

Report from Dagstuhl Seminar 23121

Pattern Avoidance, Statistical Mechanics and Computational Complexity

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Abstract

This report documents the program and the outcomes of Dagstuhl Seminar 23121 “Pattern Avoidance, Statistical Mechanics and Computational Complexity”.

Seminar March 19–24, 2023 – <https://www.dagstuhl.de/23121>

2012 ACM Subject Classification Information systems → Data structures; Theory of computation → Formal languages and automata theory; Mathematics of computing → Combinatoric problems; Theory of computation → Algebraic complexity theory; Computing methodologies → Modeling and simulation

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1 Executive Summary

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The Dagstuhl Seminar took place from March 19, 2023 to March 24, 2023. It had 36 participants, who were researchers in theoretical computer science, combinatorics, and statistical mechanics. The aftermath of COVID made the planning of the seminar more challenging than usual; for instance, researchers from China, Australia and New Zealand were still extremely reluctant to travel. However, in the end we succeeded in bringing together a geographically diverse group of researchers. The participants came from 12 countries, from Canada, the Czech Republic, Denmark, Finland, France, Germany, Hungary, Iceland, Italy, Turkey, the United Kingdom, and the United States. The seminar featured 21 talks, four of which were hour-long talks, and two open problem sessions.

Several collaborative projects have been started. For example, David Bevan started a collaboration with Mathilde Bouvel on the scaling limit of permutation classes avoiding a pattern with extremal first or last point. Jessica Striker, Mathilde Bouvel and Rebecca Smith started working on the open questions in Striker’s talk on six-vertex configurations. Colin Defant, Rebecca Smith, Miklós Bóna and Justin Troyka discussed new observations on pattern avoiding permutations whose squares are also pattern avoiding. Jessica Striker and

* Editor / Organizer



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Sergi Elizalde started working together on promotion in cylindric tableaux. Natasha Blitvić and Sergi Elizalde explored why counting permutations avoiding a pattern seems to yield moment sequences, and how to interpret that phenomenon.

The open problem sessions were especially successful. In earlier seminars, we held one open problem session. However, this time there was so much interest in presenting open problems that we decided to hold two open problem sessions. Six families of open questions were presented at each of them.

Numerous participants expressed their pleasure with the seminar and its sequence of talks. The prevailing view was that while the participants came from three different fields, they were all open to the other two fields, and therefore, they all learned about results that they would not have learned otherwise. Therefore, we have all the reasons to believe that the seminar was a success, and we would like to repeat something similar sometime in the future. We have even discussed specific plans for a possible future seminar.

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3 Overview of Talks

3.1 Saturation for permutation matrices

Benjamin Aram Berendsohn (FU Berlin, DE)

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Main reference Benjamin Aram Berendsohn: “An exact characterization of saturation for permutation matrices”.
Combinatorial Theory, 3(1), 2023.

URL <https://doi.org/10.5070/C63160430>

A 0-1 matrix M contains a 0-1 matrix *pattern* P if we can obtain P from M by deleting rows and/or columns and turning arbitrary 1-entries into 0s. The saturation function $\text{sat}(P, n)$ for a 0-1 matrix pattern P indicates the minimum number of 1s in an $n \times n$ 0-1 matrix that does not contain P , but where changing any 0-entry into a 1-entry creates an occurrence of P .

Saturation for 0-1 matrices was introduced by Brualdi and Cao [arXiv 2020]. Fulek and Keszegh [SIAM J. Discret. Math. 2021] started a systematic study, and showed that each pattern has a saturation function that is either bounded or linear. They found large classes of patterns with linear saturation function, but only a single pattern with bounded saturation function.

Subsequently, Geneson [Electron. J. Comb. 2021] showed that almost all *permutation matrices* have bounded saturation functions. In this talk, we outline how to complete the classification of permutation matrices using a construction based on oscillations in indecomposable permutation matrices.

3.2 Mesh patterns in random permutations

David Bevan (University of Strathclyde – Glasgow, GB)

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Joint work of David Bevan, Jason Smith

We say that the *likelihood* of a mesh pattern is the asymptotic probability that a random permutation contains an occurrence of the pattern. In this talk we investigate the likelihood of a variety of patterns, determining their values for every vincular pattern. For bivincular patterns, the Small Anchors Theorem distinguishes between those patterns whose likelihood equals zero, those whose likelihood is positive but less than 1, and those whose likelihood equals 1. We also determine the (rational) likelihood of any bivincular pattern formed of what we call *anchored trees*. Other bivincular patterns, such as the small ascent and small descent, have irrational likelihoods, whose values can be established using the Chen–Stein method.

3.3 Combinatorial moment sequences and permutation patterns

Natasha Blitvic (Queen Mary University of London, GB)

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Joint work of Natasha Blitvic, Mohammed Slim Kammoun, Einar Steingrímsson

Main reference Natasha Blitvic, Einar Steingrímsson: “Permutations, Moments, Measures” in Transactions of the American Mathematical Society, Vol. 374, Number 8, August 2021, pp. 5473–5509.

URL <https://doi.org/10.1090/tran/8330>

Take your favorite integer sequence. Is it a sequence of moments of some positive Borel measure on the real line? The necessary and sufficient condition for a real sequence $(a_n)_{n \geq 0}$ to be a moment sequence was determined by Hamburger over a hundred years ago: namely, the Hankel matrices $(a_{i+j})_{0 \leq i, j \leq n}$ must all be positive semidefinite. In practice, when dealing with combinatorial sequences, positivity can be difficult to evaluate. In fact, starting with a given sequence, it can be surprisingly tricky to predict (or intuit) whether it will turn out to be a moment sequence. For example, take a_n to count the permutations on n letters avoiding the consecutive permutation pattern 123 and b_n the permutations avoiding the consecutive pattern 132. Only one of these is a moment sequence – can you predict which one?

In [1], we present a combinatorial framework, in the form of a fourteen-parameter continued fraction, that turns a hard question (that of establishing positivity) into a much easier one (that of decomposing combinatorial statistics into certain elementary building blocks). It builds on the rich tradition of work on combinatorial interpretations of continued fractions (see references in [1]). Applied to permutation patterns, the framework allows us to classify all (vincular) permutation patterns of length three according to whether they give rise to moment sequences.

While several teams of researchers in this area believe that avoiders of classical permutation patterns uniformly give moment sequences (see the Open Problem presented by this author at the same conference), consecutive permutation patterns, as per the example given above, are significantly worse behaved. Nevertheless, in [2], we identify the “correct lens” through which consecutive permutation patterns appear to enjoy analogous positivity properties. This positivity result is still conjectural but, thanks to enumerative results in [2], is strongly supported by numerical evidence.

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3.4 Non-uniform permutations biased according to their records

Mathilde Bouvel (LORIA – Nancy, FR)

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Joint work of Nicolas Auger, Mathilde Bouvel, Cyril Nicaud, Carine Pivoteau

In this talk, we study a non-uniform distribution on permutations (of any given size), where the probability of a permutation is proportional to θ^{rec} where rec denotes the number of records (a.k.a. left-to-right maxima). The motivation for defining this model of non-uniform random permutations is the analysis of algorithms. Indeed, when analyzing algorithms

working on arrays of numbers (modeled by permutations), the uniform distribution on the set of possible inputs is usually assumed. However, the actual data on which these algorithms are used is rarely uniform, and often displays a bias towards “sortedness”. Our model has this bias towards “sortedness” while remaining tractable from the point of view of the analysis of algorithms. Our results on this model are of three types. First, we exhibit several efficient random samplers of permutations under this distribution. Second, we analyze the behavior of some classical permutation statistics, some of which with applications to the analysis of algorithms. Finally, we describe the “typical shape” of permutations in our model, by means of their (deterministic) permuton limit.

3.5 On the problem of Hertzsprung and similar problems

Anders Claesson (University of Iceland – Reykjavik, IS)

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Main reference Anders Claesson: “From Hertzsprung’s problem to pattern-rewriting systems”, *Algebraic Combinatorics*, 5(6):1257–1277, 2022.

URL <https://doi.org/10.5802/alco.202>

Drawing on a problem posed by Hertzsprung in 1887 (sometimes called the n -kings problem), we say that a permutation w contains the Hertzsprung pattern u if there is factor of w that differ only by a constant from u in the sense that there is a factor $w(d+1)w(d+2)\dots w(d+k)$ such that $w(d+1) - u(1) = \dots = w(d+k) - u(k)$. Using a combination of the Goulden-Jackson cluster method and the transfer-matrix method we determine the joint distribution of occurrences of any set of Hertzsprung patterns, thus substantially generalizing earlier results by Jackson et al. on the distribution of ascending and descending runs in permutations. We apply our results to the problem of counting permutations up to pattern-replacement equivalences, and using pattern-rewriting systems – a new formalism similar to the much studied string-rewriting systems – we solve a couple of open problems raised by Linton et al. in 2012.

3.6 Permutations and Exclusion Processes

Sylvie Corteel (Université Paris Cité, FR)

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We will review results on the combinatorics of the exclusion process which is a classical model in statistical physics. We will explain why permutations and the pattern 31-2 appear when we study the exclusion process with open boundaries. We will then propose a series of open problems on the combinatorics of more general processes.

3.7 On complex roots of the independence polynomial

Péter Csikvári (Alfréd Rényi Institute of Mathematics – Budapest, HU)

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Joint work of Ferenc Bencs, Péter Csikvári, Piyush Srivastava, Jan Vondrák
Main reference Ferenc Bencs, Péter Csikvári, Piyush Srivastava, Jan Vondrák: “On complex roots of the independence polynomial”, in Proc. of the 2023 ACM-SIAM Symposium on Discrete Algorithms, SODA 2023, Florence, Italy, January 22-25, 2023, pp. 675–699, SIAM, 2023.
URL <https://doi.org/10.1137/1.9781611977554.ch29>

The independence polynomial of a graph is the generating polynomial of all its independent sets. Formally, given a graph G , its independence polynomial $Z_G(\lambda)$ is given by $\sum_I \lambda^{|I|}$, where the sum is over all independent sets I of G . The independence polynomial has been an important object of study in both combinatorics, statistical physics and computer science. In particular, the algorithmic problem of estimating $Z_G(\lambda)$ for a fixed positive λ on an input graph G is a natural generalization of the problem of counting independent sets, and its study has led to some of the most striking connections between computational complexity and the theory of phase transitions. More surprisingly, the independence polynomial for negative and complex values of λ also turns out to be related to problems in statistical physics and combinatorics. In particular, the locations of the complex roots of the independence polynomial of bounded degree graphs turn out to be very closely related to the Lovász local lemma, and also to the questions in the computational complexity of counting. In this talk we give new geometric criteria for establishing zero-free regions as well as for carrying out semi-rigorous numerical explorations. We then provide several examples of the (rigorous) use of these criteria, by establishing new zero-free regions. Joint work with Ferenc Bencs, Piyush Srivastava and Jan Vondrák.

3.8 The complexity of computing immanants

Radu Curticapean (IT University of Copenhagen, DK)

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Main reference Radu Curticapean: “A full complexity dichotomy for immanant families”, CoRR, Vol. abs/2102.04340, 2021.
URL <https://arxiv.org/abs/2102.04340>

Immanants are matrix functions that generalize determinants and permanents. Given an irreducible character X_λ of S_n for some partition λ of n , the immanant associated with λ is a sum-product over permutations π in S_n , much like the determinant, but with $X_\lambda(\pi)$ playing the role of $\text{sgn}(\pi)$.

Hartmann showed in 1985 that immanants can be evaluated in polynomial time for sign-ish characters. More precisely, for a partition λ of n with s parts, let $b(\lambda) := n - s$ count the boxes to the right of the first column in the Young diagram of λ . The immanant associated with λ can be evaluated in $n^{O(b(\lambda))}$ time.

Since this initial result, complementing hardness results have been obtained for several families of immanants derived from partitions with unbounded $b(\lambda)$. This includes permanents, immanants associated with hook characters, and other classes. In this talk, we complete the picture of hard immanant families: Under a standard assumption from parameterized complexity, we rule out polynomial-time algorithms for well-behaved immanant families with unbounded $b(\lambda)$. For immanant families in which $b(\lambda)$ even grows polynomially, we establish hardness for $\#P$ and VNP.

3.9 Three Topics in Pattern Avoidance

Colin Defant (MIT – Cambridge, US)

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Joint work of Colin Defant, Amanda Burcroff, Noah Kravitz, Ashwin Sah

This talk will survey three topics related to pattern avoidance, each motivated by an interesting question:

1. Given a class C of objects and a set P of patterns from C , what is the smallest size of an object in C that contains all of the patterns in P ? We will focus on recent developments in which C is either a class of words/permutations or a class of trees. The discussion of words/permutations is mostly taken from the survey article [9] as well as the more recent articles [6, 11]. The discussion of trees is taken from my article [8], which was written with Noah Kravitz and Ashwin Sah.
2. How does permutation pattern avoidance interact with the group structure of the symmetric group? The first part of this topic will concern pattern-avoiding permutations with particular cycle types; the second part will concern permutations whose powers are required to avoid a pattern. This portion of the talk will draw from the articles [1, 3, 10, 2, 4, 5].
3. If your socks come out of the laundry all mixed up, how should you sort them? We will discuss a novel foot-sorting algorithm that uses feet to attempt to sort a sock ordering; one can view this algorithm as an analogue of Knuth's stack-sorting algorithm for set partitions. This part of the talk is based on my article [7], which was written with Noah Kravitz.

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3.10 Walks in simplices, cylindric tableaux, and asymmetric exclusion processes

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Main reference Sergi Elizalde: “Walks in Simplices, Cylindric Tableaux, and Asymmetric Exclusion Processes”, Séminaire Lotharingien de Combinatoire, 86B.8 (2022), 12 pp.

URL <https://www.mat.univie.ac.at/~slc/wpapers/FPSAC2022/8.html>

We describe bijections between three classes of combinatorial objects that have appeared in different contexts: lattice walks in simplicial regions as introduced by Mortimer–Prellberg (in a previous Dagstuhl Seminar), standard cylindric tableaux as introduced by Gessel–Krattenthaler and Postnikov, and sequences of states in the totally asymmetric simple exclusion process. Our perspective gives new insights into these objects, providing a vehicle to translate enumerative results and certain symmetries from one setting to another. As an example, we use a cylindric analogue of the Robinson–Schensted correspondence to give an alternative bijective proof of a recent result of Courtiel, Elvey Price and Marcovici relating forward and backward walks in simplices.

3.11 Fighting fish and pattern avoiding permutations

Luca Ferrari (University of Firenze, IT)

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Joint work of Lapo Cioni, Luca Ferrari, Corentin Henriot

Fighting fish are combinatorial objects recently introduced by Duchi, Guerrini, Rinaldi and Schaeffer. They are polyomino-like objects which can branch out of the plane into independent substructures. The main motivation for considering such structures lies in their remarkable probabilistic properties. In particular, it is known that the average area of fighting fish having semiperimeter n is of order $n^{5/4}$, which is a rather non-standard behaviour. The previously mentioned authors have discovered a lot of interesting combinatorics related to fighting fish. In particular, they have shown that the number of fighting fish of semiperimeter $n + 1$ is given by $\frac{2}{(n+1)(2n+1)} \binom{3n}{n}$, which is the same as the number of (West)-two-stack sortable permutations of size n . This result spurred much research to better understand the relationships between these objects (and others also counted by the same sequence). Wenjie Fang was able to describe a bijection between two-stack-sortable permutations and fighting fish which accounts for the above enumerative result (and also preserves a lot of other statistics). However, his bijection is recursive, and no direct bijection is known yet. In my talk I will present a general construction which maps any permutation to a certain labelled tree, which in turn encodes a unique fighting fish. Such a construction is not bijective in general, but it becomes a bijection when restricted to two-stack-sortable permutations, thus defining the (almost) direct bijection that was missing. As a matter of fact, this construction is equivalent to Fang’s one (in other words, it is a more direct version of the recursive procedure of Fang). Our hope would then be to use our construction to understand what the parameter area (on fighting fish) is on permutations, but we have not been successful yet. We have however some partial results, such as a description of those permutations (of fixed size) whose associated fighting fish has minimum area.

The sequence enumerating fighting fish with respect to semiperimeter also counts another class of permutations, namely those avoiding the two vincular patterns 3-1-4-2 and 2-41-3. Such permutations have been investigated by Claesson, Kitaev and Steingrímsson, in particular they provide a bijection with so-called $\beta(1, 0)$ -trees. These are trees whose recursive structure encodes somehow more naturally the recursive structures of fighting fish. This leads us to think that this class of permutations could be more useful to have a better combinatorial understanding of the area of fighting fish.

3.12 Length-4 Pattern Avoidance in Inversion Sequences

Carina Letong Hong (University of Oxford, GB)

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Joint work of Carina Letong Hong, Rupert Li

Main reference Letong Hong, Rupert Li: “Length-Four Pattern Avoidance in Inversion Sequences”, *Electron. J. Comb.*, Vol. 29(4), 2022.

URL <https://doi.org/10.37236/10948>

Pattern avoidance for permutations is a robust and well-established branch of enumerative combinatorics since the systematic study of Simeon and Schmidt in 1985. This talk summarizes recent progress on pattern avoidance in inversion sequences, including Mansour and Shattuck (2015), Corteel, Martinez, Savage, and Weselcouch (2016), Mansour and Yildirim (2022), and Testar (2022). Their work completely enumerated length-3 pattern avoidance except the case 120.

Recently, we completely classified all eight length-4 Wilf equivalences: $1011 = 1101 = 1110$, $2110 = 2101 = 2011$, $0221 = 0212$, $0312 = 0321$, $1102 = 1012$, $2201 = 2210$, $2301 = 2310$, $3201 = 3210 = 3012$. This talk mentions key lemmas that help establish the proofs that can be generalized to avoidance of longer patterns. Furthermore, there are four length-4 patterns whose enumeration appears in the OEIS by computer experiments: in addition to 0012 conjectured by Lin and Ma and proved by Chern, we resolve the cases 0000 and 0111 by proving the formulas for general $00\dots 0$ and $01\dots 1$ cases, giving bijections to bounded-degree label-increasing trees, and give a conjecture for 0021 which is then subsequently solved by Chern, Fu, and Lin and Mansour in 2022.

3.13 What makes permutation patterns hard to match?

Vít Jelínek (Charles University – Prague, CZ)

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Joint work of Vít Jelínek, Michal Opler, Jakub Pekárek

Main reference Michal Opler: “Structural and Algorithmic Properties of Permutation Classes”, PhD thesis, Faculty of Mathematics and Physics, Charles University in Prague, 2022.

URL <http://hdl.handle.net/20.500.11956/179844>

Permutation Pattern Matching (or PPM) is a fundamental decision problem in the study of permutations. Its input is a pair permutations P (the “pattern”) and T (the “text”), and the goal is to determine whether P is contained in T . While PPM is NP-hard on general inputs, it often becomes tractable when P or T are restricted to a proper hereditary permutation class.

In my talk, I will present some results and conjectures that attempt to describe the boundary between tractable and hard cases of such restrictions of PPM and other related problems. Specifically, I will focus on the relationships between these three topics: computational complexity of PPM restricted to a given permutation class; structural properties of a permutation class, such as the presence of large grid-like substructures; and the growth of width parameters, like tree-width or grid-width, within a given permutation class.

3.14 Pattern avoidance: algorithmic connections

László Kozma (FU Berlin, DE)

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Main reference Benjamin Aram Berendsohn, László Kozma, Dániel Marx: “Finding and Counting Permutations via CSPs”, *Algorithmica*, Vol. 83(8), pp. 2552–2577, 2021.

URL <https://doi.org/10.1007/s00453-021-00812-z>

Already from the beginnings of the field, pattern-avoidance has been studied in tandem with algorithmic applications. For instance, Knuth’s study of permutation-patterns was motivated by connections to stack-sorting. In this talk I will survey results from the last few years on two related lines of work: (1) the study of algorithms, in various models of computation, for detecting, counting, or enumerating patterns, and (2) the study of pattern-avoidance as a source of algorithmic “easiness”: how pattern-avoidance of the input affects the complexity of seemingly unrelated algorithmic tasks.

3.15 Sorting Genomes by Prefix Double-Cut-and-Joins

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Joint work of Guillaume Fertin, Géraldine Jean, Anthony Labarre

Main reference Guillaume Fertin, Géraldine Jean, Anthony Labarre: “Sorting Genomes by Prefix Double-Cut-and-Joins”, in *Proc. of the String Processing and Information Retrieval – 29th International Symposium, SPIRE 2022, Concepción, Chile, November 8-10, 2022, Proceedings, Lecture Notes in Computer Science*, Vol. 13617, pp. 178–190, Springer, 2022.

URL https://doi.org/10.1007/978-3-031-20643-6_13

A double cut-and-join (DCJ for short) is an operation that replaces two edges u, v and w, x in a graph with either u, x, v, w or u, w, v, x . This operation is of interest in a biological context, as it generalises several other well-studied mutations that are known to happen in genomes, e.g. (possibly signed) reversals or (block-)transpositions.

I consider two different graph models for representing genomes – namely, paths and perfect matchings – and study DCJs under the “prefix restriction”, which forces one of the cut edges to contain the first element of the genome. The talk focuses on sorting problems using these operations, which is a popular approach to reconstructing evolutionary scenarios between species, but also has applications in the field of interconnection network design, from which the prefix restriction originates. I will present some recent results on sorting genomes using variants of prefix DCJs using both models. Namely:

- new lower bounds on sorting genomes using prefix DCJs or prefix reversals;
- a polynomial-time algorithm for sorting signed genomes by prefix DCJs; and
- a $3/2$ -approximation algorithm for sorting unsigned genomes by prefix DCJs.

The latter algorithm is the first polynomial-time approximation algorithm with a ratio smaller than 2 for a prefix sorting problem not known to be in P.

3.16 Computational complexity of counting and sampling

István Miklós (ELKH – Budapest, HU)

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Joint work of Carina Letong Hong, István Miklós

Main reference Letong Hong, István Miklós: “A Markov chain on the solution space of edge colorings of bipartite graphs”, *Discret. Appl. Math.*, Vol. 332, pp. 7–22, 2023.

URL <https://doi.org/10.1016/j.dam.2023.01.029>

Whenever we can ask if a certain mathematical object with prescribed properties exists, we can also ask how many such objects exist and how to generate a random one. We can also talk about the computational complexity of these counting and sampling problems, that is, how the running time of computer programs solving these problems increases with the input size. It turns out that counting and sampling are equally hard for a large class of computational problems. Equal hardness also frequently holds for estimating weighted sums and sampling from a distribution proportional to these weights. For example, in statistical physics, these problems are sampling from the Boltzmann distribution and estimating the partition function. In this talk, we give an overview of sampling and counting complexity, then we will focus on the most powerful approach to sampling, the Markov chain Monte Carlo method.

3.17 Scaling Limits of Some Restricted Permutations

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Joint work of Borgia, Jacopo; Duchi, Enrica; Hoffman, Christopher; Rizzolo, Douglas;

Main reference Jacopo Borgia, Enrica Duchi, Erik Slivken: “Almost square permutations are typically square”, *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, Vol. 57, 2021.

URL <https://doi.org/10.1214/20-AIHP1135>

Main reference Christopher Hoffman, Douglas Rizzolo, Erik Slivken: “Scaling limits of permutations avoiding long decreasing sequences”, *CoRR*, Vol. abs/1911.04982, 2023.

URL <https://arxiv.org/abs/1911.04982>

Suppose we take a large permutation that is chosen uniformly at random and conditioned to satisfy some restriction. What does this permutation look like? The answer depends on the choice of restriction and how one decides to scale the permutation. We introduce a few scaling limits that prove useful in answering this type of question for a variety of restrictions, especially in the case of pattern-avoiding permutations. We will explore what various scaling limits say about these objects and some associated statistics (like the number of fixed points of the permutation)

3.18 Two new bijections on six-vertex configurations

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Joint work of Daoji Huang, Jessica Striker

Main reference Daoji Huang, Jessica Striker: “A pipe dream perspective on totally symmetric self-complementary plane partitions”, CoRR, Vol. abs//2303.10463, 2023, (Preprint).

URL <https://arxiv.org/abs/2303.10463>

The six-vertex model is an exactly solvable model in statistical mechanics that has been widely studied by both physicists and combinatorialists for its many lovely properties.

In this talk in two parts, we first describe joint work with Daoji Huang [2] giving a bijection between alternating sign matrices (a combinatorial manifestation of six-vertex configurations) and totally symmetric self-complementary plane partitions in the reduced, 1432-avoiding case. Finding such a bijection in full generality has been an open problem for nearly 40 years; it will be interesting to see whether this new sub-bijection may lead to further (positive or negative) results on a full bijection.

We then introduce a symmetrized version of the six-vertex model that adapts nicely from the square lattice to arbitrary 4-regular graphs embedded in a disk. By modifying certain vertex configurations, we transform these to nearly planar bipartite graphs that have regions corresponding to dimer covers of hexagonal lattices. Our underlying motivation and main result is a bijection between equivalence classes of these graphs and 4-row tableaux in such a way that promotion corresponds to rotation, yielding a web basis for SL_4 . This is joint work with Christian Gaetz, Oliver Pechenik, Stephan Pfannerer, and Joshua Swanson [1].

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3.19 Extremal theory of vertex- and edge-ordered graphs

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Turán-type extremal graph theory has been generalized in many directions. In this talk I survey the extremal theories of vertex- and edge-ordered graphs. In the vertex-ordered case the vertices of the host graph are linearly ordered and we only forbid a subgraph with a specified vertex order. As in the classical extremal graph theory, we are still looking for the maximal number of edges in a host graph on n vertices avoiding the forbidden subgraph. The analogous extremal theory for edge ordered graphs was introduced recently in a paper of Gerbner, Methuku, Nagy, Pálvölgyi, T., Vizer (2023). In contrast, the vertex ordered theory has a much richer history going back to related extremal matrix problems studied by Füredi and Hajnal in 1992.

Both theories are rich in specific results: the extremal functions of some small forbidden ordered graphs. Some of these results found applications in combinatorial geometry. Other specific forbidden patterns lead to interesting open problems.

Another direction is to find analogues of general results from classical extremal graph theory. The analogues of the Erdős-Stone-Simonovits theorem has been found in both theories: the key is to find the “correct” version of the chromatic number that applies. For vertex-ordered graphs, this is the simple notion of interval chromatic number, for edge-ordered graphs, however, the corresponding notion is surprisingly rich.

In the classical (unordered) theory we have a very simple dichotomy: forests have linear extremal functions, while other graphs have far-from-linear extremal functions. The search for the analogue of this simple observation yielded several nice results, conjectures and open problems about vertex- and edge-ordered graphs.

3.20 Pattern-avoiding affine permutations

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Joint work of Neal Madras, Justin M. Troyka

Main reference Neal Madras, Justin M. Troyka: “Bounded affine permutations I. Pattern avoidance and enumeration”, *Discret. Math. Theor. Comput. Sci.*, Vol. 22(2), 2021.

URL <https://doi.org/10.46298/dmtcs.6178>

An *affine permutation of size n* is a bijection $\pi: \mathbb{Z} \rightarrow \mathbb{Z}$ satisfying certain properties, including that $\pi(i+n) = \pi(i) + n$ for all i . The affine permutations of size n have been much studied as a Coxeter group, but our perspective is pattern avoidance. To have a finite set to count, we can require also that $|\pi(i) - -i| < n$ for each i ; such π we call a *bounded affine permutation*. We give the asymptotic number of bounded affine permutations of size n that avoid $k \dots 1$; in particular, they have the same growth rate as the ordinary permutations avoiding $k \dots 1$. We also show several results about affine permutation classes with the property that every element is a shift of the infinite direct sum of an ordinary permutation, from which we obtain exact enumerations for several affine permutation classes. This talk covers joint work with Neal Madras, published in 2021 in *Discrete Math. Theor. Comput. Sci.* and in *Ann. Comb.* Our work, especially the boundedness condition, is motivated by the fruitful concept of periodic boundary conditions in statistical physics.

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3.21 Exploring Permutation Classes with TileScope

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Joint work of Henning Ulfarsson, Michael Albert, Christian Bean, Anders Claesson, Émile Nadeau, Jay Pantone

In the world of combinatorics, there are numerous sets of objects that are in a one-to-one correspondence with sets of permutations possessing specific properties, commonly characterized by pattern avoidance. This talk will delve into the TileScope algorithm and

demonstrate its usefulness in comprehending permutation sets avoiding a finite list of patterns. Additionally, we will examine the algorithm's output, including polynomial counting formulas, systems of equations, uniform random generation, and more. Finally, we will highlight the algorithm's ability to discover bijections automatically, examine the atomicity of permutation classes, and preview planned future advancements. Visit www.permpal.com to see successful applications of the algorithm in action.

3.22 Generating Tree Method and Pattern Avoiding Inversion Sequences

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Joint work of Gökhan Yildirim, Ilias Kotsireas, Toufik Mansour

Main reference Ilias S. Kotsireas, Toufik Mansour, Gökhan Yildirim: “An algorithmic approach based on generating trees for enumerating pattern-avoiding inversion sequences”, *J. Symb. Comput.*, Vol. 120, p. 102231, 2024.

URL <https://doi.org/10.1016/j.jsc.2023.102231>

An inversion sequence of length n is an integer sequence $e = e_1 \cdots e_n$ such that $0 \leq e_i < i$ for each $0 \leq i \leq n$. We use I_n to denote the set of inversion sequences of length n . Any word τ of length k over the alphabet $[k] := \{0, 1, \dots, k-1\}$ is called a pattern. For a given pattern τ , we use $I_n(\tau)$ to denote the set of all τ -avoiding inversion sequences of length n .

Pattern-avoiding inversion sequences were systematically studied first by Mansour and Shattuck [2] for the patterns of length three with non-repeating letters and by Corteel et al. [1] for repeating and non-repeating letters. Since then, researchers have obtained several interesting results for these combinatorial objects.

We provide an algorithmic approach based on generating trees for enumerating the pattern-avoiding inversion sequences. First, by using this algorithmic approach, we determine the generating trees for many pattern classes such as $I_n(100)$, $I_n(011, 201)$, $I_n(021, 0112)$, \dots . Then we obtain enumerating formulas for them through generating functions and the kernel method.

G. Yildirim was partially supported by TUBITAK-Ardeb 120F352.

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4 Open problems

4.1 (More on) Combinatorial moment sequences and permutation patterns

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Joint work of Natasha Blitvic, Einar Steingrímsson

Several years ago, a strange idea was hatched by a probabilist and a combinatorialist, drawing on intuition from noncommutative probability and the types of constructions found there. Namely, could the sequences arising from the study of classical permutation patterns be moment sequences of probability measures on the real line? Here was our central conjecture. Given a classical permutation pattern π , let $G_\pi^{(n)}$ be the generating function of the number of *occurrences* of π in permutations on n letters, that is,

$$G_\pi^{(0)}(q) := 1 \quad \text{and} \quad G_\pi^{(n)}(q) := \sum_{\sigma \in S_n} q^{\#\text{occ}_\pi(\sigma)}.$$

► **Conjecture.** For any permutation pattern π , there is some real interval containing the origin, such that for any q in that interval, $(G_\pi^{(n)}(q))_{n \geq 0}$ is a moment sequence of some probability measure on the real line.

Given we (still) have very little numerical data, let alone very few examples of permutation patterns π for which G_π has a known expression (see the references in [1] for the two known examples of vincular patterns for which G_π is available), this might be seen as a bold conjecture. Remarkably, independently and around the same time but motivated by the numerical aspects, Tony Guttmann and Andrew Elvey Price were conjecturing the above for $q = 0$, that is, focusing on the suspected positivity of the *avoiders* of classical permutation patterns. Moreover, these appeared to give moments of probability measures on the *positive* real line.

For sequences counting avoiders of classical permutation patterns, some explicit results are available, from which positivity can be deduced using a variety of techniques [3], as well as more data thanks to state-of-the-art numerical algorithms developed for this type of enumeration (see [4] and the references therein). However, we are still far from resolving this conjecture, even in the $q = 0$ case. (Interestingly, see [2] for a non-trivial “consecutive pattern analogue” of the conjecture.) To tackle the general version given above, two distinct flavors of combinatorial results would be helpful:

- Explicit distributional results, that is closed or semi-closed expressions for G_π , for some examples of classical permutation patterns.
- Improved algorithms to enumerate the distributions of the number of occurrences of a given classical permutation pattern, in order to test the above conjecture.

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4.2 Distribution of sets of descent tops and descent bottoms on permutations avoiding patterns of length 4

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We conjecture the Destop-Wilf equivalence classes and (Destop, Desbot)-Wilf equivalence classes of permutation patterns of length 4. Specifically, we conjecture that the nontrivial (i.e. non-singleton) Destop-Wilf equivalence classes in S_4 are $\{1243, 3412\}$, $\{1423, 2413\}$, $\{2143, 3421\}$, $\{2314, 3124\}$, $\{2431, 3142, 3241, 4132\}$, and the only nontrivial (Destop, Desbot)-Wilf equivalence class in S_4 is $\{3142, 3241, 4132\}$. This has been verified for avoiders of size up to 10.

The conjectures in this note concern the distribution of some descent-related statistics on some pattern-avoiding permutation classes.

For basic definitions regarding patterns in permutations, we refer to Bevan [3]. We will need the following additional definitions, the first of which refines Wilf-equivalence, and the second defines additional descent-related statistics.

► **Definition 1.** Let f be a permutation statistic. We say that patterns σ and τ are f -Wilf-equivalent if there is a bijection $\Theta : \text{Av}_n(\sigma) \rightarrow \text{Av}_n(\tau)$ for all $n \geq 0$ that preserves the f statistic, i.e. $f = f \circ \Theta$.

► **Definition 2.** Given a permutation σ , and a position i such that $\sigma(i) > \sigma(i+1)$, we call i a *descent* of σ , $\sigma(i)$ a *descent top* of σ , and $\sigma(i+1)$ a *descent bottom* of σ . Likewise, if $\sigma(i) < \sigma(i+1)$, then we call i an *ascent* of σ , $\sigma(i)$ an *ascent bottom* of σ , and $\sigma(i+1)$ an *ascent top* of σ .

► **Notation 3.** For a permutation σ , we define the following sets (which can also be thought of as permutation statistics):

- $\text{Des}(\sigma) = \{i \mid \sigma(i) > \sigma(i+1)\}$, the descent set of σ ,
- $\text{Destop}(\sigma) = \{\sigma(i) \mid i \in \text{Des}(\sigma)\}$, the descent top set of σ ,
- $\text{Desbot}(\sigma) = \{\sigma(i+1) \mid i \in \text{Des}(\sigma)\}$, the descent bottom set of σ .

We also define the sets $\text{Asc}(\sigma)$, $\text{Ascbot}(\sigma)$, $\text{Asctop}(\sigma)$ of ascents, ascent bottoms, and ascent tops of σ similarly.

It is straightforward to show that patterns 132 and 231 are both Des-Wilf-equivalent and Destop-Wilf equivalent, but not (Des, Destop)-Wilf equivalent. Indeed, define maps $\phi, \psi : \text{Av}(132) \rightarrow \text{Av}(231)$ as follows. Given a nonempty permutation $\sigma \in \text{Av}(132)$, we can write $\sigma = 231[\sigma', 1, \sigma'']$ for some $\sigma', \sigma'' \in \text{Av}(132)$. Then let

$$\phi(\sigma) = 132[\phi(\sigma'), 1, \phi(\sigma'')] \quad \text{and} \quad \psi(\sigma) = 132[\psi(\sigma''), 1, \psi(\sigma')] \quad \text{if } \sigma', \sigma'' \neq \emptyset,$$

and $\phi(\sigma) = \psi(\sigma) = \sigma$ if $\sigma' = \emptyset$ or $\sigma'' = \emptyset$ or $\sigma = \emptyset$. Then

$$\text{Des}(\phi(\sigma)) = \text{Des}(\sigma) \quad \text{and} \quad \text{Destop}(\psi(\sigma)) = \text{Destop}(\sigma).$$

West [8], Stankova [7], and Bóna [5] together established the Wilf-equivalence classes of permutation patterns of length 4. Later, some of the Wilf-equivalences were shown to have the same distribution of various permutation statistics. For example, Bloom [4] showed that patterns 3142 and 4132 are Des-Wilf-equivalent.

The principal motivation for our conjectures comes from the following. A *Dumont permutation of the first kind* is a permutation σ of an even length $2n$ such that $\text{Destop}(\sigma) = \{2, 4, 6, \dots, 2n\}$. Burstein and Jones [6] and Archer and Lauderdale [2] together conjectured Wilf-equivalences of patterns of length 4 on Dumont permutations of the first kind. We generalize this by claiming that those are exactly the nontrivial Destop-Wilf equivalences for patterns of length 4 on all permutations.

► **Conjecture 4.** *The non-singleton Destop-Wilf equivalence classes for permutation patterns of length 4 are:*

- $1243 \sim 3412$,
- $1423 \sim 2413$,
- $2143 \sim 3421$,
- $2314 \sim 3124$,
- $2431 \sim 3142 \sim 3241 \sim 4132$.

Each of the remaining 12 patterns of length 4 is in a Destop-Wilf-equivalence class by itself. For some of the above patterns, we have an even stronger conjecture.

► **Conjecture 5.** *Patterns 3142, 3241, 4132 are (Destop, Desbot)-Wilf equivalent.*

It is easy to see by comparing the descent tops and descent bottoms of the patterns themselves that this is the only nontrivial (Destop, Desbot)-Wilf equivalence class for patterns of length 4.

Conjecture 5 can be restated by partitioning the set of all values in each permutation $\sigma \in S_n$ into four mutually disjoint parts.

- $\text{Desruntop}(\sigma) = \text{Destop}(\sigma) \setminus \text{Desbot}(\sigma)$, the set of descent run tops of σ ,
- $\text{Desrunbot}(\sigma) = \text{Desbot}(\sigma) \setminus \text{Destop}(\sigma)$, the set of descent run bottoms of σ ,
- $\text{Desrunmid}(\sigma) = \text{Destop}(\sigma) \cap \text{Desbot}(\sigma)$, the set of descent run middles of σ ,
- $\text{Ascrunmid}(\sigma) = [n] \setminus (\text{Destop}(\sigma) \cup \text{Desbot}(\sigma))$, the set of ascent run middles of $\sigma^+ = (0, \sigma, \infty)$. This includes ascent run middles of σ together with $\sigma(1)$ if $\sigma(1) < \sigma(2)$ and $\sigma(n)$ if $\sigma(n-1) < \sigma(n)$.

► **Conjecture 6.** *Patterns 3142, 3241, 4132 are (Desruntop, Desrunbot, Desrunmid, Ascrunmid)-Wilf equivalent.*

Both Conjectures 4 and 5 have been verified for avoiders of length $n \leq 10$ with the help of Michael Albert's *PermLab* [1] software.

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4.3 A curious mesh pattern

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Main reference Anders Claesson: “From Hertzsprung’s problem to pattern-rewriting systems”, Algebraic Combinatorics, Vol. 5(6), pp. 1257–1277, The Combinatorics Consortium, 2022.

URL <https://doi.org/10.5802/alco.202>

I would like to bring attention to a particular mesh pattern. It is of length 3 and enumerating the permutations avoiding it is an open problem. Moreover, this pattern shares some features with the Hertzsprung patterns, yet it does not fall into that category.

Are there nontrivial mesh-patterns p other than the Hertzsprung patterns for which there is a rational function $R(x)$ such that

$$\sum_{n \geq 0} |\mathcal{S}(p)| x^n = \sum_{m \geq 0} m! R(x)^m ?$$

It appears that the answer is yes.

► **Conjecture 1.** *It holds that*

$$\sum_{n \geq 0} |\mathcal{S}_n(p)| x^n = \sum_{m \geq 0} m! \left(\frac{x}{1+x^2} \right)^m, \text{ where } p = \begin{array}{c} \begin{array}{|c|c|c|} \hline \text{///} & \text{///} & \text{///} \\ \hline \bullet & \bullet & \bullet \\ \hline \text{///} & \text{///} & \text{///} \\ \hline \end{array} \\ \text{---} \\ \bullet \\ \text{---} \end{array}$$

This conjecture is based on computing the numbers $|\mathcal{S}_n(p)|$ for $n \leq 14$:

$$1, 1, 2, 5, 20, 103, 630, 4475, 36232, 329341, 3320890, 36787889, \\ 444125628, 5803850515, 81625106990$$

At the time of writing this sequence is not in the OEIS.

4.4 Universal Set Partitions

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In 2000, Klazar introduced a natural notion of pattern containment/avoidance for set partitions [1, 2]. Let (A, B) be one of the following pairs of phrases:

- (set partition, set partition);
- (set partition, noncrossing partition);
- (set partition, nonnesting partition);
- (noncrossing partition, noncrossing partition);
- (nonnesting partition, nonnesting partition).

For each positive integer k , what is the smallest size of an A that contains all B ’s of size k as patterns?

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4.5 Two conjectures on quasi-kernels

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Let $D = (V, \vec{E})$ be (finite or infinite) directed graph. An independent vertex subset $A \subset V$ is a *quasi-kernel* (also known as *semi-kernel*) iff for each point v there is a path of length at most 2 from some point of A to v . Similarly, the independent vertex subset $B \subset V$ is a *quasi-sink*, iff for each point v there is a path of length at most 2 from v to some point of A .

It is a well-known fact, that every finite (table-tennis) tournament has a (single-point) quasi-kernel (quasi-sink). In 1973 the following nice generalization was proved:

► **Theorem 1** (V. Chvátal – L. Lovász [1]). *Every finite directed graph contains a quasi-kernel (quasi-sink).*

In 1976 the following conjecture was stated:

► **Conjecture 2** (P.L. Erdős – L. A. Székely). *Assume that in the finite digraph $D = (V, \vec{E})$ for each vertex $v \in V$ the indegree $d^-(v) \geq 1$. Then there exists a quasi-kernel A with the property $|A| \leq |V|/2$. For example the disjoint union of oriented C_4 's satisfies this with equality.*

Theorem 1 does not hold in case of infinite directed graphs: for example if Z denotes the directed graph of all integers where each edge is directed “upwards”, clearly there is nor quasi-kernel neither quasi-sink. However its vertex set can be easily partitioned into two subsets, such that one spanned sub-tournament contains a quasi-kernel, while the other one contains a quasi-sink.

► **Conjecture 3** (P.L. Erdős – L. Soukup (2008) [2]). *Every (countable) infinite directed graph D can be partitioned into two vertex classes, such that one spanned subgraph contains a quasi-kernel, while the other one contains a quasi-sink.*

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4.6 Sorting by parallel block reversals

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Joint work of Vít Jelínek, Michal Opler, Jakub Pekárek

Suppose that we are given a sequence of n numbers, which we want to sort into ascending order by using the following iterative procedure: in each round, we partition the current sequence arbitrarily into disjoint blocks of entries in consecutive positions, not necessarily of the same length, and then in a single step we reverse the order of entries within each block. What is the smallest number of rounds needed to sort any input of length n ? Equivalently:

what is the smallest number $K(n)$ such that any permutation of length n can be obtained by composing $K(n)$ layered permutations? A counting argument shows that $\Omega(\log n)$ rounds are sometimes needed, and I can show that $O(\log^2 n)$ rounds suffice, so the problem is to close this gap.

4.7 Füredi-Hajnal limits of permutations

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Joint work of Josef Cibulka, Jan Kynčl

Main reference Josef Cibulka, Jan Kynčl: “Better upper bounds on the Füredi-Hajnal limits of permutations”, CoRR, Vol. abs/1607.07491v3, 2019

URL <https://arxiv.org/abs/1607.07491v3>

Füredi–Hajnal limits of permutations

A *binary matrix* is a matrix with entries from the set $\{0, 1\}$. We say that a binary matrix A *contains* a binary matrix S if S can be obtained from A by removal of some rows, some columns, and changing some 1-entries to 0-entries. If A does not contain S , we say that A *avoids* S . A *k -permutation matrix* P is a binary $k \times k$ matrix with exactly one 1-entry in every row and one 1-entry in every column.

The Füredi–Hajnal conjecture, proved by Marcus and Tardos, states that for every permutation matrix P , there is a (smallest) constant c_P such that for every $n \in \mathbb{N}$, every $n \times n$ binary matrix A with at least $c_P \cdot n$ 1-entries contains P .

The proof by Marcus and Tardos [3] implies the upper bound $c_P \leq 2k^4 \binom{k^2}{k}$ for every k -permutation matrix P . Fox [2] improved the upper bound to $c_P \leq 3k2^{8k}$. Our current best upper bound is $c_P \leq \frac{8}{3}(k+1)^2 \cdot 2^{4k}$ [1].

For the lower bound, Fox [2] gave a randomized construction showing that for every k , there are k -permutation matrices P with $c_P \geq 2^{\Omega(k^{1/2})}$.

For random k -permutation matrices, we can prove that $c_P \leq 2^{O(k^{2/3} \log^{7/3} k / (\log \log k)^{1/3})}$ asymptotically almost surely [1]. In particular, we can prove subexponential upper bounds for so-called *scattered* permutation matrices and a few specific examples of non-scattered permutation matrices. However, there are still many cases where we do not have better than exponential upper bound on c_P .

A *two-diagonal* matrix is a binary matrix whose all 1-entries lie on two parallel diagonals, which may be arbitrarily far apart.

Let P be k -permutation matrix contained in a two-diagonal matrix. Is $c_P \leq 2^{o(k)}$? Is c_P polynomial in k ?

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4.8 Number of Successful Pressing sequences of black-and-white lines

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Main reference Eliot Bixby, Toby Flint, István Miklós: “Proving the Pressing Game Conjecture on Linear Graphs”, CoRR, Vol. abs/1303.6799, 2013.

URL <http://arxiv.org/abs/1303.6799>

Let a line graph L be given whose vertices are colored by black and white. Black vertices can be pressed. Pressing a black vertex has an effect that the black vertex is deleted, the neighbors of the black vertex change color and become neighbors (when there are two neighbors). A successful pressing sequence of L is a permutation of its vertices such that the vertices can be pressed in that order and the result is the empty graph (that is, the last step must be pressing the remaining single black vertex). The open question is the computational complexity of counting the successful pressing sequences of a black-and-white line graph. That is, is there a polynomial running time algorithm to compute this number or is the problem $\#P$ -complete? It is also an interesting problem to prove rapid mixing of an irreducible Markov chain on successful pressing sequences, see Bixby *et al.* (that paper also tells the motivation of the problem, which is actually a genome rearrangement problem).

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4.9 Particle systems for the longest pattern-avoiding subsequences for permutations

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The longest increasing subsequence problem for permutations can be rephrased as the longest 21-avoiding subsequence. This leads us to generalize the problem to the longest τ -avoiding subsequence for any given pattern τ [1, 2].

Hammersley’s interacting particle process on the unit interval $[0,1]$ corresponds to the longest 21-avoiding subsequence. This particle process is in KPZ universality class. It provides an efficient algorithm to numerically understand the distributional properties of the longest 21-avoiding subsequence problem under the uniform measure (21-case is solved theoretically).

Can we find particle processes corresponding to the longest τ -avoiding subsequence for a given pattern τ ? Specifically for patterns of length 3 except 321, which is known [3].

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4.10 Linear temporal spanners in temporal cliques

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A *temporal graph* \mathcal{G} is a pair (G, λ) , where $G = (V, E)$ is a simple undirected graph and λ is a function that assigns to every edge e of G a finite non-empty set of natural numbers. The temporal graph is *simple* if (1) every edge of G is assigned exactly one time label, i.e., $|\lambda(e)| = 1$ for every $e \in E$; and (2) no two edges get assigned the same time label. Without loss of generality, we will assume that if \mathcal{G} is simple, then λ is a bijection between E and $\{1, 2, \dots, |E|\}$. For this problem we focus only on simple temporal graphs.

A *temporal* (u, v) -*path* or a *temporal path* from u to v in \mathcal{G} is a path $u = u_0, u_1, \dots, u_\ell = v$ in G such that $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_\ell$, where $\lambda_i \in \lambda(u_{i-1}u_i)$.

The temporal graph \mathcal{G} is *temporally connected* if each vertex can reach every other vertex by a temporal path. A temporal graph $\mathcal{G}' = (G', \lambda')$ is a *temporal subgraph* of \mathcal{G} if G' is a subgraph of G and λ' is the restriction of λ to the edges of G' . Furthermore, if $V(G') = V(G)$ and \mathcal{G}' is temporally connected, then \mathcal{G}' is called a *temporal spanner* of \mathcal{G} .

► **Problem 1.** Let \mathcal{G} be an arbitrary simple temporal clique on n vertices, i.e. $\mathcal{G} = (K_n, \lambda)$, where K_n is a complete graph on n vertices. Is it true that \mathcal{G} has a temporal spanner with $O(n)$ edges?

It is known that any temporal clique on n vertices has a temporal spanner with $O(n \log n)$ edges [1]. Whether this bound is order optimal or not is open.

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