The Calculus of Temporal Influence

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Abstract

We present the Calculus of Temporal Influence, a simple logical calculus that allows reasoning about the behaviour of real-valued functions over time by making assertions that bound their values or the values of their derivatives. The motivation for the design of such a proof system comes from the need to provide the background computational machinery for tools that support learning in experimental subjects in secondary-education classrooms. The end goal is a tool that allows school pupils to formalise hypotheses about phenomena in natural sciences, such that their validity with respect to some formal experiment model can be checked automatically. The Calculus of Temporal Influence provides a language for formal statements and the mechanisms for reasoning about valid logical consequences. It extends (and deviates in parts from) previous work introducing the Calculus of (Non-Temporal) Influence by integrating the ability to model temporal effects in such experiments. We show that reasoning in the calculus is sound with respect to a natural formal semantics, that logical consequence is at least semi-decidable, and that one obtains polynomial-time decidability for a natural stratification of the problem.

2012 ACM Subject Classification Theory of computation → Theory and algorithms for application domains; Theory of computation → Automated reasoning; Applied computing → Interactive learning environments

Keywords and phrases temporal reasoning, formal models, continuous functions, polynomial decidability

Digital Object Identifier 10.4230/LIPIcs.TIME.2023.10

1 Introduction

Digitalisation is an ongoing process that aims at providing better solutions in all kinds of areas in industry, science, society and everyday's life. The work presented here is motivated by efforts to provide digital technology in particular learning environments, mainly secondary-education classrooms in natural sciences like biology, physics, chemistry. Digitalisation in school classrooms is a well-studied topic in educational sciences, but it often just deals with the employment of digital equipment in order to enhance learning environments, like electronic whiteboards, tablets, the internet as an online source of information, or at most the acquisition of competences to use digital (software) tools.

An aspect that is fundamental to science education is the promotion of scientific literacy which targets "*the skills to use scientific knowledge, ask questions, and draw evidence-based conclusions to understand and make decisions about the natural world and the changes humans are making to it*" [11]. The *Circle of Inquiry* asks learners to form phenomena-based hypotheses and test them by experiment [6]. The curricula of natural science subjects therefore often contain experimental studies where pupils start from a given *research question* like "*does temperature influence the growth of yeast?*" which prompts them to formulate

30th International Symposium on Temporal Representation and Reasoning (TIME 2023).
Editors: Alexander Artikis, Florian Bruse, and Luke Hunsberger; Article No. 10; pp. 10:1–10:19
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

a hypothesis like "*yeast grows when it is warmer*" or, better, "*yeast growth is maximal at temperatures between 32° and 38°, and higher temperatures influence the growth negatively.*" This is typically followed by a validation process in which they assemble and physically conduct an experiment in order to check the validity of their hypothesis, thus learning some form of scientific reasoning.

The design and use of learning tools, in the form of specialised software for instance, can take this a step further by making use of general digital *technology* rather than just equipment, see strategies devised by ministries and departments of education.[1] The next step is then of course to use computational resources like algorithms, formal models, etc. in order to further advance the digitalisation process in such areas.

Technology-based learning makes it possible to create differentiated learning methods and to enable individualised learning of subject-specific competencies using feedback [9]. It comes with several advantages.

- It is *resource efficient*, making it possible to run experiments more often and at smaller running costs after investing into generic and reusable hardware.
- It is *more scalable* since digital environments can easily be multiplied, depending only on the availability of digital hardware, thus making it possible to run experiments by smaller groups of pupils.
- It makes experimental lessons *more widely applicable* as it eliminates risk in handling dangerous substances, lifts the restriction to objects of manageable size, and allows time to be scaled up or down in experiments that would otherwise take a very long or too short time to be observed in reality.
- It *enhances learning efforts* by reducing the influence of *human factors*: broadly speaking, it forces pupils to concentrate on the learning material when interacting with a learning tool, rather than to look for clues to the right answers by interacting with a teacher directly, cf. [8].

In order to be effectively used by school pupils, a digitalised learning environment for experimental lessons needs to combine different kinds of digital technology, including intuitive web interfaces, secure and stable network communications, etc. Here we are concerned with the logic that is needed in order to automatically check the correctness of a formulated hypothesis w.r.t. some background knowledge about a modelled experiment. In previous work we have introduced the *Calculus of Influence* [4] which provides

- a simple language for making statements of the form

  "*variable A influences variable B* $\big[$ *on data range* $[x, y]$ $\big]$ $\big[$ *into values in* $[x', y']$ $\big]$
  $\big[$ *showing monotonic / antitonic / constant / arbitrary behaviour* $\big]$"

  where variables are partially ordered entities, determined by the experiment (for instance temperature and yeast), and $[x, y]$ and $[x', y']$ are intervals over the reals providing a common data type for all variables;
- a formal semantics for interpreting such statements in collections of real-valued continuous partial functions over partially ordered sets of variables, thus providing a formal model of such experiments in which influence of a variable B by a variable A is modelled by the existence of a function associated with the pair (A, B);

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[1] See e.g. `https://www.kmk.org/themen/bildung-in-der-digitalen-welt/strategie-bildung-in-der-digitalen-welt.html` (in German) for a joint strategy paper on eduction in a digital world of the German federal states' ministers for education.

- a simple proof system for derivability of a hypothesis from a set of statements representing the aforementioned background knowledge about the experiment. It turned out to be sound and polynomially decidable, and it was shown to be complete for a restricted class of formal experiment models.

The framework can be used to formalise known facts about the underlying experiment through sets of statements in the above sense, called an *influence scheme*. The pupils' task is then to formulate a hypothesis in the form of another statement, detecting some form of (simple) influence between the variables involved in the scheme. The underlying tool should then verify, based on the notion of logical consequence, whether the hypothesis follows from the facts given in the scheme. If this is the case, then the pupils could be given the possibility to further interact with the tool, for instance by providing an explanation for the discovered influence (which could be checked for correctness by matching it against the found proof). If it is not the case, then pupils can be guided towards a correct hypothesis by presenting them with a counterexample, i.e. a realisation of experimental behaviour that satisfies all the facts in the scheme but not the hypothesis, derived from the failed proof attempt in the underlying tool. Note that the proof rules and strategy are not open to interaction for the pupils but are hidden away in an automated decision procedure. Likewise, the formalisation of experiments as influence schemes is not something that the pupils are tasked with; this needs to be done by someone with further expertise, or it can be automatically extracted from sets of data points.

Note that influence in the above model is always *static*: it is possible to state that a growing temperature between 10° and 20° causes increasing yeast activity between $0.14h^{-1}$ and $0.26h^{-1}$, but it is *not* possible to state how yeast activity causes an incressing volume of dough, as this keeps growing at a particular rate *over time*. Note that time cannot be modelled as a variable in this scenario like temperature or yeast activity, since neither of them cause particular values of time, and time alone does not cause particular volumes. Instead, the influence between yeast activity and dough volume is *dynamic*, more specifically it is time-dependent in the sense that a particular value of yeast activity causes particular volumes over a particular time interval.

In this paper we introduce the *Calculus of Temporal Influence* in order to provide a formal model for capturing further experiments and to allow hypotheses to be made about such time-dependent influences. In Sect. 2 we first introduce a simple formal language for formalising statements about background knowledge or hypotheses, incorporating time-dependency by allowing statements that assert an influence of a variable onto another one, resp. its *derivative* (w.r.t. time). This does not only extend the calculus by incorporating time as a special entity s.t. values of any variable are always implicitly time-dependent; it also enhances its ability to make more refined statements about the nature of influence, for instance asserting that the *gradient* of some growth rate – i.e. the values of the derivative function – falls into an interval $[l, u]$. For instance, monotonicity then corresponds to growth rates in the range $[0, \infty)$. Note, however, that the implicit time-dependency makes time play a special role in this setting. Moreover, in the Calculus of Temporal Influence, there are no direct formalisations available to express that one variable influences another, but rather indirect ways tightly connected to the fact that we model the behaviour of variables over time.

We then interpret such statements in collections containing a function of type $\mathbb{R}^{\geq 0} \to \mathbb{R}$ for each variable, modelling its behaviour over time in the experiment. As it turns out, the introduction of time-dependency alleviates the need for ordering the variables as it is done in the non-temporal Calculus of Influence, thus extending it in this respect as well.

The formal semantics immediately gives rise to the question of logical consequence, explaining when a statement, resp. hypothesis H is seen to follow from a set of statements, resp. scheme $\mathcal{C}$. This provides the mathematical ground for an automatic correctness check for hypotheses H when $\mathcal{C}$ models an experiment as a collection of known facts about time-dependent variable influences in it.

In Sect. 3 we present a simple proof system as the computational ground for such automatic correctness checks. The ultimate aim is polynomial decidability since the employment in a learning tool requires such correctness checks to be carried out efficiently – usually running on mobile devices – in order to provide instantaneous feedback to pupils formulating hypotheses. Another requirement posed by this application restricts the proof rules to formalisations of intuitive reasoning principles since an effective learning tool needs to be able to provide feedback to pupils about *why* a hypothesis might be false, i.e. why it cannot be inferred from the experiment model. This leads to a natural stratification of logical consequence w.r.t. the number of "difficult" rule applications, and we obtain polynomial-time decidability for each stratum in Sect. 4. We briefly report on a prototypical implementation of the deciding algorithm, implemented in Python, in Sect. 5.

Sect. 6 concludes with remarks on further work in this direction.

2 Modeling Influence over Time

Variables and Statements. Let $\mathcal{V} = \{a, b, \dots\}$ be a finite set of so-called *variables* (like *temperature*, some bacteria's *growth rate*, etc.). We introduce a small language for formalising statements about the behaviour of such variables over time. Below we introduce a formal semantics based on certain partial and continuous functions of the reals. This puts some underlying assumptions about such behaviours in place which is best motivated here in order to then proceed to the definition of the formal statements.

- There is a special variable $\mathtt{t}$ representing time which is not included in $\mathcal{V}$.
- Each variable – including $\mathtt{t}$ – is *real-valued*. This is in accordance with what is done in many areas in natural sciences, in particular in physics with the only exception of quantum physics perhaps. For simplicity we do not state units like seconds, kilograms, cubic meters, etc.; they are assumed to be given implicitly.
- The behaviour of a variable over time is a partial function whose domain is a single interval, i.e. values of variable a may only be defined after time point t_1, but if such values exist for time points t_1 and $t_2 > t_1$, then they also exist for all time points t with $t_1 \le t \le t_2$. It is arguable that this puts restrictions on the structure of experiments to be modelled here, but there is also a benefit to it: it means that experiment models can be obtained from measurements at discrete time points, and while we may not know the exact values in between the measurements, we can assume them to exist and even make some reasonable assumption about their values. The domain of $\mathtt{t}$ is always $\mathbb{R}^{\ge 0}$.
- The behaviour of a variable over time is a continuous function on its domain. This is in line with typical behaviour in nature which is rarely discontinuous.
- The behaviour of a variable over time is a derivable function on its domain, and the derivative in time is also continuous. Again, this is in accordance with typical behaviour in nature; it is also a necessary consequence of introducing time into the study of influence between such variables. We want to allow statements that prescribe values of a variable at certain *later* moments, possibly depending on values at a current moment. A simple way to do this is to assert that the values of the derivative of the variable's function are bounded, and for this to be well-founded such functions need to be derivable. We identify a variable a with an underlying function $\mathtt{t} \to a$ and write $\dot{a}$ for its first derivative.

Such functions of type $\mathtt{t} \to a$ can of course easily be shown graphically. We remark, though, that they are only used here for a formal semantics for simple statements about temporal influences. Such statements – to be defined next – are the main objects of concern, as they will be used to symbolically reason about such functions.

Intervals of reals with rational bounds are written as $[x, y]$ for $x, y \in \mathbb{Q} \cup \{\infty, -\infty\}$ with $x \leq y$. We simply write $[5.3, \infty]$ instead of $[5.3, \infty)$ in order to avoid unnecessary case distinctions. Intervals are only (semi-)open on an infinite interval bound. We restrict ourselves to rational bounds to keep them easily representable. A *time interval* is one in which the bounds are non-negative.

▶ **Definition 1.** Let $\mathcal{V}$ be given, $a, b \in \mathcal{V}$, $[l, u]$ and $[l', u']$ be intervals in the sense above, and $[t_1, t_2]$ be a time interval.

- A *time-value statement* (TVS) is of the form $\mathtt{t} \xhookrightarrow{[t_1,t_2],[l,u],[l',u']} a$. It states that the value of a is within $[l, u]$ at time point t_1, is within $[l', u']$ at time point t_2, and is within some interval $[l'', u'']$ at any time point $t' \in [t_1, t_2]$ that is obtained by linearly transforming $[l, u]$ into $[l', u']$ along the interval $[t_1, t_2]$. For details, see Def. 4 below.
  Thus, such a TVS intuitively states that the portion of the graph of the function a in the interval $[t_1, t_2]$ is contained in the trapezoid that has left vertical edge $[l, u]$ at t_1 and right vertical edge $[l', u']$ at t_2.
  Note that every rectangle is a trapezoid (with equal left and right vertical edges). We will write the special case of a TVS $\mathtt{t} \xhookrightarrow{[t_1,t_2],[l,u],[l,u]} a$ also simply as $\mathtt{t} \xhookrightarrow{[t_1,t_2],[l,u]} a$. Moreover, if $t_2 = \infty$ then the "right edge" of the trapezoid is ill-defined, and we will assume that in such a case, the trapezoid degenerates to a rectangle in this sense, i.e. in a statement $\mathtt{t} \xhookrightarrow{[t_1,t_2],[l,u],[l',u']} a$ with $t_2 = \infty$ we will always have $[l, u] = [l', u']$ and consequentially write it in its abbreviated form. A similar case arises with degenerate trapezoids of the form $\mathtt{t} \xhookrightarrow{[t_1,t_2],[-\infty,\infty],[-\infty,\infty]} a$ which we also write as $\mathtt{t} \xhookrightarrow{[t_1,t_2],[-\infty,\infty]} a$.
- A *time-derivative statement* (TDS) is of the form $\mathtt{t} \xhookrightarrow{[t_1,t_2],[l,u]} \dot{a}$. It states that the gradient of the portion of the graph of the function a is bounded from below by l and from above by u, likewise that the portion of the *derivative* of the function a in this interval is contained in the rectangle that is formed by the horizontal interval $[t_1, t_2]$ and the vertical interval $[l, u]$.
- A *value-derivative statement* (VDS) is of the form $a \xhookrightarrow{[l,u],[t_1,t_2],[l',u']} \dot{b}$. It states: if at some point t in time, the function a has a value within $[l, u]$, then in the interval $[t + t_1, t + t_2]$ the derivative of the function b has values between l' and u'.

A $\mathcal{V}$-statement is either a TVS, a TDS or a VDS. Note that TVS and TDS assert that something holds in time, while a VDS is more reminiscent of a temporal logic formula like $\Box(\varphi \to \Diamond\psi)$.

Note the special role that time plays in the above definitions: the time variable $\mathtt{t}$ only ever appears in the context of descriptions of the behaviour of other variables over time, and the time axis only serves as the domain of functions, but never appears in the range of any function. The only exception are VDS, where time also appears in the context of a delay after which a certain statement is supposed to hold.

▶ **Definition 2.** A *temporal $\mathcal{V}$-influence scheme* $\mathcal{C}$ is a finite set of $\mathcal{V}$-statements.

This introduces the main tool for formal modelling of temporal influence experiments. Intuitively, a temporal influence scheme collects abstract information about the way that the variables of an experiment behave over time and influence each other, in particular influence each other's growth rate. We simply speak of statements and influence schemes if $\mathcal{V}$ is clear from the context.

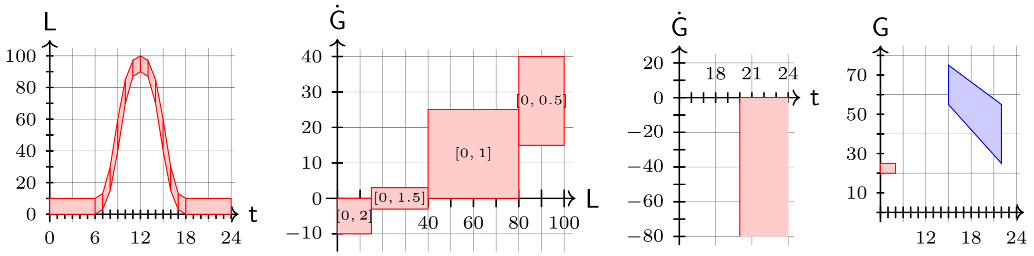

Figure 1 Graphical presentation of some statements about temporal influence in the experiment on photosynthesis and cellular respiration, as explained in Ex. 3.

▶ **Example 3.** We consider a phenomenon that is routinely discussed in biology classes: photosynthesis produces glucose (G) under the influence of light (L), cell respiration consumes glucose. We model this as a temporal influence scheme over $\mathcal{V} = \{\mathsf{L}, \mathsf{G}\}$.[2]

A typical pattern of light intensity during a 24h day is shown in Fig. 1 left. It is modelled using 14 TVS like $\mathsf{t} \xhookrightarrow{[9,10],[40,60],[75,85]} \mathsf{L}$ that can be visualised as trapezoids in the plane for influences of type $\mathsf{t} \to \mathsf{L}$. The unit on the t-axis is hours, the one on the L-axis is percent.

Photosynthesis causes the production of glucose under light. On the other hand, cellular respiration consumes glucose at a fairly standard rate (which we assume here to be constant). We model this phenomenon by dividing the range of light intensity into four categories, leading to four VDS as follows.

- In lowest light or darkness (say 0–15%), glucose is solely consumed by cellular respiration and therefore decreases in overall availability: $\mathsf{L} \xhookrightarrow{[0,15],[0,2],[-10,0]} \dot{\mathsf{G}}$ actually states that the gradient of light over the next time interval of 2h lies within the range of staying constant to dropping by 10, say $\mu g/h$.
- At a lower light intensity (15–40%), the production and consumption of glucose in the processes of photosynthesis and cellular respiration balance each other out, and there is at most a small increase or decrease: $\mathsf{L} \xhookrightarrow{[15,40],[0,1.5],[-3,3]} \dot{\mathsf{G}}$. Here we choose a shorter time interval – and then even shorter below – as bright light may be more prone to sudden changes as low light.
- At a higher intensity, production wins over consumption, and the growth rate is positive: $\mathsf{L} \xhookrightarrow{[40,80],[0,1],[0,20]} \dot{\mathsf{G}}$.
- At highest intensity, the growth rate of glucose is even higher: $\mathsf{L} \xhookrightarrow{[80,100],[0,0.5],[15,40]} \dot{\mathsf{G}}$.

Such VDS are less easy to depict graphically as they include *three* independent entities: a value range of L, a time interval, and a range for the gradient that values of G can follow over time. We introduce the graphical notation shown in Fig. 1 second to left, presenting those four statements from above.

A TDS seems to be of little use here in order to model the photosynthesis experiment; they are included in general because of common reasoning principles: if time influences light intensity, and light intensity influences the way that glucose levels change, then time indirectly influences the way that glucose levels change. Hence, a statement like "*glucose levels are falling after 20h*" – formalised as the TDS $\mathsf{t} \xhookrightarrow{[20,\infty],[-\infty,0]} \dot{\mathsf{G}}$ and shown in Fig. 1 second to right, actually saying that the levels are not rising instead of falling – may or may not be a valid conclusion from the TVS and VDS discussed beforehand.

[2] Clearly, such metabolism processes could be modelled at arbitrarily higher levels of detail, taking into account all sorts of involved biochemical agents. The view of the interaction focussing on these three agents is consistent with what can be done in biology school classes for instance. It already leaves out further factors like carbon dioxide, water and oxygen, which can either be seen as by-products or as constantly available.

At last, if time exerts an influence over the derivative of G, then it must also influence G. I.e. the picture of what kinds of influence between variables at certain value ranges or at certain time intervals exist in this model or can be derived from known facts in there, will not be complete without the possibility to include TVS for G. One such statement is shown in Fig. 1 right, in blue to distinguish it from the others as a hypothesis, i.e. a statement with which we associate the question "*Is this a correct logical consequence from the statements in the influence scheme?*" It predicts a glucose concentration whose expected value is falling slightly in the hours between 15 and 22.

It should be clear that in order to be able to answer this question we would need to fix initial values for G which can be done using another TVS like $\mathsf{t} \xhookrightarrow{[0,2],[20,25]} \mathsf{G}$ shown in Fig. 1 right in red, and fixing a range of initial glucose concentration.

The problem of deciding whether a given hypothesis (in the form of a statement H) is *correct* w.r.t. some experiment is modelled here as the question whether H follows logically from the temporal influence scheme $\mathcal{C}$ modelling knowledge about the way that the variables of the experiment behave over time and possibly influence each over. For this to be well-defined we introduce a formal semantics for statements and schemes next.

Semantics of Influence Schemes. An *influence* is a function $f\colon \mathbb{R}^{\geq 0} \to \mathbb{R}$ s.t. its domain $dom(f)$ is a non-unit interval as per above, and f is derivable on its entire domain. An influence is used to specify the behaviour of a variable $a \in \mathcal{V}$ over time. We write $f(t) = \bot$ if $t \notin dom(f)$. Note that necessarily also $t \notin dom(\dot{f})$.

▶ **Definition 4.** Let $\mathcal{V}$ be a set of variables. A $\mathcal{V}$*-influence experiment* is a collection $\mathcal{F}$ of influences containing exactly one influence $\mathcal{F}_a$ for each variable $a \in \mathcal{V}$.

- $\mathcal{F}$ satisfies the TVS $S = \mathsf{t} \xhookrightarrow{[t_1,t_2],[l,u],[l',u']} a$, written $\mathcal{F} \models S$, if
$$l + (l'-l)\cdot\frac{t-t_1}{t_2-t_1} \;\leq\; \mathcal{F}_a(t) \;\leq\; u + (u'-u)\cdot\frac{t-t_1}{t_2-t_1} \tag{1}$$
for all $t \in [t_1, t_2]$. In the special case where $t_2 = \infty$ (and $[l', u'] = [l, u]$ by convention) this is to be interpreted as $t \in [l, u]$ for all $t \geq t_1$.
- $\mathcal{F}$ satisfies the TDS $S = \mathsf{t} \xhookrightarrow{[t_1,t_2],[l,u]} \dot{a}$, also written $\mathcal{F} \models S$, if $l \leq \dot{\mathcal{F}}_a(t) \leq u$ for all $t \in [t_1, t_2]$.
- $\mathcal{F}$ satisfies the VDS $S = a \xhookrightarrow{[l,u],[t_1,t_2],[l',u']} \dot{b}$, also written $\mathcal{F} \models S$, if $l' \leq \dot{F}_b(z) \leq u'$ for all z such that there is t with $\mathcal{F}_a(t) \in [l, u]$ and $z \in [t + t_1, t + t_2]$.

$\mathcal{F}$ satisfies a $\mathcal{V}$-influence scheme $\mathcal{C}$, written $\mathcal{F} \models \mathcal{C}$, if $\mathcal{F} \models S$ for all $S \in \mathcal{C}$.

An influence scheme $\mathcal{C}$ is called *satisfiable* if there is some $\mathcal{F}$ such that $\mathcal{F} \models \mathcal{C}$. We say that a statement S follows from an influence scheme $\mathcal{C}$, written $\mathcal{C} \models S$, if $\mathcal{F} \models S$ for all $\mathcal{F}$ such that $\mathcal{F} \models \mathcal{C}$. Hence, a temporal influence experiment models concrete behaviour in terms of particular, real-valued functions; a temporal influence scheme models this abstractly by collecting (a finite amount) of information about bounds on the expected temporal behaviour of variables and on the influence they exert on each other leading to such temporal behaviour. Likewise, a temporal influence scheme can be seen as a finite representation of a (typically infinite and even uncountable) number of influence experiments with behaviours within the bounds included in the statements of the scheme.

We also obtain a notion of logical equivalence for influence schemes: we say that $\mathcal{C}$ and $\mathcal{C}'$ are *equivalent*, written $\mathcal{C} \equiv \mathcal{C}'$ iff for all influence experiments $\mathcal{F}$ we have $\mathcal{F} \models \mathcal{C}$ iff $\mathcal{F} \models \mathcal{C}'$. Note that in this case, for all statements S we have $\mathcal{C} \models S$ iff $\mathcal{C}' \models S$. Hence, equivalent models can be seen as forming abstract representations of the same experimental setup that may differ syntactically.

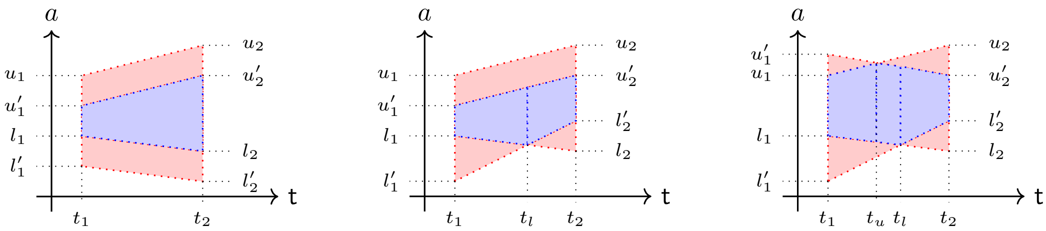

Figure 2 Intersecting two trapezoids (in red) over the same time interval into one, two or three adjacent trapezoids (in blue).

3 The Calculus of Temporal Influence

Geometric Considerations. We consider several subproblems – relaxation of a TVS, intersection and join of two TVS, and the effect of derivatives on TVS – arising with the problem of deciding (via a sound proof system) whether a given hypothesis follows from a given scheme, e.g. when two TVS make assertions about the same interval in time. Given the geometric interpretation of statements introduced in Ex. 3, the considerations carried out here can be seen as simple geometric principles.

Let $S = \mathsf{t} \xhookrightarrow{[t_1,t_w],[l_1,u_1],[l_2,u_2]} a$ be a TVS. By Def. 4, S implies that for all $t \in [t_1, t_2]$, the function $\mathcal{F}_a : \mathsf{t} \to a$ satisfies the inequalities in Eq. 1. Clearly, S also implies conditions on subintervals of $[t_1, t_2]$, and less restrictive conditions w.r.t. the vertical extent of the two edges of the trapezoid implied by it. However, it is generally not correct to simply make the vertical intervals larger and to shorten the time interval. Instead, relaxation is formalised as follows.

▶ **Definition 5.** Let S be as above. The set of *relaxed TVS* w.r.t. S, written $\mathsf{relax}(S)$, is the set of all TVS $\mathsf{t} \xhookrightarrow{[t_1',t_2'],[l_1',u_1'],[l_2',u_2']} a$ s.t.

- $t_1 \leq t_1' < t_2' \leq t_2$, and
- $l_1' \leq l_1 + (l_2 - l_1) \cdot \frac{t_1'-t_1}{t_2-t_1}$ and $u_1' \geq u_1 + (u_2 - u_1) \cdot \frac{t_1'-t_1}{t_2-t_1}$, and
- $l_2' \leq l_1 + (l_2 - l_1) \cdot \frac{t_2'-t_1}{t_2-t_1}$ and $u_2' \geq u_1 + (u_2 - u_1) \cdot \frac{t_2'-t_1}{t_2-t_1}$.

Let $S = \mathsf{t} \xhookrightarrow{[t_1,t_2],[l_1,u_1],[l_2,u_2]} a$ and $S' = \mathsf{t} \xhookrightarrow{[t_1,t_2],[l_1',u_1'],[l_2',u_2']} a$ be two TVS over the same time interval. Note that both induce a trapezoid in the graphical representation of the function a, and the horizontal range is $[t_1, t_2]$ for both of these trapezoids. Clearly, if e.g. $l_1 \leq l_1'$ and $l_2 \leq l_2'$, then the second lower bound implies the first, and similarly for the upper bounds. However, if $l_1 < l_1'$ and $l_2 > l_2'$, then there is a unique inner point $t_l \in [t_1, t_2]$ at which the two lower sides of the trapezoids intersect, namely such that $l_1 + (l_2 - l_1) \cdot \frac{t_l-t_1}{t_2-t_1} = l_1' + (l_2' - l_1') \cdot \frac{t_l-t_1}{t_2-t_1}$ and $l_1 + (l_2 - l_1) \cdot \frac{t-t_1}{t_2-t_1} < l_1' + (l_2' - l_1') \cdot \frac{t-t_1}{t_2-t_1}$ for all $t \in [t_1, t_l)$, and $l_1 + (l_2 - l_1) \cdot \frac{t-t_1}{t_2-t_1} > l_1' + (l_2' - l_1') \cdot \frac{t-t_1}{t_2-t_1}$ for all $t \in (t_l, t_2]$. Hence, splitting $[t_1, t_2]$ into $[t_1, t_l]$ and $[t_l, t_2]$ would make it possible to have each lower bound be represented by a straight line in this case, the case where $l_1 > l_1'$ and $l_2 > l_2'$ works similarly. Moreover, this reasoning also applies to the upper bounds.

It should be clear that the bounds imposed by two trapezoids over the same time interval $[t_1, t_2]$ can equally be represented by one (over $[t_1, t_2]$), two (over $[t_1, t_l]$ and $[t_l, t_2]$) or three trapezoids (over $[t_1, t_l]$, $[t_l, t_u]$ and $[t_u, t_2]$, or $[t_1, t_u]$, $[t_u, t_l]$ and $[t_l, t_2]$, for some t_l, t_u determined by the two upper and lower sides of the trapezoids at hand and the order of their intersection points). A technical definition of the TVS $\mathsf{isect}_i([t_1, t_2], [l_1, u_1], [l_2, u_2], [l_1', u_1'], [l_2', u_2'])$

for $i \in \{1, 2, 3\}$ is given in Appendix A. For the following it suffices to know that these terms refer to the first, second, or third (if they exist) trapezoid resulting from the intersection of S and S', as shown in Fig. 2.

Another important reasoning principle is derived from the converse operation of joining two adjacent TVS S and S'. This is possible if the two trapezoids defined by them fit into the trapezoid defined by taking the left vertical interval of S and the right vertical interval of S' at their respective moments in time as new trapezoid-defining edges.

▶ **Definition 6.** Let $S = \mathsf{t} \xrightarrow{t_1,t_2,[l_1,u_1],[l_2,u_2]} a$ and $S' = \mathsf{t} \xrightarrow{[t_2,t_3],l'_1,u'_1],[l'_2,u'_2]} a$ be TVS. Let $l = l_1 + (l'_2 - l_1) \cdot \frac{t_2-t_1}{t'_2-t_1}$ and let $u = u_1 + (u'_2 - u_1) \cdot \frac{t_2-t_1}{t'_2-t_1}$. If $u \geq u_2, u \geq u'_1$ and $l \leq l_2, l \leq l'_1$ then set $\mathsf{join}(t_1, t_2, t_3, [l_1, u_1], [l_2, u_2], [l'_1, u'_1], [l'_2, u'_2]) = \mathsf{t} \xrightarrow{[t_1,t_2],[l_1,u_1],[l'_2,u'_2]} a$, otherwise set $\mathsf{join}(t_1, t_2, t_3, [l_1, u_1], [l_2, u_2], [l'_1, u'_1], [l'_2, u'_2]) = S$ to make it always defined.

Let $S_1 = \mathsf{t} \xrightarrow{[t_1,t_2],[l_1,u_1],[l_2,u_2]} a$ be a TVS and $\mathcal{F} \models S_1$, i.e. $\mathcal{F}_a : \mathsf{t} \to a$ satisfies $\mathcal{F}_a(t_1) \in [l_1, u_1]$ and $\mathcal{F}_a(t_2) \in [l_2, u_2]$. Let furthermore $S_2 = \mathsf{t} \xrightarrow{[t'_1,t_1],[l_3,u_3]} \dot{a}$ and $S_3 = \mathsf{t} \xrightarrow{[t_2,t'_2],[l_4,u_4]} \dot{a}$ be TDS. They imply $\dot{\mathcal{F}}_a(t) \in [l_3, u_3]$ for all $t \in [t'_1, t_1]$ and that $\dot{\mathcal{F}}_a(t) \in [l_4, u_4]$ for all $t \in [t_2, t'_2]$. This entails $l_1 + (t_1 - t) \cdot u_3 \leq \mathcal{F}_a(t) \leq u_1 - (t_1 - t) \cdot l_3$ for all $t \in [t'_1, t_1]$ and $l_2 + (t - t_2) \cdot l_4 \leq \mathcal{F}_a(t) \leq u_2 + (t - t_2) \cdot u_4$ for all $t \in [t_2, t'_2]$.

▶ **Definition 7.** Let S_1, S_2 and S_3 be as above. Define $\mathsf{lderivative}([t_1, t_2], [l_1, u_1], [l_3, l_4])$ as the TVS $\mathsf{t} \xrightarrow{[t'_1,t_1],[l_1+(t_1-t'_1)\cdot u_3,u_1+(t_1-t'_1)\cdot l_3],[l_1,u_1]} a$ and $\mathsf{rderivative}([t_1, t_2], [l_2, u_2], [l_4, u_4])$ as the TVS $\mathsf{t} \xrightarrow{[t_2,t'_2],[l_2,u_2],[l_2+(t'_2-t_2)\cdot l_4,u_2+(t'_2-t_2)\cdot u_4]} a$.

Note the inverted role of u_3 and l_3 in $\mathsf{lderivative}$, which is due to the reasoning from right to left that happens here.

The Calculus. We say that a $\mathcal{V}$-statement S is *provable* in the Calculus of Temporal Influence (CTI) w.r.t. an influence scheme $\mathcal{C}$, written $\mathcal{C} \vdash S$, if there is a finite proof for S in the proof system whose rules are shown in Fig. 3. We say that $\mathcal{C}$ is *consistent* in CTI if there are no statements S, S' derivable from $\mathcal{C}$ s.t. rules (S_{TVS}) or (S_{TDS}), when applied to S, S' as premises, would yield an ill-defined TVS or TDS, i.e. one where $u < l$ in a vertical interval $[l, u]$. We briefly explain each rule and argue why it is sound.

- (F): This rule stipulates that any statement S that is already contained in $\mathcal{C}$ is derivable. This is evidently sound.
- (G_{TVS}) & (G_{TDS}): These rules close gaps in the domain of a function or its derivative by asserting that the function, resp. its derivative be defined in the gap. This is sound due to the stipulation that functions and their derivatives are defined on intervals.
- (W_{TVS}): This weakening rule stipulates the following kind of reasoning. Suppose we know that in the interval I_1, the function a is contained in the trapezoid generated by I_2 and I_3. Then in any subinterval of I_1 it is contained in any trapezoid that is larger in the vertical dimension, using the reasoning outlined before Def. 5. Note that this rule can be used to split a TVS in half on the time axis. The rule is easily seen to be sound already due to geometric reasoning.
- (W_{TDS}): This weakening rule stipulates that if $\dot{a}$ is contained in some rectangle, it must be contained in any rectangle that is smaller on the horizontal axis or larger on the vertical axis. Soundness is also by geometric reasoning.
- (J_{TVS}) & (J_{TDS}): These rules join two TVS or two TDS via the machinery introduced before Def. 6, resp. simple geometric reasoning. Soundness is by invoking weakening first to adjust vertical extents and then by straightforward joining of two trapezoids, resp. two rectangles.

$$(\mathsf{F})\ \frac{}{S}\ \text{if } S \in \mathcal{C} \qquad (\mathsf{G}_{\mathrm{TVS}})\ \frac{\mathsf{t} \xhookrightarrow{[t_1,t_2],I_2,I_3} a \qquad \mathsf{t} \xhookrightarrow{[t'_1,t'_2],I'_2,I'_3} a}{\mathsf{t} \xhookrightarrow{[t_2,t'_1],[-\infty,\infty]} a}\ \text{if } t_2 < t'_1$$

$$(\mathsf{G}_{\mathrm{TDS}})\ \frac{\mathsf{t} \xhookrightarrow{[t_1,t_2],I_2} \dot{a} \qquad \mathsf{t} \xhookrightarrow{[t'_1,t_2],I'_2} \dot{a}}{\mathsf{t} \xhookrightarrow{[t_2,t'_1],[-\infty,\infty]} \dot{a}}\ \text{if } t_2 < t'_1 \qquad (\mathsf{W}_{\mathrm{TVS}})\ \frac{S}{S'}\ \text{if } S' \in \mathsf{relax}(S)$$

$$(\mathsf{W}_{\mathrm{TDS}})\ \frac{\mathsf{t} \xhookrightarrow{I_1,I_2} \dot{a}}{\mathsf{t} \xhookrightarrow{I'_1,I'_2} \dot{a}}\ \text{if } I'_1 \subseteq I_1, I_2 \subseteq I'_2 \qquad (\mathsf{J}_{\mathrm{TVS}})\ \frac{\mathsf{t} \xhookrightarrow{[t_1,t_2],I_2,I_3} a \qquad \mathsf{t} \xhookrightarrow{[t_2,t_3],I'_2,I'_3} a}{\mathsf{join}(t_1,t_2,t_3,I_2,I_3,I'_2,I'_3)}$$

$$(\mathsf{J}_{\mathrm{TDS}})\ \frac{\mathsf{t} \xhookrightarrow{[t_1,t_2],I_2} \dot{a} \qquad \mathsf{t} \xhookrightarrow{[t_2,t_3],I'_2} \dot{a}}{\mathsf{t} \xhookrightarrow{[t_1,t_3],I_2 \cup I_3} \dot{a}} \qquad (\mathsf{VD})\ \frac{\mathsf{t} \xhookrightarrow{I_1,I_2,I_3} a}{\mathsf{t} \xhookrightarrow{I_1,[-\infty,\infty]} \dot{a}}$$

$$(\mathsf{DV})\ \frac{\mathsf{t} \xhookrightarrow{I_1,I_2} \dot{a}}{\mathsf{t} \xhookrightarrow{I_1,[-\infty,\infty]} a} \qquad (\mathsf{S}_{\mathrm{TVS}})\ \frac{\mathsf{t} \xhookrightarrow{I_1,I_2,I_3} a \qquad \mathsf{t} \xhookrightarrow{I_1,I'_2,I'_3} a}{\mathsf{isect}_i(I_1,I_2,I_3,I'_2,I'_3)}\ \text{if } i \in \{1,2,3\}$$

$$(\mathsf{S}_{\mathrm{TDS}})\ \frac{\mathsf{t} \xhookrightarrow{I_1,I_2} \dot{a} \qquad \mathsf{t} \xhookrightarrow{I'_1,I'_2} \dot{a}}{\mathsf{t} \xhookrightarrow{I_1 \cap I'_1, I_2 \cap I'_2} \dot{a}} \qquad (\mathsf{Der})\ \frac{\mathsf{t} \xhookrightarrow{[t_1,t_2],I_1} a \qquad a \xhookrightarrow{I_1,[t'_1,t'_2],I_2} \dot{b}}{\mathsf{t} \xhookrightarrow{[t_1+t'_1,t_2+t'_2],I_2} \dot{b}}$$

$$(\mathsf{CDL})\ \frac{\mathsf{t} \xhookrightarrow{[t_2,t_3],I_1,I_2} a \qquad \mathsf{t} \xhookrightarrow{[t_1,t_2],I_3} \dot{a}}{\mathsf{lderivative}([t_1,t_2],I_1,I_3)} \qquad (\mathsf{CDR})\ \frac{\mathsf{t} \xhookrightarrow{[t_1,t_2],I_1,I_2} a \qquad \mathsf{t} \xhookrightarrow{[t_2,t_3],I_3} \dot{a}}{\mathsf{rderivative}([t_2,t_3],I_2,I_3)}$$

Figure 3 Proof rules for correctness of a statement w.r.t. an influence scheme $\mathcal{C}$.

- (VD) & (DV): These rules assert that a function is defined on some interval if its derivative is defined there, and vice versa. Soundness is by the definition of an influence.
- ($\mathsf{S}_{\mathrm{TVS}}$) & ($\mathsf{S}_{\mathrm{TDS}}$): The former rule can be used to derive up to three distinct TVS from two TVS defined on the same time interval, via the considerations outlined after Def. 5. Soundness follows from the discussion there; the technical definition is in Def. 13 in Appendix A. The latter rule is similar and for TDS instead, and, hence much simpler and readily seen to be sound.
- (Der): This core rule of the calculus can be thought of as a form of modus ponens. The right premise is a VDS, which always has the form of an implication that, if the graph of the function a is contained in the vertical interval $[x, y]$ at some point, then the derivative of the function b is contained in the interval I_2 during some interval dictated by $[t'_1, t'_2]$. The left premise of this rule is the assertion that a has the property demanded in the right premise of the rule, whence the conclusion of the right premise must hold, i.e. the derivative of b has the given properties in some interval. Soundness is due to the semantics of a VDS, applied to the whole interval $[t_1, t_2]$.
- (CDL) & (CDR): These rules combine the information that the graph of a is contained in some trapezoid (left premise) together with assertions on the derivative of a to the left (CDL) or right (CDR), in order to derive new trapezoids left, resp. right of the trapezoid given by the left premise. Soundness is due to the discussion before Def. 7.

Note that the left premise of rule (Der) is a rectangle-shaped TVS. It would be possible to formulate a similar rule more specialised to general trapezoids, but this would make it even more unwieldy, whence we refrain from this.

The following is a consequence of soundness of all the rules. It also entails that an inconsistent influence scheme is not satisfiable.

▶ **Theorem 8** (Soundness). *For any temporal influence scheme $\mathcal{C}$ and and statement S we have: if $\mathcal{C} \vdash S$ then $\mathcal{C} \models S$.*

4 Decidability For Fixed Strata in the Consequence Relation

Let $\mathcal{C}$ be an influence scheme and let S be a statement such that $\mathcal{C} \vdash S$. We say that a proof of S from $\mathcal{C}$ has (Der)-depth k if, on any path from S to an axiom, there are at most k invocations of rule (Der). We say that S has (Der)-*depth* k w.r.t. $\mathcal{C}$ if there is a proof of $\mathcal{C} \models S$ with (Der)-depth at most k. Let $\mathcal{C}$ be consistent. We write $\mathrm{CTI}[k](\mathcal{C})$ for the set of statements S such that $\mathcal{C} \vdash S$ and S has (Der)-depth k w.r.t. $\mathcal{C}$. If $\mathcal{C}$ is clear from context, we simply speak of (Der)-depth. We single out rule (Der) here since it is conceptually the most complicated, in particular for a pupil in secondary education. The stratification induced by the above definition also is very natural, as witnessed by Lemma 11.

Of course, if $\mathcal{C}$ is not consistent, $\mathcal{C}$ is not satisfiable, whence $\mathcal{C} \models S$ trivially, but the proof of any statement witnessing inconsistency, for example by containing empty vertical intervals, might have (Der)-depth k' with k' being much greater than k. This is the reason for the restriction to consistent influence schemes. We now define two notions of normalisation.

▶ **Definition 9.** Let $\mathcal{C}$ be a consistent $\mathcal{V}$-influence scheme, let $a \in \mathcal{V}$, let $k \in \mathbb{N}$ and let $\mathcal{S} = \{S_1, \dots, S_n\}$ be a set of TDS of (Der)-depth k or less, where $S_j = \mathsf{t} \xhookrightarrow{[t_1^j, t_2^j],[l^j,u^j]} \dot{a}$. We call $\mathcal{S}$ k-*normalised* if the following hold:

- $\mathcal{S}$ is *separated*, i.e. for all $j < k$, we have $t_2^j = t_1^{j+1}$.
- $\mathcal{S}$ is *minimal*, i.e. for all TDS $\mathsf{t} \xhookrightarrow{[t_1,t_2],[l,u]} \dot{a}$ of (Der)-depth (w.r.t. $\mathcal{C}$) of k or less, if $t_1 < t_2$ and there are $j \leq j'$ such that $t_1^j \leq t_1 < t_2 \leq t_2^{j'}$, then for all j'' s.t. $j \leq j'' \leq j'$, we have $l^{j''} \geq l$ and $u^{j''} \leq u$.
- $\mathcal{S}$ is *representative*, i.e. for all TDS $S = \mathsf{t} \xhookrightarrow{I_1,I_2} \dot{a}$ of (Der)-depth (w.r.t. $\mathcal{C}$) of k or less, S is derivable from $\mathcal{S}$ via applications of rules ($\mathsf{W}_{\mathrm{TDS}}$) and ($\mathsf{J}_{\mathrm{TDS}}$) alone.

We call a set of TDS k-normalised if it is k-normalised for each individual variable.

The intuition here is that being separated removes temporal overlap between the individual TDS and that the union of their temporal domains forms an interval. Minimality means that $\mathcal{S}$ cannot be strengthened any further without invocations of rule (Der), and being representative means that $\mathcal{S}$ is a complete representation of $\mathrm{CTI}[k](\mathcal{C})$ w.r.t. the derivative of the function a in the sense that any TDS from the latter set can be derived from the former by very simple rules. In fact, derivability of such a TDS S can be decided by visual inspection: we have $\mathcal{S} \vdash S$ iff the corridor defined by $\mathcal{S}$ stretches at least over S's time interval and is entirely surrounded by the rectangle defined by S there.

▶ **Definition 10.** Let $\mathcal{C}$ be a consistent $\mathcal{V}$-influence scheme, let $a \in \mathcal{V}$, let $k \in \mathbb{N}$ and let $\mathcal{S} = \{S_1, \dots, S_n\}$ be a set of TVS of (Der)-depth k or less, where $S_j = \mathsf{t} \xhookrightarrow{[t_1^j, t_2^j],[l^j,u^j],[l'^j,u'^j]} a$. We call $\mathcal{S}$ k-*pre-normalised* if the following hold:

- $\mathcal{S}$ is *separated*, i.e. for all $j < k$, we have $t_2^j = t_1^{j+1}$.
- $\mathcal{S}$ is *minimal*, i.e. for all TVS $\mathsf{t} \xhookrightarrow{[t_1,t_2],[l',u'],[l',u']} a$ of (Der)-depth (w.r.t. $\mathcal{C}$) of k or less, if $t_1 < t_2$ and there are $j \leq j'$ such that $t_1^j \leq t_1 < t_2 \leq t_2^{j'}$, then for all j'' s.t. $j \leq j'' \leq j'$, and for all $t \in [t_1^{j''}, t_2^{j''}]$ we have that

$$l + (l - l') \cdot t' \;\leq\; l^{j''} + (l'^{j''} - l^{j''}) \cdot t'' \;\leq\; u^{j''} + (u'^{j''} - u^{j''}) \cdot t'' \;\leq\; u + (u - u') \cdot t'$$

where $t' = \frac{t-t_1}{t_2-t_1}$ and $t'' = \frac{t-t_1^{j''}}{t_2^{j''}-t_1^{j''}}$.

- $\mathcal{S}$ is *representative*, i.e. for all TVS $S = \mathsf{t} \xhookrightarrow{I_1,I_2,I_3} a$ of (Der)-depth (w.r.t. $\mathcal{C}$) of k or less, S is derivable from $\mathcal{S}$ via applications of rules ($\mathsf{W}_{\mathrm{TVS}}$) and ($\mathsf{J}_{\mathrm{TVS}}$) alone.

Moreover, we call $\mathcal{S}$ *k-normalised* if it is k-pre-normalised and the following holds:

- $\mathcal{S}$ is *derivative-reduced*, i.e. for all TDS $\mathsf{t} \xhookrightarrow{I,[l,u]} \dot{a}$ of (Der)-depth (w.r.t. $\mathcal{C}$) of k or less, and for all $j \leq k$, the following holds: if $I \cap [t_1^j, t_2^j] \neq \emptyset$, then (I) if $l^j \neq -\infty$ and $l'^j \neq -\infty$, then $l \leq \frac{l'^j - l^j}{t_2^j - t_1^j} \leq u$, and (II) if $u^j \neq \infty$ and $u'^j \neq \infty$, then $l \leq \frac{u'^j - u^j}{t_2^j - t_1^j} \leq u$.
- $\mathcal{S}$ is (Der)-*ready*, i.e. if S can be obtained from some S_j via ($\mathsf{W}_{\mathrm{TVS}}$), and there is a VDS S' such that S and S' are possible premises for an application of rule (Der), then S and S_j agree on their time interval.

We call a set of TVS k-normalised if it is k-normalised w.r.t. each variable.

The intuition for the first three items is the same as in Def. 9, but the formulation of minimality is more complicated due to the more complex geometry involved. Minimality in geometric terms requires that any of the S_j that overlaps with any TVS S of (Der)-depth k or less must be contained in the trapezoid defined by S for the time interval of the overlap. Hence, S_j (or any TVS derived by splitting it) could not be strengthened via ($\mathsf{S}_{\mathrm{TVS}}$) using S.

The intuition for being derivative-reduced is the following: when we know that the graph of function a must pass through a given trapezoid that is bounded from e.g. above, and we know that the derivative of that function is bounded from above by u and from below by l, then unless the slope of the upper edge of the trapezoid is between l and u, there will be parts of the trapezoid that a cannot pass through without violating the bounds l and u on its derivative, and this fact can be derived via rules (CDL) or (CDR) after potentially splitting the TDS in question via ($\mathsf{W}_{\mathrm{TDS}}$), followed by ($\mathsf{S}_{\mathrm{TVS}}$). Similar reasoning applies for lower bounds or if the derivative is only properly bounded from one side. Note that for two neighbouring TVS such that the derivative in question is bounded from both above and below on the point in time of their intersection, being derivative-reduced entails that the upper and lower bounds of the TVS match due to continuity.

Finally, the point of being (Der)-ready is to ensure that a TVS can serve as the left premise of a given VDS either in its horizontal extent, or not at all.

▶ **Lemma 11.** *Let $\mathcal{C}$ be a consistent $\mathcal{V}$-influence scheme, let $a \in \mathcal{V}$, let $k \in \mathbb{N}$. Let $\mathcal{S}$ be a finite set of TVS and let $\mathcal{S}'$ be a finite set of TDS s.t. all TVS and TDS of* (Der)*-depth (w.r.t. $\mathcal{C}$) k or less can be derived from $\mathcal{S} \cup \mathcal{S}'$ without using rule* (Der)*. Then there are $\mathcal{S}_{\mathsf{norm}}$ and $\mathcal{S}'_{\mathsf{norm}}$ that both are k-normalised (and therefore representative, ensuring equivalence to $\mathcal{S}$ and $\mathcal{S}'$ w.r.t. $\vdash$). Moreover, $\mathcal{S}_{\mathsf{norm}}$ and $\mathcal{S}'_{\mathsf{norm}}$ can be computed from $\mathcal{S}$ and $\mathcal{S}'$ in polynomial time.*

The proof has been moved to Appendix B for space considerations. This lemma yields a polynomial-time decision procedure for CTI$[k](\mathcal{C})$:

▶ **Theorem 12.** *Let k be fixed. Let $\mathcal{C}$ be a consistent $\mathcal{V}$-influence scheme and let S be $\mathcal{V}$-statement. It is decidable in polynomial time whether $S \in$ CTI$[k](\mathcal{C})$.*

Proof. If S is a VDS then there is nothing to prove since no proof rule has a VDS as its conclusion. So assume that S is a TVS or a TDS. $\mathcal{C}$ trivially contains subsets $\mathcal{S}, \mathcal{S}'$ of TVS, resp. TDS that satisfy the premises of Lemma 11 for $k = 0$. Let $\mathcal{S}^0_{\mathsf{norm}}, \mathcal{S}'^0_{\mathsf{norm}}$ be the 0-normalised, polynomially-sized sets from said lemma. If $k = 0$ we are done.

Now assume that we have obtained j-normalised polynomially-sized sets $\mathcal{S}^j_{\mathsf{norm}}$ and $\mathcal{S}'^j_{\mathsf{norm}}$ for $j \geq 0$. We obtain sets $\mathcal{S}^{j+1} \supseteq \mathcal{S}^j_{\mathsf{norm}}$ and $\mathcal{S}'^{j+1} \supseteq \mathcal{S}^j_{\mathsf{norm}}$ by extending $\mathcal{S}^j_{\mathsf{norm}}$ and $\mathcal{S}'^j_{\mathsf{norm}}$ the following way: (I) First we apply rule ($\mathsf{W}_{\mathrm{TVS}}$) to sets in $\mathcal{S}^j_{\mathsf{norm}}$ to obtain premises for all instances of rule (Der) that can be obtained this way. Since $\mathcal{S}^j$ is (Der)-ready, this is at most one instance per pair of TVS in $\mathcal{S}^j$ and VDS in $\mathcal{C}$, since only vertical weakening is necessary. (II) Then we obtain all possible TDS that can be derived via rule (Der) from these pairs.

By the above, the sets $\mathcal{S}^{j+1}$ and $\mathcal{S}'^{j+1}$ are of polynomial size. Moreover, they satisfy the conditions of Lemma 11 for $k = j+1$: let S' be a TVS or a TDS in $\mathrm{CTI}[j+1](\mathcal{C})$. If also $S' \in \mathrm{CTI}[j](\mathcal{C})$, then due to j-normalisation of $\mathcal{S}^j_{\mathsf{norm}}$ and $\mathcal{S}'^j_{\mathsf{norm}}$, we are done. Otherwise, in any proof of S' from $\mathcal{C}$ that witnesses $S' \in \mathrm{CTI}[j+1](\mathcal{C})$, there are finitely many top-level applications of rule (Der), i.e. those such that on the path from the root S' to the application of (Der) in the proof tree, there is no second application of (Der). Let T be a conclusion of such a rule application, let T_l and T_r be its left- and right-hand premises. If we can show that T is derivable from $\mathcal{S}^{j+1} \cup \mathcal{S}'^{j+1}$, we are done since T is arbitrary.

By definition, $T_l \in \mathrm{CTI}[j](\mathcal{C})$ and, hence, T_l can be derived from $\mathcal{S}^j_{\mathsf{norm}}$ via applications of ($\mathsf{J}_{\mathrm{TVS}}$) and ($\mathsf{W}_{\mathrm{TVS}}$). W.l.o.g. the applications of ($\mathsf{J}_{\mathrm{TVS}}$) happen last in the proof tree of T_l, i.e. there are $T_1, \dots, T_m$ s.t. T_i can be obtained from $\mathcal{S}^j_{\mathsf{norm}}$ via ($\mathsf{W}_{\mathrm{TVS}}$) for $1 \le i \le m$. By (Der)-readiness of $\mathcal{S}^j_{\mathsf{norm}}$, each of the T_i is a valid premise for rule (Der) together with T_r, and the conclusions $T'_1, \dots, T'_m$ are all in $\mathcal{S}'j+1$. It is easily verified that T can be obtained from $T'_1, \dots, T'_m$ via rules ($\mathsf{W}_{\mathrm{TDS}}$) and ($\mathsf{J}_{\mathrm{TDS}}$). Since T was arbitrary, S' is provable from $\mathcal{S}^{j+1} \cup \mathcal{S}'^{j+1}$ without using rule (Der).

Using Lemma 11 on $\mathcal{S}^{j+1}$ and $\mathcal{S}'^{j+1}$ yields, in polynomial time, j+1-normalised $\mathcal{S}^{j+1}_{\mathsf{norm}}$ and $\mathcal{S}'^{j+1}_{\mathsf{norm}}$. Continuing this sequence of mass-applications of rule (Der) and re-normalisations yields, in polynomial time, k-normalised $\mathcal{S}^k_{\mathsf{norm}}$ and $\mathcal{S}'^k_{\mathsf{norm}}$ from which it follows directly whether $S \in \mathrm{CTI}[k](\mathcal{C})$. ◀

As a corollary, we obtain the following: given a consistent influence scheme $\mathcal{C}$ and a statement S, it is semi-decidable whether $\mathcal{C} \vdash S$. If $\mathcal{C} \vdash S$, then there is a proof of S from $\mathcal{C}$, and it has (Der)-depth k for some k. Hence, by checking consecutively whether $S \in \mathrm{CTI}[k'](\mathcal{C})$ for $k' = 0, \dots$ will yield a positive result when reaching $k' = k$ at the latest. In fact, this procedure is polynomial for k given in unary.

5 A Prototypical Implementation

We have implemented the proof search algorithm given in Thm. 12 in Python.[3] The program accepts statements and hypotheses as lists of tuples, or from a CSV file.

The solver module can run in different modes: it can either generate all derivable statements up to a certain (Der)-depth, until it has covered a certain point in time with statements, or until it runs out of new statements to derive, which, in practice, happens quite frequently due to compounding imprecision growing with the distance from the initial values. This is an effect that is known from numerical methods for solving differential equations, cf. [12, Chp. 5]. It remains to be seen whether such effects are acceptable for the foreseen application in a digital learning tool, and whether mathematical methods can be employed to reduce such effects.

The implementation also comes with a plotter module to visualise the statements that were derived. Full implementation into a classroom application is still to be done. Finally, the implementation comes with some pre-implemented example problems, including the photosynthesis problem from Ex. 3.

[3] Available at `https://github.com/SoerenMoeller/timed_influence_solver`.

Preliminary results show that the implementation is fast enough to be able to solve didactically meaningful problems in seconds even on mobile devies, which matches similar results for the Calculus of (Non-Temporal) Influence [4].

The implementation differs in some details from the variant presented in the proof. For example, the implementation keeps sets of TVS and TDS associated to a variable normalised, i.e. after each additional statement that is derived w.r.t. a given variable, its entire set of associated statements is re-normalised immediately, instead of doing this in bulk after exhausting certain derivation rules. The reason for this is that keeping a representation of derived statements normalised is advantageous form a computational point of view, but the structure of the proof above becomes simpler if normalisation is done in bulk. It is not hard to see that both approaches yield the same results.

6 Conclusion

This work introduced the Calculus of Temporal Influence with the aim to extend previously started work on the formalisation of experiments and phenomena in nature, specifically in natural science classes. The focus here is on allowing processes to be modelled that are largely driven by time.

There are various other proposals of formalisms that also provide means to model dynamic systems evolving in time, like timed automata [2], timed Petri nets [13], hybrid automata [7], etc. In fact, modelling biological, chemical or physics phenomena by means of discrete and/or continuous mathematical formalisms is an active field of research, cf. [15, 1, 10, 14, 5, 3]. The need to develop a new formalism is driven by its application in an environment that is largely determined by didactical considerations. For example, hybrid automata allow for very precise modelling which is needed in verification; the price to pay is undecidability, high complexity and potentially only approximative analysis. For the hypotheses that are typically created by secondary-education pupils, precision is much less relevant, and so the Calculi of (Temporal and Non-Temporal) Influence are designed to allow reasoning about imprecise models.

It is important to note that the Calculus of Temporal Influence is a suggestion; it will have to prove worthy under exactly those circumstances. This will require much further work, starting with a clear categorisation of natural science experiments to which a hierarchy of modelling formalisms w.r.t. expressiveness can be aligned. An important point here is that it is not necessarily desirable to be able to model every such phenomenon, since more complicated argument structures may not be suitable for secondary-education level. In particular for lower grades, explainability of the results and ease of use of the learning framework are considerably more important.

However, being able to model more phenomena is certainly interesting alongside a different axis, i.e. the scientific one. Something that immediately comes to mind is the ability to make statements about compound functions, e.g. on the sum of the values of two variables a and b. Apart from this and the obvious question of whether (polynomial) decidability could be obtained for $\vdash$, perhaps based on some pumping argument, there are also further aspects of future work, perhaps of more technical nature, that will have to be considered like completeness of the calculus: does $\mathcal{C} \models H$ imply $\mathcal{C} \vdash H$? In the non-temporal case, completeness does not hold, but it is possible to retain it for a large class of experiment models [4]. We suspect the obstacles incurring in the temporal version to be even bigger.

We are also planning integration of our implementation in the wider context of a digital tool that supports learning and teaching in experimental science classes. Beyond the obvious problems of providing students with an answer whether their hypothesis was correct, based on the Calculis of Temporal Influence, this includes also tasks such as visualisation of the data in question, automatic translations from textual statements into formal ones, and teacher support.

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A The Technical Definition of Intersecting Two Trapezoids

▶ **Definition 13.** Let $S = \mathsf{t} \xhookrightarrow{[t_1,t_2],[l_1,u_1],[l_2,u_2]} a$ and $S' = \mathsf{t} \xhookrightarrow{[t_1,t_2],[l'_1,u'_1],[l'_2,u'_2]} a$ be two TVS.

- If $l_1 < l'_1$ and $l_2 > l'_2$, or if $l_1 > l'_1$ and $l_2 < l'_2$, let t_l be the unique point in (t_1, t_2) such that $\mathsf{lo}_l = l_1 + (l_2 - l_1) \cdot \frac{t_l - t_1}{t_2 - t_1} = l'_1 + (l'_2 - l'_1) \cdot \frac{t_l - t_1}{t_2 - t_1}$. Let $\mathsf{hi}_l = \min(u_1 + (u_2 - u_1) \cdot \frac{t_l - t_1}{t_2 - t_1}),\ u'_1 + (u'_2 - u'_1) \cdot \frac{t_1 - t_2}{t_2 - t_1})$.
- If $u_1 < u'_1$ and $u_2 > u'_2$, or if $u_1 > u'_1$ and $u_2 < u'_2$, let t_u be the unique point in (t_1, t_2) such that $\mathsf{hi}_u = u_1 + (u_2 - u_1) \cdot \frac{t_u - t_1}{t_2 - t_1}) = u'_1 + (u'_2 - u'_1) \cdot \frac{t_u - t_2}{t_2 - t_1}$. Let $\mathsf{lo}_u = \max(l_1 + (l_2 - l_1) \cdot \frac{t_u - t_1}{t_2 - t_1},\ l'_1 + (l'_2 - l'_1) \cdot \frac{t_u - t_1}{t_2 - t_1})$.

We now define three TVS $S_i = \mathsf{isect}_i([t_1, t_2], [l_1, u_1], [l_2, u_2], [l'_1, u'_1], [l'_2, u'_2])$ for $i \in \{1, 2, 3\}$, paremeterized in the parameters of S, S' via:

- If both (I) $l_1 \geq l'_1$ and $l_2 \geq l'_2$ or $l_1 \leq l'_1$ and $l_2 \leq l'_2$, and (II) $u_1 \geq u'_1$ and $u_2 \geq u'_2$, or $u_1 \leq u'_1$ and $u_2 \leq u'_2$ hold, then $S_1 = \mathsf{t} \xhookrightarrow{[t_1,t_2],[\max(l_1,l'_1),\min(u_1,u'_1)],[\max(l_2,l'_2),\min(u_2,u'_2)]} a$ and $S_2 = S_3 = S_1$.
- If both (I) $l_1 \geq l'_1$ and $l_2 \geq l'_2$ or $l_1 \leq l'_1$ and $l_2 \leq l'_2$, and (II) $u_1 > u'_1$ and $u_2 < u'_2$, or $u_1 < u'_1$ and $u_2 > u'_2$ hold, then $S_1 = \mathsf{t} \xhookrightarrow{[t_1,t_u],[\max(l_1,l'_1),\min(u_1,u'_1)],[\mathsf{lo}_u,\mathsf{hi}_u)]} a$ and $S_2 = \mathsf{t} \xhookrightarrow{[t_u,t_2],[\mathsf{lo}_u,\mathsf{hi}_u)],[\max(l_2,l'_2),\min(u_2,u'_2)]} a$, and $S_3 = S_2$.
- If both (I) $l_1 < l'_1$ and $l_2 > l'_2$ or $l_1 > l'_1$ and $l_2 < l'_2$, and (II) $u_1 \leq u'_1$ and $u_2 \leq u'_2$, or $u_1 \geq u'_1$ and $u_2 \geq u'_2$ hold, then $S_1 = \mathsf{t} \xhookrightarrow{[t_1,t_l],[\max(l_1,l'_1),\min(u_1,u'_1)],[\mathsf{lo}_l,\mathsf{hi}_l)]} a$ and $S_2 = \mathsf{t} \xhookrightarrow{[t_l,t_2],[\mathsf{lo}_l,\mathsf{hi}_l)],[\max(l_2,l'_2),\min(u_2,u'_2)]} a$, and $S_3 = S_2$.
- If both (I) $l_1 < l'_1$ and $l_2 > l'_2$ or $l_1 > l'_1$ and $l_2 < l'_2$, and (II) $u_1 < u'_1$ and $u_2 > u'_2$, or $u_1 < u'_1$ and $u_2 > u'_2$ hold, then both t_l and t_u are defined. If $t_l < t_u$, then $S_1 = \mathsf{t} \xhookrightarrow{[t_1,t_l],[\max(l_1,l'_1),\min(u_1,u'_1)],[\mathsf{lo}_l,\mathsf{hi}_l)]} a$ and $S_2 = \mathsf{t} \xhookrightarrow{[t_l,t_u],[\mathsf{lo}_l,\mathsf{hi}_l)],[\mathsf{lo}_u,\mathsf{hi}_u]} a$, and $S_3 = \mathsf{t} \xhookrightarrow{[t_u,t_2],[\mathsf{lo}_u,\mathsf{hi}_u)],[\max(l_2,l'_2),\min(u_2,u'_2)]} a$. The case for $t_u < t_l$ is defined analogously. If $t_l = t_u$, then $S_1 = \mathsf{t} \xhookrightarrow{[t_1,t_l],[\max(l_1,l'_1),\min(u_1,u'_1)],[\mathsf{lo}_l,\mathsf{hi}_l)]} a$ and $S_2 = \mathsf{t} \xhookrightarrow{[t_l,t_2],[\mathsf{lo}_l,\mathsf{hi}_l)],[\max(l_2,l'_2),\min(u_2,u'_2)]} a$ and $S_3 = S_2$.

By setting $S_3 = S_2$ or $S_1 = S_2 = S_3$, we make sure that, both for $i = 2$ and $i =$, the TVS $\mathsf{isect}_i([t_1, t_2], [l_1, u_1], [l_2, u_2], [l'_1, u'_1], [l'_2, u'_2])$ are defined, even if the intersection produces less than three distinct TVS.

B Proof of Lemma 11

Before we begin with the proof of Lemma 11, we study the interplay between TVS and TDS on adjacent time intervals. Let $S_1 = \mathsf{t} \xhookrightarrow{[t_1,t_2],[l_1,u_1],[l'_1,u'_1]} a$ and $S_2 = \mathsf{t} \xhookrightarrow{[t_2,t_3],[l_2,u_2],[l'_2,u'_2]} a$ be two adjacent TVS and let $S_3 = \mathsf{t} \xhookrightarrow{[t_1,t_2],[l_3,u_3]} \dot{a}$ and $S_4 = \mathsf{t} \xhookrightarrow{[t_2,t_3],[l_4,u_4]} \dot{a}$ be two adjacent TDS, all over the same variable a and with pairwise matching time intervals. This is a situation that appears in sets of separated TVS and TDS. The synchronisation of the time intervals can be achieved by using rules ($\mathsf{W}_{\mathrm{TVS}}$) and ($\mathsf{W}_{\mathrm{TDS}}$).

We call an individual TVS *derivative-reduced*, if it satisfies the conditions laid out in Def. 10 for derivative-reducedness of an entire set of TVS. Obviously, a set of TVS is derivative-reduced if all the TVS in it are so. By the above, we can assume that, for separated sets of TVS and separated and minimal TDS, the individual points where time intervals touch each other are the same. This can be achieved by splitting a TVS if two TDS for the same variable touch in an interior point of its time interval, and vice versa. Clearly, this produces polynomial blowup at most. Hence, for a given TVS, there is a unique strictest TDS in question that dictates whether the TVS is derivative-reduced. For our considerations, we assume that this is S_3 for S_1 and S_4 for S_2.

We can now make S_1 derivative-reduced by using rule $(\mathsf{S}_{\mathrm{TVS}})$ to obtain TVS for the unit intervals $[t_1, t_1]$ and $[t_2, t_2]$ and using them as left premises for rules (CDR), resp. (CDL). The former of these rules produces an upper bound on the slope of the upper edge of the trapezoid generated by S_1, and a lower bound on the slope of the lower edge of said trapezoid. The latter rule produces a lower bound on the slope of the upper edge, and an upper bound on the slope of the lower edge. Notably, the actual slopes of the edges of the trapezoid defined by S_1 can violate at most one of these bounds per slope. It follows that the intersection of the three TVS in question, i.e. S_1, the trapezoid obtained by using rule (CDR), and the one obtained by using rule (CDL), do not produce intersecting upper and lower edges unless there is inconsistency: The trapezoid obtained by using the former rule shares the right edge with the one defined by S_1, and the trapezoid obtained by using the latter rule shares a left edge with the one defined by S_1. Hence, neither of these produces a nontrivial intersection with S_1, and they cannot intersect with each other inside the trapezoid defined by S_1 for simple geometric reasons unless S_1 and S_3 together are inconsistent. Hence, if we assume consistency, combining these three trapzoids using rule $(\mathsf{S}_{\mathrm{TVS}})$ produces a single new trapezoid $S'_1 = \mathsf{t} \xrightarrow{[t_1,t_2],[l_1^d,u_1^d],[l_1'^d,u_1'^d]} a$. Moreover, S'_1 shares at least two of l_1, u_1, l'_1, u'_1, since S_1 violates at most one bound on the slope of its upper edge, and at most one on the slope of its lower edge. We write $\mathsf{reduce}(S_1, S_3)$ for the TVS S'_1 obtained this way. We say that a TVS S is reduced w.r.t. a TDS S' if $\mathsf{reduce}(S, S') = S$, i.e. if the trapezoid defined by it already satisfies the conditions on its slopes induced by S'. As outlined above, in a separated, minimal setting, w.l.o.g., for each TVS S there is a unique TDS S' such that S is derivative-reduced if it is derivative-reduced w.r.t. S'. The above also works if S_3 contains infinite upper and lower bounds; in this case there are simply less restrictions.

Now image that we obtain $S'_1 = \mathsf{t} \xrightarrow{[t_1,t_2],[l_1^d,u_1^d],[l_2^d,u_2^d]} a = \mathsf{reduce}(S_1, S_3)$ and $S'_2 = \mathsf{t} \xrightarrow{[t_2,t_3],[l_1'^d,u_1'^d],[l_2'^d,u_2'^d]} a = \mathsf{reduce}(S_2, S_4)$. If $[l_2^d, u_2^d] = [l_1'^d, u_1'^d]$ then we are done. Otherwise we obtain the TVS $\mathsf{t} \xrightarrow{[t_2,t_2],[\max(l_2^d,l_1'^d),\min(u_2^d,u_1'^d)]} a$, and we can apply rules (CDL) and (CDR) together with S_2 resp. S_4 to further restrict the function $\mathcal{F}_a$ on the intervals $[t_1, t_2]$ and $[t_2, t_3]$. Hence, making a single TVS derivative-reduced is rather straightforward, but already with two, it is not clear that the process of making both of them derivative-reduced at the same time ever halts. The following is a crucial observation for the process of making a set of TVS derivative-reduced.

▶ **Observation 14.** *Let $S' = \mathsf{t} \xrightarrow{[t_1,t_2],[l_1,u_1],[l_2,u_2]} a$ be the result of $\mathsf{reduce}(S, S'')$ for some TDS S''. Let $[l'_1, u'_1] \subseteq [l_1, u_1]$ and $[l'_2, u'_2] \subseteq [l_2, u_2]$ and let $R_1 = \mathsf{t} \xrightarrow{[t_1,t_1],[l'_1,u'_1]} a$ and $R_2 = \mathsf{t} \xrightarrow{[t_2,t_2],[l'_2,u'_2]} a$ be two TVS over unit intervals. Then there are unique, maximal intervals $[l''_1, u''_1] \subseteq [l_1, u_1]$ and $[l''_2, u''_2] \subseteq [l_2, u_2]$ s.t. $S'_l = \mathsf{t} \xrightarrow{[t_1,t_2],[l'_1,u'_1],[l''_2,u''_2]} a$ and $S'_r = \mathsf{t} \xrightarrow{[t_1,t_2],[l''_1,u''_1],[l'_2,u'_2]} a$ are derivative-reduced w.r.t. S''. Moreover, they can be obtained using rules* (CDR)*, resp. rule* (CDL) *with the R_1 resp. R_2 as premises and then using* (S_{TVS})*.*

The reason for this is that S is already derivative-reduced w.r.t. S'', i.e. the slopes of the upper and lower edges of the trapezoid defined by it are already within the bounds given by S''. Hence, making one of the vertical intervals smaller will only result in the other interval shrinking when making the TVS derivative-reduced again. We write $\mathsf{reduce}(S', S'', [l'_1, u'_1])$ or $\mathsf{reduce}(S', S'', [l'_2, u'_2])$ for the TVS S'_l and S'_r obtained this way, tacitly assuming that it is clear on which side the stricter vertical interval goes. Note that all of this also works if one of the TVS has infinite upper or lower bounds.

These re-reduced TVS are important since they give us an easy termination argument for the process of making a separated sequence of TVS derivative-reduced: Use the two-argument version of reduce to obtain a derivative-reduced version of the leftmost TVS. Let $[l, u]$ be it's right vertical interval, and let $[l', u']$ be the left interval of the second TVS from the left.

Then use the three-argument version together with $[l,u] \cap [l',u']$ to make the second TVS from the left derivative-reduced. This might yield a new left interval $[l'',u'']$ for said second TVS, so we use the three-argument version of reduce on the first TVS to adjust it to the new interval bounds. Crucially, this will not change the right vertical interval of the first TVS, whence the process terminates.

The general procedure of making an entire separated sequence of TVS derivative-reduced works as follows: assume that the first k TVS are already derivative-reduced. Let $[l,u]$ be the right vertical interval of the kth such TVS, and let $[l',u']$ be the left vertical interval of it's right neighbour, i.e. the $k{+}1$st TVS. Use the three-argument version of reduce on this from the left, together with the interval $[l,u] \cap [l',u']$. This makes the $k{+}1$st TVS derivative-reduced, and potentially introduces an even stricter interval $[l'',u'']$ for the vertical interval between the kth and $k{+}1$st TVS. Use the three-argument version of reduce to adjust the kth TVS. This potentially yields a new interval bound between the kth and $k{-}1$st TVS, so use the three-argument version of reduce to adjust the $k{-}1$st TVS as well. Since the new, stricter intervals only propagate to the left, this process terminates when the first TVS is re-reduced, since it does not have a left neighbour. Now the first $k+1$ TVS are derivative-reduced. Clearly this process works in polynomial time, since it propagates to the left at most once per TVS, which, in turn, only happens once for each of them.[4]

It follows that a sequence of separated TVS can be made derivative-reduced in polynomial time.

We are now ready to prove the following.

▶ **Lemma 11.** *Let $\mathcal{C}$ be a consistent $\mathcal{V}$-influence scheme, let $a \in \mathcal{V}$, let $k \in \mathbb{N}$. Let $\mathcal{S}$ be a finite set of TVS and let $\mathcal{S}'$ be a finite set of TDS s.t. all TVS and TDS of* (Der)*-depth (w.r.t. $\mathcal{C}$) k or less can be derived from $\mathcal{S} \cup \mathcal{S}'$ without using rule* (Der)*. Then there are $\mathcal{S}_{\mathsf{norm}}$ and $\mathcal{S}'_{\mathsf{norm}}$ that both are k-normalised (and therefore representative, ensuring equivalence to $\mathcal{S}$ and $\mathcal{S}'$ w.r.t. $\vdash$). Moreover, $\mathcal{S}_{\mathsf{norm}}$ and $\mathcal{S}'_{\mathsf{norm}}$ can be computed from $\mathcal{S}$ and $\mathcal{S}'$ in polynomial time.*

Proof. We begin by transforming $\mathcal{S}'$ into a set $\mathcal{S}'_c$ of TDS that is almost k-normalised for every variable. Note that the only rules that have a TDS as their conclusion are rules (Der), ($\mathsf{G}_{\mathrm{TDS}}$), ($\mathsf{W}_{\mathrm{TDS}}$), ($\mathsf{S}_{\mathrm{TDS}}$) and (VD). Obviously, rule (Der) plays no role here.

As a first step, we generate a candidate set $\mathcal{S}'_c$ from $\mathcal{S}'$ by making it separated for every variable. This is done by removing non-unit overlap between TDS with different time intervals using rule ($\mathsf{W}_{\mathrm{TDS}}$). We use this rule to split a TDS that overlaps with another w.r.t. their time intervals such that any two TDS in the set either have the same time interval, or time intervals that overlap in at most one point. Since each TDS overlaps at most with all the others in $\mathcal{S}'$, this produces at most polynomial blowup in the number of TDS.

We then use rule ($\mathsf{S}_{\mathrm{TDS}}$) if several TDS still overlap at non-unit intervals. By assumption, any such overlapping TDS have the same time interval, so it is easy to see that this produces at most one TDS per time interval with nontrivial overlap. We use rule ($\mathsf{G}_{\mathrm{TDS}}$) to close any uncovered gaps in the horizontal dimension. Hence, $\mathcal{S}'_c$ is now separated. It is also minimal: Any witness to the contrary must be derivable from $\mathcal{S}' \cup \mathcal{S}$ by assumption. Rules ($\mathsf{G}_{\mathrm{TDS}}$) and (VD) certainly cannot help to produce such a counterexample, as they produce TDS with infinite lower and upper bounds. Rule ($\mathsf{W}_{\mathrm{TDS}}$) can also be omitted since its premise would already be a counterexample. However, we just used rule ($\mathsf{S}_{\mathrm{TDS}}$) to its maximum extent, so $\mathcal{S}'_c$ is minimal for each variable. It is not yet representative though, since there might

[4] In fact, the process can be optimised to only propagate to the left once. We simply state this here without an argument.

be a TDS derivable using rule (VD), but the premise of that rule is not in $\mathcal{S}$. However, by the reasoning above it is almost representative in the sense that these are the only TDS of (Der)-depth k or less not yet derivable from $\mathcal{S}'_c$.

Hence, we now transform $\mathcal{S}$ into a candidate set $\mathcal{S}_c$ that is almost k-pre-normalised for each variable. Again, we make $\mathcal{S}_c$ separated by using rule ($\mathsf{W}_{\mathrm{TVS}}$) to reduce overlaps to statements with either unit intervals, or the same time interval, and then ($\mathsf{S}_{\mathrm{TVS}}$) to reduce overlap to unit intervals only. Note that rule ($\mathsf{S}_{\mathrm{TVS}}$) may produce shorter time intervals due to the mechanics of intersecting trapezoids, however these are still only polynomially many, since each overlap produces at most three trapezoids. Using rules ($\mathsf{G}_{\mathrm{TVS}}$) and ($\mathsf{VD}$), we make $\mathcal{S}_c$ separated for each variable. We then synchronise the time intervals for the TVS in $\mathcal{S}_c$ and TDS $\mathcal{S}'_c$ by splitting them, if necessary.

Using the procedure outlined after Obs. 14, we make $\mathcal{S}_c$ derivative-reduced. Since $\mathcal{S}'_c$ is minimal except for TDS with infinite upper and lower vertical bounds, the TDS in $\mathcal{S}'_c$ do contain the strictest bounds on the respective derivative functions, so the result of the procedure is really derivative-reduced. Note that the procedure may have produced gaps in the sequence of TVS, if the time intervals in some of the TDS are large. We use rule ($\mathsf{G}_{\mathrm{TVS}}$) to close these gaps, and rule (VD) to transport new information on the domains of the functions to the TDS. It is not hard to verify that both $\mathcal{S}_c$ and $\mathcal{S}'_c$ are now separated, minimal, and representative. Moreover, $\mathcal{S}_c$ is derivative-reduced. We make it (Der)-ready by splitting intervals, if necessary, for, again, at most polynomial blowup.

Hence, the sets $\mathcal{S}_c$ and $\mathcal{S}'_c$ are now both k-normalised, and are our desired sets $\mathcal{S}_{\mathsf{norm}}$ and $\mathcal{S}'_{\mathsf{norm}}$. ◀