible future directions below.

- The expressive completeness for FO$[<,+1]$ is achieved with a family of modalities that enable one to specify the relative orders of the fractional parts of the points involved. This begs the question of whether this feature is really necessary; in other words, is PGQMLO strictly less expressive than FO$[<,+1]$?
- Is it possible to add (or perhaps restricted versions of) the modalities $\mathcal{F}^{\mathit{frac}}$ and $\mathcal{P}^{\mathit{frac}}$ to GQMLO while retaining the decidability of the satisfiability problem?

- It is known that the pointwise and continuous interpretations of $\mathsf{FO}[<, +1]$ are actually equally expressive [9], if one considers a special "timed word' form of signals [5, 7, 28]. Does a similar result hold for $\mathsf{Q2MLO}$ as well?
- There are some existing SMT-based tools for checking the satisfiablity of CMITL in the continuous semantics (e.g., [4]), although they require a predetermined bound k on the variability of signals. In light of the recent developments in back-end algorithms [10, 11], it would be interesting to see how a timed-automata-based implementation compares in terms of practical performance.

References

1 Rajeev Alur, Tomás Feder, and Thomas A. Henzinger. The benefits of relaxing punctuality. *Journal of the ACM*, 43(1):116–146, 1996.

2 Rajeev Alur and Thomas A. Henzinger. Back to the future: Towards a theory of timed regular languages. In *FOCS*, pages 177–186. IEEE Computer Society Press, 1992.

3 Rajeev Alur and Thomas A. Henzinger. Real-time logics: Complexity and expressiveness. *Information and Computation*, 104(1):35–77, 1993. doi:10.1006/inco.1993.1025.

4 Marcello M. Bersani, Matteo Rossi, and Pierluigi San Pietro. A tool for deciding the satisfiability of continuous-time metric temporal logic. *Acta Informatica*, 53(2):171–206, 2016. doi:10.1007/s00236-015-0229-y.

5 Patricia Bouyer, Fabrice Chevalier, and Nicolas Markey. On the expressiveness of TPTL and MTL. *Information and Computation*, 208(2):97–116, 2010.

6 Patricia Bouyer, François Laroussinie, Nicolas Markey, Joël Ouaknine, and James Worrell. Timed temporal logics. In *Models, Algorithms, Logics and Tools - Essays Dedicated to Kim Guldstrand Larsen on the Occasion of His 60th Birthday*, volume 10460 of *LNCS*, pages 211–230. Springer, 2017.

7 Deepak D'Souza and Pavithra Prabhakar. On the expressiveness of mtl in the pointwise and continuous semantics. *International Journal on Software Tools for Technology*, 9(1):1–4, 2007.

8 Deepak D'Souza and Nicolas Tabareau. On timed automata with input-determined guards. In Yassine Lakhnech and Sergio Yovine, editors, *FORMATS/FTRTFT*, volume 3253 of *LNCS*, pages 68–83. Springer, 2004.

9 Deepak D'Souza and Raveendra Holla. Equivalance of pointwise and continuous semantics of fo with linear constraints. In *ICLA*, pages 40–45, 2021.

10 R Govind, Frédéric Herbreteau, and Igor Walukiewicz. Abstractions for the local-time semantics of timed automata: a foundation for partial-order methods. In *LICS*, pages 1–14. ACM/IEEE, 2022.

11 Frédéric Herbreteau, B Srivathsan, and Igor Walukiewicz. Checking timed büchi automata emptiness using the local-time semantics. In *CONCUR*, volume 243 of *LIPIcs*, pages 12:1–12:24. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022.

12 Yoram Hirshfeld and Alexander Rabinovich. A framework for decidable metrical logics. In *ICALP*, volume 1644 of *LNCS*, pages 422–432. Springer, 1999.

13 Yoram Hirshfeld and Alexander Rabinovich. Expressiveness of metric modalities for continuous time. *Logical Methods in Computer Science*, 3(1):1–11, 2007.

14 Yoram Hirshfeld and Alexander Rabinovich. Continuous time temporal logic with counting. *Information and Computation*, 214:1–9, 2012.

15 Yoram Hirshfeld and Alexander Moshe Rabinovich. Logics for real time: Decidability and complexity. *Fundamenta Informaticae*, 62(1):1–28, 2004.

16 Yoram Hirshfeld and Alexander Moshe Rabinovich. An expressive temporal logic for real time. In *MFCS*, volume 4162 of *LNCS*, pages 492–504. Springer, 2006.

17 Hsi-Ming Ho. On the expressiveness of metric temporal logic over bounded timed words. In *RP*, volume 8762 of *LNCS*, pages 138–150. Springer, 2014.

18 Hsi-Ming Ho. Revisiting timed logics with automata modalities. In *HSCC*, pages 67–76. ACM, 2019.

19 Hsi-Ming Ho and Khushraj Madnani. When do you start counting? revisiting counting and pnueli modalities in timed logics. In *DCM*. EPTCS, 2023. To appear.

20 Paul Hunter. When is metric temporal logic expressively complete? In *CSL*, volume 23 of *LIPIcs*, pages 380–394. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2013.

21 Paul Hunter, Joël Ouaknine, and James Worrell. Expressive completeness for metric temporal logic. In *LICS*, pages 349–357. IEEE Computer Society Press, 2013.

22 Ron Koymans. Specifying real-time properties with metric temporal logic. *Real-Time Systems*, 2(4):255–299, 1990.

23 Shankara Narayanan Krishna, Khushraj Madnani, Manuel Mazo Jr., and Paritosh Pandya. From non-punctuality to non-adjacency: A quest for decidability of timed temporal logics with quantifiers. *Formal Aspects of Computing*, 2022.

24 Shankara Narayanan Krishna, Khushraj Madnani, and Paritosh K. Pandya. Metric temporal logic with counting. In *FoSSaCS*, volume 9634 of *LNCS*, pages 335–352. Springer, 2016.

25 Shankara Narayanan Krishna, Khushraj Madnani, and Paritosh K. Pandya. Making metric temporal logic rational. In *MFCS*, volume 83 of *LIPIcs*, pages 77:1–77:14. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2017.

26 Shankara Narayanan Krishna, Khushraj Madnani, and Paritosh K. Pandya. Logics meet 1-clock alternating timed automata. In *CONCUR*, volume 118 of *LIPIcs*, pages 39:1–39:17. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018.

27 Kim G. Larsen, Paul Pettersson, and Wang Yi. Uppaal in a nutshell. *International Journal on Software Tools for Technology Transfer*, 1(1-2):134–152, 1997.

28 Joël Ouaknine, Alexander Rabinovich, and James Worrell. Time-bounded verification. In *CONCUR*, volume 5710 of *LNCS*, pages 496–510. Springer, 2009.

29 Joël Ouaknine and James Worrell. On metric temporal logic and faulty turing machines. In *FoSSaCS*, volume 3921 of *LNCS*, pages 217–230. Springer, 2006.

30 Paritosh K. Pandya and Simoni S. Shah. On expressive powers of timed logics: Comparing boundedness, non-punctuality, and deterministic freezing. In *CONCUR*, volume 6901 of *LNCS*, pages 60–75. Springer, 2011.

31 Amir Pnueli. The temporal logic of programs. In *FOCS*, pages 46–57. IEEE, 1977.

32 Saharon Shelah. The monadic theory of order. *The Annals of Mathematics*, 102(3):379, November 1975.

33 Thomas Wilke. Specifying timed state sequences in powerful decidable logics and timed automata. In *FTRTFT*, volume 863 of *LNCS*, pages 694–715. Springer, 1994.

34 Pierre Wolper and Moshe Y. Vardi. Reasoning about infinite computations. *Information and Computation*, 115(1):1–37, 1994.

35 Yuchen Zhou, Dipankar Maity, and John S. Baras. Timed automata approach for motion planning using metric interval temporal logic. In *ECC*, pages 690–695. IEEE, 2016.