A Sound and Complete Tableau System for Fuzzy Halpern and Shoham's Interval Temporal Logic

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Abstract

Interval temporal logic plays a critical role in various applications, including planning, scheduling, and formal verification; recently, interval temporal logic has also been successfully applied to learning from temporal data. Halpern and Shoham's interval temporal logic, in particular, stands out as a very intuitive, yet expressive, interval-based formalism. To address real-world scenarios involving uncertainty and imprecision, Halpern and Shoham's logic has been recently generalized to the fuzzy (many-valued) case. The resulting language capitalizes on many-valued modal logics, allowing for a range of truth values that reflect multiple expert perspectives, but inherits the bad computational behaviour of its crisp counterpart. In this work, we investigate a sound and complete tableau system for fuzzy Halpern and Shoham's logic, which, although possibly non-terminating, offers a semi-decision procedure for the finite case.

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1 Introduction

Temporal logic is an essential framework for representing and reasoning about time. To accurately represent time, it is crucial to adopt suitable primitive ontological entities, usually categorized into point-based and interval-based ones. In this work, we take intervals as primary semantic objects. *Halpern and Shoham's Modal Logic for Time Intervals* (*HS*) [14] is one of the most influential logical languages for time intervals, providing a robust and expressive formalism for reasoning about temporal relations between events with duration. Its applications range from planning, to scheduling, to formal verification; more recently, it has been shown how transparent and explainable interval temporal logic theories can be extracted from temporal data by exploiting the integration of HS in machine learning systems, including decision trees (e.g., see [6, 19]) and random forests (e.g., see [15, 16]).

As it turns out, the satisfiability problem for HS is undecidable in all interesting cases of underlying linear order. Various strategies have been studied to obtain fragments of HS with better computational behaviour, such as restricting the set of modal operators [1, 3], constraining the underlying temporal structure [17], restricting the propositional power of the languages [5], and considering coarser logics based on relations that describe a less precise relationship between intervals [18]. On a more practical side, a few attempts to devise practical reasoning systems for HS and its fragments have been made; among them there are sound and complete procedures, respectively for the fragment of HS known as *PNL* [12, 4], and the coarser version of HS called HS_3 [18], among a few others; moreover, it has also been devised a similar procedure for the interval temporal logic *CDT*, introduced in [20], which generalizes HS to the case of ternary relations [11].

A unifying aspect of the work on interval temporal logic as we have presented it is the crisp (that is, based on the classic two-values Boolean algebra) semantics of all mentioned logics. In order to enhance the applicability and effectiveness in addressing real-world scenarios, it has been recently proposed to generalize the syntax and semantics of HS to accommodate the inherent uncertainty and imprecision when dealing with real-world data, including (multivariate) time series. A natural way to accomplish such a generalization is following the pioneering work of Fitting on fuzzy modal logics [8], in which both propositions and accessibility relations are no longer just true or false but can have different truth values. As a consequence, the definition of a fuzzy logic rests on a specific class of algebras. Typical choices are Heyting algebras and Łukasiewicz algebra; in the former case, the resulting logic embodies the perspectives of multiple experts whose opinions may not necessarily be independent, while in the latter case the idea is to represent intrinsic vagueness of data.

Fuzzy Halpern and Shoham's Modal Logic for Time Intervals (*FHS*) [7] is precisely the Fitting-style generalization of HS in the case of Heyting algebras. FHS inherits the bad computational behaviour of its crisp counterpart. In particular, in the case of chain Heyting algebras and the class of all (fuzzy) linearly ordered sets, the (fuzzy generalisation of the) satisfiability problem for FHS is undecidable, as well as in the case of all finite (fuzzy) linearly ordered sets, and it is believed so in the other two natural sub-classes of Heyting algebras, that is, the class of finite and the class of Boolean Heyting algebras. However, satisfiability of interval temporal logic formulas is much less studied in the fuzzy case than it is in its crisp counterpart. In this sense, there is a general lack of reasoning tools that are able to deal with fuzzy interval temporal logics, and with FHS in particular.

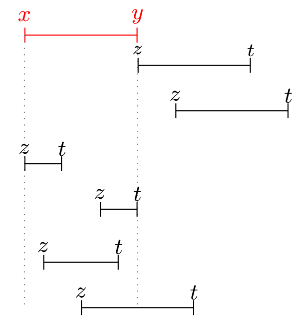
This work is a first step towards filling in this gap. In particular, we consider FHS in the case of finite Heyting algebras, and, following (again) Fitting [9], we study a tableau system for FHS in the case of all (fuzzy) linear orders. We shall prove that our tableau system, which generalizes tableau systems for crisp interval logics such as those proposed in [11, 12], is sound and complete for satisfiability (at a certain degree of truth or more), and that it is a semi-decision procedure for the case of all finite (fuzzy) linear orders.

This paper is organized as follows. In Section 2 we give some necessary background on HS, Heyting algebras and their properties, and FHS. Then, in Section 3 we present our tableau system, and prove its soundness and completeness, before concluding.

2 Background

While several different interval temporal logics have been proposed in the recent literature [13], *Halpern and Shoham's Modal Logic for Time Intervals (HS)* [14] is certainly the formalism that has received the most attention. Let $\mathbb{D} = \langle D, < \rangle$ be a linear order with

Table 1 Allen's interval relations and HS modalities.

HS modality	Definition w.r.t. the interval structure	Example
$\langle A\rangle$ (after)	$[x,y]R_A[z,t] \quad \Leftrightarrow \quad y=z$	
$\langle L\rangle$ (later)	$[x,y]R_L[z,t] \quad \Leftrightarrow \quad y<z$	
$\langle B\rangle$ (begins)	$[x,y]R_B[z,t] \quad \Leftrightarrow \quad x=z \wedge t<y$	
$\langle E\rangle$ (ends)	$[x,y]R_E[z,t] \quad \Leftrightarrow \quad y=t \wedge x<z$	
$\langle D\rangle$ (during)	$[x,y]R_D[z,t] \quad \Leftrightarrow \quad x<z \wedge t<y$	
$\langle O\rangle$ (overlaps)	$[x,y]R_O[z,t] \quad \Leftrightarrow \quad x<z<y<t$	

domain D; in the following, we shall use D and $\mathbb{D}$ interchangeably. A *strict interval* over $\mathbb{D}$ is an ordered pair $[x,y]$, where $x,y\in\mathbb{D}$ and $x<y$. If we exclude the identity relation, there are 12 different binary ordering relations between two strict intervals on a linear order, often called *Allen's interval relations* [2]: the six relations R_A (*adjacent to*), R_L (*later than*), R_B (*begins*), R_E (*ends*), R_D (*during*) and R_O (*overlaps*), depicted in Tab. 1, and their *inverses*, that is, $R_{\overline{X}}=(R_X)^{-1}$, for each $X\in\{A,L,B,E,D,O\}$. We interpret interval structures as Kripke structures, with Allen's relations playing the role of accessibility relations. Thus, we associate an *existential modality* $\langle X\rangle$ with each Allen's relation R_X. Moreover, for each $X\in\{A,L,B,E,D,O\}$, the *transpose* of modality $\langle X\rangle$ is the modality $\langle\overline{X}\rangle$ corresponding to the inverse relation $R_{\overline{X}}$ of R_X. Now, let $\mathcal{X}=\{A,\overline{A},L,\overline{L},B,\overline{B},E,\overline{E},D,\overline{D},O,\overline{O}\}$; well-formed HS formulas are built from a set of *propositional letters* $\mathcal{P}$, the classical connectives $\vee$ and $\neg$, and a modality for each Allen's interval relation, as follows:

$$\varphi ::= p \mid \neg\varphi \mid \varphi\vee\varphi \mid \langle X\rangle\varphi,$$

where $p\in\mathcal{P}$ and $X\in\mathcal{X}$. The other propositional connectives and constants (i.e., $\psi_1\wedge\psi_2\equiv\neg\psi_1\vee\neg\psi_2$, $\psi_1\rightarrow\psi_2\equiv\neg\psi_1\vee\psi_2$ and $\top=p\vee\neg p$), as well as, for each $X\in\mathcal{X}$, the *universal modality* $[X]$ (e.g., $[A]\varphi\equiv\neg\langle A\rangle\neg\varphi$), can be derived in the standard way. The set of all subformulas of a given HS formula φ is denoted by $sub(\varphi)$.

The strict semantics of HS is given in terms of *interval models* of the type $M=\langle\mathbb{I}(\mathbb{D}),V\rangle$, where $\mathbb{D}$ is a linear order, $\mathbb{I}(\mathbb{D})$ is the set of all strict intervals over $\mathbb{D}$, and V is a *valuation function* $V:\mathcal{P}\rightarrow 2^{\mathbb{I}(\mathbb{D})}$ which assigns to every atomic proposition $p\in\mathcal{P}$ the set of intervals $V(p)$ on which p holds. The truth of a formula φ on a given interval $[x,y]$ in an interval model M, denoted by $M,[x,y]\Vdash\varphi$, is defined by structural induction on the complexity of formulas, as follows:

$$\begin{array}{lll}
M,[x,y]\Vdash p & \text{if and only if} & [x,y]\in V(p),\ \text{for each } p\in\mathcal{AP},\\
M,[x,y]\Vdash\neg\psi & \text{if and only if} & M,[x,y]\not\Vdash\psi,\\
M,[x,y]\Vdash\psi_1\vee\psi_2 & \text{if and only if} & M,[x,y]\Vdash\psi_1 \text{ or } M,[x,y]\Vdash\psi_2,\\
M,[x,y]\Vdash\langle X\rangle\psi & \text{if and only if} & \text{there exists } [w,z] \text{ s.t. } [x,y]R_X[w,z] \text{ and } M,[w,z]\Vdash\psi,
\end{array}$$

where $X\in\mathcal{X}$. Given a model $M=\langle\mathbb{I}(\mathbb{D}),V\rangle$ and a formula φ, we say that M *satisfies* φ if there exists an interval $[x,y]\in\mathbb{I}(\mathbb{D})$ such that $M,[x,y]\Vdash\varphi$. A formula φ is *satisfiable* if there exists an interval model that satisfies it. Moreover, a formula φ is *valid* if it is satisfiable at every interval of every (interval) model or, equivalently, if its negation $\neg\varphi$ is *unsatisfiable*.

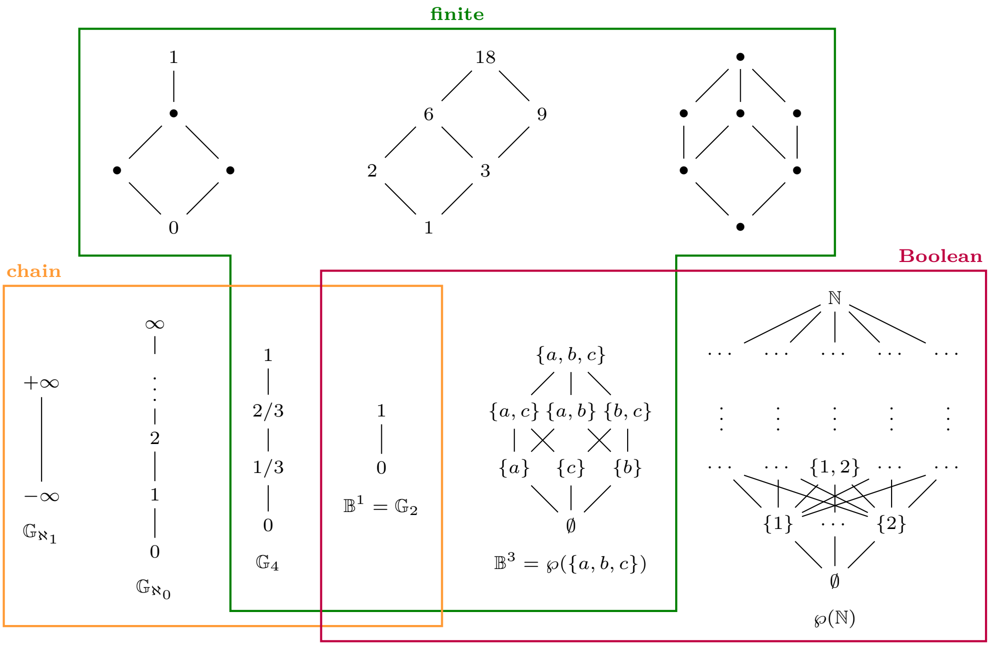


Figure 1 Graphical representation of finite, Boolean, and chain algebras. Some examples have well-known names: $\mathbb{G}$s are Göedel algebras, $\mathbb{B}$s are Boolean algebras, and $\wp(\mathbb{N})$ is the powerset of $\mathbb{N}$.

A *Heyting algebra* is a structure of the type

$$\mathcal{H} = (H, \cap, \cup, \hookrightarrow, 0, 1),$$

where $(H, \cap, \cup, 0, 1)$ is a bounded lattice with *domain* H, with top (resp., bottom) element 1 (resp., 0); in the following, we shall use H and $\mathcal{H}$ interchangeably. Recall that a bounded lattice is a set with internal operations $\cap$ (*meet*[1]) and $\cup$ (*join*), both commutative, associative, and connected by the absorption law, in which a partial order can be defined, as follows:

$$\alpha \preceq \beta \text{ iff } \alpha \cap \beta = \alpha \text{ iff } \alpha \cup \beta = \beta.$$

It is well-known that Heyting algebras are always distributive. In the following we use $\bigcap$ (resp., $\bigcup$) to indicate the generalized $\cap$ (resp., $\cup$), and we assume them to have the lowest priority in algebraic expressions; moreover, we omit the quantification domains when it is clear from the context. The symbols 0 and 1 denote, respectively, least and the greatest elements of $\mathcal{H}$. In other words, a Heyting algebra is a bounded distributive lattice in which the *relative pseudo-complement* of α w.r.t. β, defined as

$$\bigcup\{\gamma \mid \alpha \cap \gamma \preceq \beta\},$$

and denoted by $\alpha \hookrightarrow \beta$ (it is also called *Heyting implication*), exists for every α and β [10]. For instance, consider the Heyting algebra $\mathbb{B}^3$ in Fig. 1. Then, as expected, $0 \hookrightarrow 0 = 1$ and $1 \hookrightarrow 0 = 0$, where 0 is $\emptyset$ and 1 is $\{a, b, c\}$ (and, in general, this is true for every Heyting algebra, since it generalizes the Boolean case); moreover, we have that, for example, $\{a, c\} \hookrightarrow 0 = \{b\}$. A Heyting algebra is said to be *complete* if for every subset $H' \subseteq H$, both its least upper

[1] This is the classical nomenclature in lattice theory, and it should not be confused with Allen's relation *meets*.

bound $\bigcup H'$ and its greatest lower bound $\bigcap H'$ exist. Moreover, a Heyting algebra is *finite* if its domain is finite, *Boolean* if it is isomorphic to a nonempty set of subsets of a given set closed under the set operations of union, intersection, and complement relative to that set, and a *chain* if $\preceq$ is total. Graphical examples of such algebras can be found in Figure 1.

As in [7], assuming that $\mathcal{H}$ is a complete Heyting algebra with domain H we define an *adequate fuzzy strictly linearly ordered set* as a structure of the type

$$\widetilde{\mathbb{D}} = \langle D, \tilde{<}, \tilde{=} \rangle,$$

where D is a *domain* (and, again, we identify D with $\mathbb{D}$) enriched with two functions $\tilde{<}, \tilde{=} : D \times D \mapsto \mathcal{H}$, for which the following conditions apply for every x, y, and z:

$$\begin{array}{l}
\tilde{=}(x,y) = 1 \text{ iff } x = y,\\
\tilde{=}(x,y) = \tilde{=}(y,x),\\
\tilde{<}(x,x) = 0,\\
\tilde{<}(x,z) \succeq \tilde{<}(x,y) \cap \tilde{<}(y,z),\\
\text{if } \tilde{<}(x,y) \succ 0 \text{ and } \tilde{<}(y,z) \succ 0 \text{ then } \tilde{<}(x,z) \succ 0,\\
\text{if } \tilde{<}(x,y) = 0 \text{ and } \tilde{<}(y,x) = 0 \text{ then } \tilde{=}(x,y) = 1,\\
\text{if } \tilde{=}(x,y) \succ 0 \text{ then } \tilde{<}(x,y) \prec 1.
\end{array}$$

An adequate fuzzy linear order is *finite* when D is finite. The above conditions are called *adequate fuzzy linear order axioms.*

Now, let us fix a complete Heyting algebra $\mathcal{H}$. Similarly to the crisp case, well-formed *Fuzzy Halpern and Shoham's Modal Logic for Time Intervals* (*FHS*) formulas are built from a set of propositional letters $\mathcal{P}$, the classical connectives $\vee$ and $\neg$, and a modality for each Allen's interval relation, as follows:

$$\varphi ::= \alpha \mid p \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \varphi \to \psi \mid \langle X \rangle \varphi \mid [X]\varphi,$$

where $\alpha \in \mathcal{H}$, $p \in \mathcal{P}$, and, as in the crisp case, $X \in \mathcal{X}$. As before, the set of all subformulas of a given FHS formula φ is denoted by $sub(\varphi)$.

As for the semantics of FHS formulas, given an adequate fuzzy strictly linearly ordered set we define the set of fuzzy strict intervals in $\widetilde{\mathbb{D}}$ as

$$\mathbb{I}(\widetilde{\mathbb{D}}) = \{[x,y] \mid \tilde{<}(x,y) \succ 0\},$$

and, generalizing classical Boolean evaluation, propositional letters are directly evaluated in the underlying algebra by defining a *fuzzy valuation function*, as follows:

$$\widetilde{V} : \mathcal{P} \times \mathbb{I}(\widetilde{\mathbb{D}}) \mapsto H.$$

On top of the fuzzyfication of valuations we need to define how accessibility relations behave in the fuzzy context. The definition of fuzzy Allen's relations is obtained by generalizing the original, crisp definition, and substituting every $=$ with $\tilde{=}$ and every $<$ with $\tilde{<}$:

$$\begin{array}{rcl}
\widetilde{R}_A([x,y],[z,t]) &=& \tilde{=}(y,z),\\
\widetilde{R}_L([x,y],[z,t]) &=& \tilde{<}(y,z),\\
\widetilde{R}_B([x,y],[z,t]) &=& \tilde{=}(x,z) \cap \tilde{<}(t,y),\\
\widetilde{R}_E([x,y],[z,t]) &=& \tilde{<}(x,z) \cap \tilde{=}(y,t),\\
\widetilde{R}_D([x,y],[z,t]) &=& \tilde{<}(x,z) \cap \tilde{<}(t,y),\\
\widetilde{R}_O([x,y],[z,t]) &=& \tilde{<}(x,z) \cap \tilde{<}(z,y) \cap \tilde{<}(y,t),
\end{array}$$

and similarly for the inverse relations. Finally, we say that an $\mathcal{H}$*-valued interval model* (or *fuzzy interval model*) is a tuple of the type:

$$\widetilde{M} = \langle \mathbb{I}(\widetilde{\mathbb{D}}), \widetilde{V} \rangle$$

where $\widetilde{\mathbb{D}}$ is a fuzzy strictly linearly ordered set and $\widetilde{V}$ is a fuzzy valuation function. We interpret an FHS formula in a fuzzy interval model $\widetilde{M}$ and an interval $[x, y]$ by extending the valuation $\widetilde{V}$ of propositional letters as follows, where $X \in \mathcal{X}$ and $[z, t]$ varies in $\mathbb{I}(\widetilde{\mathbb{D}})$:

$$\begin{array}{lcl}
\widetilde{V}(\alpha, [x, y]) & = & \alpha, \\
\widetilde{V}(\varphi \wedge \psi, [x, y]) & = & \widetilde{V}(\varphi, [x, y]) \cap \widetilde{V}(\psi, [x, y]), \\
\widetilde{V}(\varphi \vee \psi, [x, y]) & = & \widetilde{V}(\varphi, [x, y]) \cup \widetilde{V}(\psi, [x, y]), \\
\widetilde{V}(\varphi \to \psi, [x, y]) & = & \widetilde{V}(\varphi, [x, y]) \hookrightarrow \widetilde{V}(\psi, [x, y]), \\
\widetilde{V}(\langle X \rangle \varphi, [x, y]) & = & \bigcup \{\widetilde{R}_X([x, y], [z, t]) \cap \widetilde{V}(\varphi, [z, t])\}, \\
\widetilde{V}([X]\varphi, [x, y]) & = & \bigcap \{\widetilde{R}_X([x, y], [z, t]) \hookrightarrow \widetilde{V}(\varphi, [z, t])\}.
\end{array}$$

We say that a formula of FHS φ is α-*satisfied* at an interval $[x, y]$ in a fuzzy interval model $\widetilde{M}$ if and only if

$$\widetilde{V}(\varphi, [x, y]) \succeq \alpha.$$

The formula φ is α-*satisfiable* if and only if there exists a fuzzy interval model and an interval in that model where it is α-satisfied. A formula is *satisfiable* if it is α-satisfiable for some $\alpha \in \mathcal{H}$, $\alpha \neq 0$. A formula is α-*valid* if it is α-satisfied at every interval in every model, and *valid* if it is 1-valid. Observe that since a Heyting algebra, in general, does not encompass classical negation, and since our definition of satisfiability is graded, instead of absolute, then the usual duality of satisfiability and validity does not hold anymore.

As shown in [7], α-satisfiable of FHS formulas is undecidable in the case of chain algebras. Such a result cannot be immediately generalized to the case of *all* Heyting algebras, but, since as crisp HS is undecidable in every class of linearly ordered sets, one can expect that FHS is too, regardless the underlying algebra.

3 A Tableau System for FHS

In this section we consider the problem of reasoning with FHS formulas. Tableau systems have been introduced in [4, 11, 12, 18] for variants, fragments, and generalizations of crisp HS, and in [9] for fuzzy modal logics; as in the latter case, we limit ourselves to the case of finite Heyting algebras as truth value algebras.

A tableau for a FHS formula is a directed tree, in which every node is associated to a *truth judgment*, to a pair formula/interval, and to a finite constraint system. In Fitting's terminology, a truth judgement, as we use it, is a *signed formula with bounding implications* [9]. Such a system represents an adequate fuzzy linearly ordered set; its constraints come from both the formula whose satisfiability has to be checked and the axioms that every adequate fuzzy linear order must meet. It may be possible that such a constraint system cannot be satisfied at some node: this will cause the branch that contain that node to be closed.

▶ **Definition 1** (fuzzy constraint system). *Given a finite Heyting algebra $\mathcal{H}$, a* fuzzy constraint system *C is a finite set of elements $\{x, y, \ldots\}$ associated to a finite set of constraints of the following types:*

$$\begin{array}{ll}
\widetilde{=}(x, y) \bowtie \alpha, & \\
\widetilde{<}(x, y) \bowtie \alpha, & \\
F & F \text{ is an adequate fuzzy linear order axiom,}
\end{array}$$

where $\alpha \in \mathcal{H}$ and $\bowtie \in \{\preceq, \succeq, \prec, \succ\}$.

For the sake of convenience, when describing a constraint system we shall omit to include adequate fuzzy linear order axioms (as they do not vary from system to system). It is immediate to see that any fuzzy constraint system can be checked for satisfiability using a first-order reasoner, and that termination is guaranteed by the fact that the system is finite. In the following, we shall use $x \in C$ to indicate that a certain element is mentioned in the fuzzy constraint system C. Intuitively, C represent a possibly incomplete adequate fuzzy linear order; if C can be satisfied, then it can be extended to a complete adequate fuzzy linear order. In the following, we say that C is *solved* if a value for $\widetilde{=}$ and a value for $\widetilde{<}$ has been chosen for every pair $x, y \in C$ in such a way that all constraints are met; moreover, we say that C is *inconsistent* if a solution cannot be found. We assume that a constraint system C (once solved) can be *queried*, so that, for example, at any given time, we can know the value of the relation $R_X([x,y],[z,t])$ for any points $x, y, z, t \in C$. In particular, for a given constraint system C, we assume that a function $o(C)$ (resp., $n(C)$) is defined that returns a list of all possible *old* intervals $[x,y]$ that can be formed with points in C (resp., all possible *new* intervals that can be formed in C using one or two points not currently in C); clearly, $o(C) \cap n(C) = \emptyset$. Finally, observe that for a given non-inconsistent system C there may be more than one solution. The set of all possible constraint systems is denoted by $\mathcal{C}$.

Because classic negation is not available in the fuzzy case, following Fitting, our tableau is designed to answer the question of whether a given formula φ can be satisfied to a degree at least α in $\mathcal{H}$-valued interval model, for a given finite Heyting algebra $\mathcal{H}$.

▶ **Definition 2** (decoration). *Given a Heyting algebra $\mathcal{H}$, an FHS formula φ, and a fuzzy constraint system C, a* decoration *is an object of the type*

$$Q(\alpha \to \varphi, [x,y], C), \text{ or } Q(\varphi \to \alpha, [x,y], C),$$

where $\alpha \in \mathcal{H}$ and $Q \in \{T, F\}$ is a judgment*. The expression $\alpha \to \varphi$ ($\varphi \to \alpha$) is an* assertion *on $[x,y] \in o(C)$. The universe of all possible decorations is denoted by $\mathcal{D}$.*

Intuitively, the assertion $\alpha \to \varphi$ (resp., $\varphi \to \alpha$) on an interval $[x,y]$ means that there exists a fuzzy model $\widetilde{M}$ with valuation function $\widetilde{V}$ such that $\widetilde{V}(\varphi,[x,y]) \succeq \alpha$ (resp., for every fuzzy model $\widetilde{M}$ and valuation functions $\widetilde{V}$ it is the case that $\widetilde{V}(\varphi,[x,y]) \preceq \alpha$); associating a judgment T (resp., F) to an assertion can be interpreted as (trying to) proving that the assertion holds (resp., does not hold).

▶ **Definition 3** (tableau for FHS). *Given an FHS formula φ and a finite Heyting algebra $\mathcal{H}$, the* tableau *τ for φ and $\alpha \in \mathcal{H}$ is an object of the type*

$$\tau = (\mathcal{V}, \mathcal{E}, d, f, c),$$

where $(\mathcal{V}, \mathcal{E})$ is a tree with vertices *(or* nodes*) in $\mathcal{V}$ and* edges *in $\mathcal{E}$. The nodes in τ are partially ordered by the relation $\lhd$ (induced by the edges) and whose set of branches is denoted by $\mathcal{B}$,*

$$d : \mathcal{V} \to \mathcal{D},$$

is a node labeling function*, which associates a decoration $Q(\psi \to \alpha, [x,y], C)$ or $Q(\alpha \to \psi, [x,y], C)$ to any node ν, where $\psi \in sub(\varphi)$ and $x, y \in C$, and*

$$f : \mathcal{V} \to \{0, 1\}$$

is a node flag function*, which determines which nodes have been already expanded,*

$$c : \mathcal{B} \to \mathcal{C}$$

is a branch labeling function*, which associates every branch to the constraint system in the decoration of its leaf, and it has been obtained starting from the* initial tableau τ_0

$$(\{\nu_0\}, \emptyset, \{(\nu_0, T(\alpha \to \varphi, [x,y], \{x, y, \widetilde{<}(x,y) \succ 0\}))\}, \{(\nu_0, 0)\}, \{(\nu_0, \{x, y, \widetilde{<}(x,y) \succ 0\})\})$$

by iteratively applying the branch expansion rule *in Fig. 2 to the closest-to-the-root node ν such that $f(\nu) = 0$ and every leaf ν' such that $\nu \lhd \nu'$, until no further application is possible or all branches have been closed. The tableau is* closed *(resp.,* open*) if all its branches (resp., at least one of its branches) are (resp., is) closed ✗ by some condition in Fig. 3 (resp., open ✓).*

Following the original terminology, the first four rules in Fig. 2 are referred to as *reverse* rules, the rules for nodes with a decoration that contain a propositional formula in the assertion are referred to as *propositional* rules, and the rules for nodes with a decoration that contain a temporal formulas in the assertion are referred to as *temporal* rules. Observe that the set actually covers all cases; those that are not covered can be treated by (the application of) a reverse rule.

The application of the branch expansion rule to a specific branch B in a tableau defined as above works as follows. First, the closest-to-the-root node ν of B, such that $f(\nu) = 0$ is chosen. Then, the consequent of the rule produces a new tree which is attached to the leaf of B; observe that the constraint system used in the application is always the one currently at the leaf. Finally, the application is not possible if the nodes produced by it are already present on the branch.

Now, we move to proving that the tableau system is sound.

▶ **Lemma 4** (soundness). *Let φ be an FHS formula and $\alpha \in \mathcal{H}$ a constant of a finite Heyting algebra. Then, if φ is α-satisfiable, then the tableau τ for φ and α is open.*

Proof. Consider an FHS formula φ. Assume that τ is the tableau for φ and $\alpha \in \mathcal{H}$, where $\mathcal{H}$ is a fixed finite Heyting algebra. We proceed contrapositively to prove that if τ is closed then φ is not α-satisfiable. Given a node ν in τ such that C is the constraint system in $d(\nu)$, we define the set

$$S(\nu) = \{\nu' \mid \nu' \lhd \nu\}$$

and we say that is $S(\nu)$ *is α-satisfiable* if and only if there is an $\mathcal{H}$-valued interval model

$$\widetilde{M} = \langle \mathbb{I}(C^*), \widetilde{V} \rangle,$$

where C^* is a fuzzy strictly linearly ordered set that extends C, such that

- for each node $\nu' \in S(\nu)$ such that $d(\nu') = T(\beta \to \psi, [x,y], C')$ (resp., $F(\psi \to \beta, [x,y], C')$), it is the case that $\widetilde{V}(\psi, [x,y]) \succeq \beta$ (resp., $\widetilde{V}(\psi, [x,y]) \succeq \gamma$, for some minimal γ not below β), and
- for each node $\nu' \in S(\nu)$ such that $d(\nu') = T(\psi \to \beta, [x,y], C')$ (resp., $F(\beta \to \psi, [x,y], C')$), it is the case that $\widetilde{V}(\psi, [x,y]) \preceq \beta$ (resp., $\widetilde{V}(\psi, [x,y]) \preceq \gamma$ for some maximal γ not above β).

Observe that $\widetilde{M}$ depends on α since $\nu_0 \in S(\nu)$, for every ν. Moreover, if $S(\nu_0)$ is α-satisfiable, then φ is α-satisfiable. Now, we prove the following stronger statement: *if every branch containing a node ν is closed, then the set $S(\nu)$ is not α-satisfiable.* Let us proceed by induction on the height h of the node ν.

$$(T\succeq)\ \frac{T(\alpha\to\psi,[x,y],C)}{F(\psi\to\gamma,[x,y],c(B))}$$
where $\alpha\neq 0$ and γ is any maximal element not above α, i.e., $\gamma\not\succeq\alpha$

$$(F\succeq)\ \frac{F(\alpha\to\psi,[x,y],C)}{T(\psi\to\beta_i,[x,y],c(B))\mid\ldots\mid T(\psi\to\beta_n,[x,y],c(B))}$$
where $\alpha\neq 0$ and $\beta_1,\ldots,\beta_n$ are all maximal elements not above α, i.e., $\beta_1,\ldots,\beta_n\not\succeq\alpha$

$$(T\preceq)\ \frac{T(\psi\to\alpha,[x,y],C)}{F(\gamma\to\psi,[x,y],c(B))}$$
where $\alpha\neq 1$ and γ is any minimal element not below α, i.e., $\gamma\not\preceq\alpha$

$$(F\preceq)\ \frac{F(\psi\to\alpha,[x,y],C)}{T(\beta_i\to\psi,[x,y],c(B))\mid\ldots\mid T(\beta_i\to\psi,[x,y],c(B))}$$
where $\alpha\neq 1$ and $\beta_1,\ldots,\beta_n$ are all minimal elements not below α, i.e., $\beta_1,\ldots,\beta_n\not\preceq\alpha$

(a) Reverse rules.

$$(T\wedge)\ \frac{T(\alpha\to(\psi\wedge\xi),[x,y],C)}{\begin{array}{c}T(\alpha\to\psi,[x,y],c(B))\\ T(\alpha\to\xi,[x,y],c(B))\end{array}}$$
where $\alpha\neq 0$

$$(F\wedge)\ \frac{F(\alpha\to(\psi\wedge\xi),[x,y],C)}{F(\alpha\to\psi,[x,y],c(B))\mid F(\alpha\to\xi,[x,y],c(B))}$$
where $\alpha\neq 0$

$$(T\vee)\ \frac{T((\psi\vee\xi)\to\alpha,[x,y],C)}{\begin{array}{c}T(\psi\to\alpha,[x,y],c(B))\\ T(\xi\to\alpha,[x,y],c(B))\end{array}}$$
where $\alpha\neq 1$

$$(F\vee)\ \frac{F((\psi\vee\xi)\to\alpha,[x,y],C)}{F(\psi\to\alpha,[x,y],c(B))\mid F(\xi\to\alpha,[x,y],c(B))}$$
where $\alpha\neq 1$

$$(T\to)\ \frac{T(\alpha\to(\psi\to\xi),[x,y],C)}{F(\gamma\to\psi,[x,y],c(B)\mid T(\gamma\to\xi,[x,y],c(B))}$$
where $\alpha\neq 0$ and γ is any element below α except 0, i.e., $0\neq\gamma\preceq\alpha$

$$(F\to)\ \frac{F(\alpha\to(\psi\to\xi),[x,y],C)}{\begin{array}{c}T(\beta_1\to\psi,[x,y],c(B))\mid\ldots\mid T(\beta_n\to\psi,[x,y]c(B))\\ F(\beta_1\to\xi,[x,y],c(B))\mid\ldots\mid F(\beta_n\to\xi,[x,y],c(B))\end{array}}$$
where $\alpha\neq 0$ and $\beta_1,\ldots,\beta_n$ are all elements below α except 0, i.e., $0\neq\beta_1,\ldots,\beta_n\preceq\alpha$

(b) Propositional rules.

$$(T\Box)\ \frac{T(\alpha\to[X]\psi,[x,y],C)}{\begin{array}{c}T((\alpha\cap\beta_1)\to\psi,[z_1,t_1],c(B))\\ \ldots\\ T((\alpha\cap\beta_n)\to\psi,[z_n,t_n],c(B))\\ T(\alpha\to[X]\psi,[x,y],c(B))\end{array}}$$
where $\beta_i=R_X([x,y],[z_i,t_i])$, $[z_i,t_i]\in o(c(B))$, $\beta_i\succ 0$, and $\alpha\cap\beta_i\neq 0$

$$(T\Diamond)\ \frac{T(\langle X\rangle\psi\to\alpha,[x,y],C)}{\begin{array}{c}T((\psi\to(\beta_1\hookrightarrow\alpha),[z_1,t_1],c(B))\\ \ldots\\ T(\psi\to(\beta_n\hookrightarrow\alpha),[z_n,t_n],c(B))\\ T(\langle X\rangle\psi\to\alpha,[x,y],c(B))\end{array}}$$
where $\beta_i=R_X([x,y],[z_i,t_i])$, $[z_i,t_i]\in o(c(B))$, $\beta_i\succ 0$, and $\beta_i\hookrightarrow\alpha\neq 1$

$$(F\Box)\ \frac{F(\alpha\to[X]\psi,[x,y],C)}{F((\alpha\cap\beta_1)\to\psi,[z_1,t_1],c(B))\mid\ldots\mid F((\alpha\cap\beta_n)\to\psi,[z_n,t_n],c(B))}$$
where $\beta_i=R_X([x,y],[z_i,t_i])$,$[z_i,t_i]\in o(c(B))\cup n(c(B))$, $\beta_i\succ 0$, and $\alpha\cap\beta_i\neq 0$

$$(F\Diamond)\ \frac{F(\langle X\rangle\psi\to\alpha,[x,y],C)}{F(\psi\to(\beta_1\hookrightarrow\alpha),[z_1,t_1],c(B))\mid\ldots\mid F(\psi\to(\beta_n\hookrightarrow\alpha),[z_n,t_n],c(B))}$$
where $\beta_i=R_X([x,y],[z_i,t_i])$,$[z_i,t_i]\in o(c(B))\cup n(c(B))$, $\beta_i\succ 0$, and $\beta_i\hookrightarrow\alpha\neq 1$

(c) Temporal rules.

Figure 2 Branch expansion rules for a branch B, to be applied to $\nu\in B$ under the conditions specified below each rule. The node flag is 0 when a rule is applied on a node with label at the top, modified into 1 after the application, and set to 0 on every produced node. When applying the rules $(F\Box)$ and $(F\Diamond)$, the constraint system C is first solved, and, queried for $o(C)$, and finally, for $n(C)$, returning all possible (old and new) intervals relevant for the application.

$$(\text{✗}1)\ \frac{T(\alpha \to \beta, [x,y], C)}{\text{✗}}\ \text{where } \alpha \not\preceq \beta \qquad (\text{✗}2)\ \frac{F(\alpha \to \beta, [x,y], C)}{\text{✗}}\ \text{where } \alpha \preceq \beta$$

$$(\text{✗}3)\ \frac{F(0 \to \psi, [x,y], C)}{\text{✗}} \qquad (\text{✗}4)\ \frac{F(\psi \to 1, [x,y], C)}{\text{✗}}$$

$$(\text{✗}5)\ \frac{\begin{array}{c} T(\beta \to \psi, [x,y], C) \\ F(\alpha \to \psi, [x,y], C) \end{array}}{\text{✗}}\ \text{where } \alpha \preceq \beta \qquad (\text{✗}6)\ \frac{Q(\cdot,\cdot,C)}{\text{✗}}\ \text{where } C \text{ is inconsistent}$$

Figure 3 Branch closing conditions.

If $h = 0$, then there is exactly one branch that contains it. Since such branch is closed, one of the following must hold. First, for some $\nu' \in S(\nu)$, $d(\nu') = T(\beta \to \gamma, [x,y], C')$ but $\beta \not\preceq \gamma$ (condition (✗1)), or $d(\nu') = F(\beta \to \gamma, [x,y], C')$ but $\beta \preceq \gamma$ (condition (✗2)). Second, for some $\nu' \in S(\nu)$, $d(\nu') = F(0 \to \psi, [x,y], C')$ (condition (✗3)), or $F(\psi \to 1, [x,y], C')$ (condition (✗4)). Third, for some $\nu' \in S(\nu)$, $d(\nu') = Q(\cdot,\cdot,C)$ but C is inconsistent (condition (✗6)). Or, fourth, for some $\nu', \nu'' \in S(\nu)$, $d(\nu') = F(\beta \to \psi, [x,y], C')$ and $d(\nu'') = T(\gamma \to \psi, [x,y], C'')$, but $\gamma \preceq \beta$ (condition (✗5)). In all such cases, $\widetilde{M}$ cannot be realized, so $S(\nu)$ is not α-satisfiable, as we wanted.

Suppose, now, that $h > 0$. First, observe that if every branch that contains ν is closed, then every branch that contains any of its successors must be closed too, so that the inductive hypothesis applies to them. Then, consider the node $\nu' \in S(\nu)$ that has been expanded when ν was a leaf, and let us analyze the possible rules that have been applied at ν'. If $d(\nu') = T(\psi \to \beta, [x,y], C')$, then the immediate successor ν'' of ν is such that $d(\nu'') = F(\gamma \to \psi, [x,y], C'')$, where γ is some minimal element of $\mathcal{H}$ such that $\gamma \not\succeq \beta$ (rule $(T \preceq)$); by inductive hypothesis, $S(\nu'')$ is not α-satisfiable, but this implies that $S(\nu)$ cannot be α-satisfiable either. If $d(\nu') = F(\beta \to \psi, [x,y], C')$, then all immediate successors ν_i of ν are such that $d(\nu_i) = T(\psi \to \gamma_i, [x,y], C_i)$, where γ_i is a maximal element of $\mathcal{H}$ such that $\gamma_i \not\succeq \beta$ (rule $(F \succeq)$); by inductive hypothesis, $S(\nu_i)$ is not α-satisfiable for any i, but this implies that $S(\nu)$ cannot be α-satisfiable either. The cases in which another reverse rule has been applied to ν are similar. If $d(\nu') = T(\beta \to (\psi \wedge \xi), [x,y], C')$, then ν has an immediate successor ν_1 with $d(\nu_1) = T(\beta \to \psi, [x,y], C_1)$, which in turn has an immediate successor ν_2 with $d(\nu_2) = T(\beta \to \xi, [x,y], C_2)$ (rule $(T\wedge)$). By inductive hypothesis, $S(\nu_2)$, in particular, is not α-satisfiable, but this implies that $S(\nu)$ is not α-satisfiable either. If $d(\nu') = F(\beta \to (\psi \wedge \xi), [x,y], C')$, then ν has two immediate successors ν_1 and ν_2 with $d(\nu_1) = F(\beta \to \psi, [x,y], C_1)$ and $d(\nu_2) = F(\beta \to \xi, [x,y], C_2)$ (rule $(F\wedge)$). By inductive hypothesis, both $S(\nu_1)$ and $S(\nu_2)$ are not α-satisfiable, but this implies that $S(\nu)$ is not α-satisfiable either. The cases in which another propositional rule has been applied to ν are similar. If $d(\nu') = T(\beta \to [X]\psi), [x,y], C')$ then ν has a chain of successors $\nu_1, \ldots, \nu_n$, such that $d(\nu_i) = T((\beta \cap \gamma_i) \to \psi, [z_i,t_i], C_i)$ and $\gamma_i = R_X([x,y],[z_i,t_i])$, for all $[z_i,t_i] \in o(c(B))$, where $1 \leq i \leq n$ (rule $(T\Box)$). Observe that asking that the evaluation of $[X]\psi$ is above β is equivalent to asking that the evaluation of ψ is above $\beta \cap \gamma_i$ on every interval $[z_i,t_i]$. Since, in particular, $S(\nu_n)$ is not α-satisfiable by inductive hypothesis, $S(\nu)$ is not α-satisfiable as well. The case in which $(T\Diamond)$ has been applied to ν is similar. Finally, if $d(\nu') = F(\langle X\rangle\psi \to \beta), [x,y], C')$ then every immediate successor ν_i of ν is such that $d(\nu_i) = F(\psi \to (\gamma_i \hookrightarrow \beta), [z_i,t_i], C_i)$

and $\gamma_i = R_X([x,y],[z_i,t_i])$, for all $[z_i,t_i] \in o(c(B)) \cup n(c(B))$, where $1 \le i \le n$ (rule $(F\Diamond)$). Observe that asking that the evaluation of $\langle X\rangle\psi$ is below β is equivalent to asking that the evaluation of ψ is below $\gamma_i \hookrightarrow \beta$ on some interval $[z_i,t_i]$. Since all $S(\nu_i)$ are not α-satisfiable by inductive hypothesis, $S(\nu)$ is not α-satisfiable as well. The case in which $(F\Box)$ has been applied to ν is similar. ◀

Finally, we turn our attention to proving completeness.

▶ **Lemma 5** (completeness). *Let φ be an FHS formula and $\alpha \in \mathcal{H}$ a constant of a finite Heyting algebra. If τ is an open tableau for φ and α, then φ is α-satisfiable.*

Proof. Consider an FHS formula φ, and assume that τ is the tableau for φ and α. Consider an open branch B in τ, let $C_\mu = \bigcup_{\nu\in B} C_\nu$, where C_ν is the constraint system in the label $d(\nu)$, and let C^* the complete extension of C_μ. Then, consider the model

$$\widetilde{M} = \langle \mathbb{I}(C^*), \widetilde{V}\rangle,$$

where $\widetilde{V}$ is the following fuzzy valuation function, defined for every propositional letter p and fuzzy strict interval $[x,y]$ in $\mathbb{I}(C^*)$:

$$\widetilde{V}(p,[x,y]) = \begin{cases} \beta & \text{if } d(\nu) = T(\beta \to p,[x,y],C), \text{ for some } \nu \in B; \\ \gamma & \text{if } d(\nu) = F(\beta \to p,[x,y],C), \text{ for some } \nu \in B \text{ and } \gamma \not\succeq \beta. \end{cases}$$

The model $\widetilde{M}$ is the direct translation of the branch B into an (fuzzy) interval model; in particular, is an coherent assignment of truth values of all propositional letters on all intervals. As much as the case of the judgment $F(\beta \to p,[x,y],C)$ is considered we need to associate any truth value γ such that $\gamma \not\succeq \beta$. We want to prove that, for every $\nu \in B$,

- if $d(\nu) = T(\beta \to \psi,[z,t],C)$, then $\widetilde{V}(\psi,[z,t]) \succeq \beta$,
- if $d(\nu) = T(\psi \to \beta,[z,t],C)$, then $\widetilde{V}(\psi,[z,t]) \preceq \beta$,
- if $d(\nu) = F(\beta \to \psi,[z,t],C)$, then $\widetilde{V}(\psi,[z,t]) \not\succeq \beta$, and
- if $d(\nu) = F(\psi \to \beta,[z,t],C)$, then $\widetilde{V}(\psi,[z,t]) \not\preceq \beta$.

Observe that the above implies that φ is α-satisfiable on $[x,y]$ in $\widetilde{M}$, that is, it is α-satisfiable. Also, observe that $\widetilde{M}$ is constructible and well-defined because B is open. Consider a node $\nu \in B$ such that $d(\nu)$ is a decoration with a judgment for a formula ψ on some interval $[z,t]$. We proceed by structural induction on ψ.

If $\psi = p$ or $\psi = \beta$, then the claim is trivial.

If $\psi = \xi \wedge \chi$, then suppose, first, that $d(\nu) = T(\beta \to (\xi \wedge \chi),[z,t],C)$. Since τ is fully expanded, rule $(T\wedge)$ has been applied to ν. It follows that B contains two nodes ν_1 and ν_2 such that $d(\nu_1) = T(\beta \to \xi,[z,t],C)$ and $d(\nu_2) = T(\beta \to \chi,[z,t],C)$. By inductive hypothesis, $\widetilde{V}(\xi,[z,t]) \succeq \beta$ and $\widetilde{V}(\chi,[z,t]) \succeq \beta$, which is equivalent to $\widetilde{V}(\psi,[z,t]) \succeq \beta$. Suppose, now, that $d(\nu) = F(\beta \to (\xi \wedge \chi),[z,t],C)$. Since τ is fully expanded, rule $(F\wedge)$ has been applied to ν. It follows that B contains a node ν' such that $d(\nu') = F(\beta \to \xi,[z,t],C)$ or $d(\nu') = T(\beta \to \chi,[z,t],C)$. By inductive hypothesis, $\widetilde{V}(\xi,[z,t]) \not\succeq \beta$ or $\widetilde{V}(\chi,[z,t]) \not\succeq \beta$, which is equivalent to $\widetilde{V}(\psi,[z,t]) \not\succeq \beta$. The remaining propositional cases are similar.

Finally, if $\psi = [X]\xi$, then suppose, first, that $d(\nu) = T(\beta \to [X]\xi,[z,t],C)$. Since τ is fully expanded, rule $(T\Box)$ has been applied to ν. This entails that, for every interval $[z_i,t_i]$ in $\mathbb{I}(C^*)$, B contains a node ν_i such that $d(\nu_i) = T((\beta \cap \gamma_i) \to \xi,[z_i,t_i],C_i)$, that is, $d(\nu_i) = T((\beta \cap R_X([z,t],[z_i,t_i])) \to \xi,[z_i,t_i],C_i)$; observe that this is guaranteed by the fact that the rule has been applied at a certain point of the construction on the n possible intervals that are constructible at that point, but then an additional $(n+1)$-th node is also created at the end of the branch with the same decoration, ensuring that, should more points be added at some later moment, the rule is applied, again, on them. By inductive hypothesis, $\widetilde{V}(\xi, [z_i, t_i]) \succeq (\beta \cap \gamma_i)$. Therefore, $\widetilde{V}(\psi, [z, t]) = \bigcap_{[z_i,t_i]}\{R_X([z, t], [z_i, t_i]) \hookrightarrow \widetilde{V}(\xi, [z_i, t_i])\} \succeq \beta$. Suppose, now, that $d(\nu) = F(\beta \to [X]\xi, [z, t], C)$. Since τ is fully expanded, rule $(F\Box)$ has been applied to ν. This entails that, for some interval $[z_i, t_i]$ in $\mathbb{I}(C^*)$, B contains a node ν_i such that $d(\nu_i) = F((\beta \cap \gamma_i) \to \xi, [z_i, t_i], C_i)$, that is, $d(\nu_i) = F((\beta \cap R_X([z, t], [z_i, t_i])) \to \xi, [z_i, t_i], C_i)$. By inductive hypothesis, $\widetilde{V}(\xi, [z_i, t_i]) \not\succeq \beta \cap \gamma_i$. Therefore, $\widetilde{V}(\psi, [z, t]) = \bigcap_{[z_i,t_i]}\{R_X([z, t], [z_i, t_i]) \hookrightarrow \widetilde{V}(\xi, [z_i, t_i])\} \not\succeq \beta$. The remaining temporal cases are similar. ◀

▶ **Theorem 6** (semi-decision procedure). *The tableau system for FHS is sound and complete. Moreover, it is also a semi-decision procedure in the case of finite domains.*

A semi-decision procedure for the problem of establishing if a given FHS formula φ is α-satisfiable for some truth value α of a given Heyting algebra, as stated by the above theorem, emerges naturally as the systematic application of the expansion rules (see Fig. 2) to the initial tableau. Termination is not guaranteed as there may exist formulas that are α-satisfiable but only on an infinite domain: in such a case, no contradiction would be found within a finite amount of time.

We conclude this part with an example of application of the tableau system, illustrated in Fig. 4. In this example, we consider the formula $\langle A\rangle p \wedge [A](p \to 0)$ and $1 \in \mathbb{G}_3$, where $\mathbb{G}_3$ is the Göedel algebra from Fig. 1. To show how the system is applied, we tested if the truth value of the above formula is at least 1, that is, $T(1 \to \langle A\rangle p \wedge [A](p \to 0), [x, y], C)$ on the interval $[x, y]$ with $C = \{\widetilde{<}(x, y) \succ 0\}$; since all branches are closed, we verified, as expected, that it cannot. In the figure, only two branches are displayed.

4 Conclusion

Interval temporal logic is a crucial tool for planning, scheduling, and formal verification, and are also particular interesting for learning tasks, especially from continuous data. To deal with the uncertainty of real data, a fuzzy (many-valued) generalization of the most representative interval temporal logic (HS), called FHS, had been recently introduced and studied. The computational properties of FHS strongly depend on the underlying algebra on which it is based. Within the context of Heyting algebras, we considered, here, the finite case, and we devised a sound and complete tableau system for it. Our method builds on previous work by Fitting, and it is the first case of an implementable deduction procedure for fuzzy interval temporal logic, which could be applied as a reasoning system, for example, on formulas learned from real data in order to combine them with expert knowledge.

As future work, we plan to design an efficient implementation of the proposed tableau system. Observe that, in particular, such an implementation would be a generalization of its crisp counterpart. The experiments that have been carried on so far seem to indicate that the best implementation strategies are those based on the naive approaches as in [18], which is essentially different from the point-based case; therefore, in order to obtain a truly useful tool, an effort should be made to optimize the construction of such a tableau system in the crisp and fuzzy case alike.

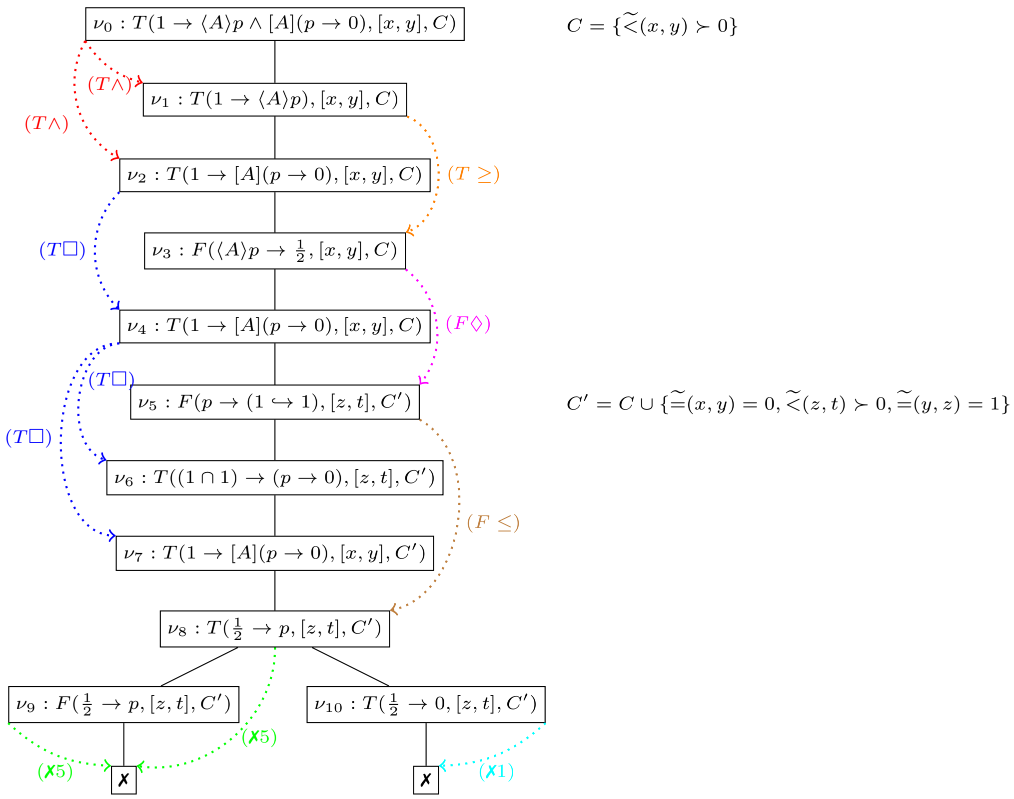


Figure 4 Some closed branches of the tableau for $\langle A\rangle p \wedge [A](p \rightarrow 0)$ and $1 \in \mathbb{G}_3$.

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